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ERGODIC ACTIONS AND SPECTRAL TRIPLES

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Dedicated to Marc A. Rieffel on the occasion of his 75th birthday.

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ABSTRACT. In this article, we give a general construction of spectral triples from certain Lie group actions on unital C^*-algebras. If the group G is compact and the action is ergodic, we actually obtain a real and finitely summable spectral triple which satisfies the first order condition of Connes' axioms. This provides a link between the "algebraic" existence of ergodic action and the "analytic" finite summability property of the unbounded selfadjoint operator. More generally, for compact G we carefully establish that our (symmetric) unbounded operator is essentially selfadjoint. Our results are illustrated by a host of examples — including noncommutative tori and quantum Heisenberg manifolds.

KEYWORDS: *Spectral triple, Lie group, ergodic action, Dirac operator, K-homology, unbounded Fredholm module.*

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1. INTRODUCTION

The Gelfand–Naimark theorem establishes an equivalence of categories between locally compact topological spaces and commutative C^*-algebras, which one takes advantage of in order to define noncommutative spaces.

In 1980, Connes introduced in [8] what he called a "differential structure" induced by a Lie group action on a C^*-algebra. This early notion was later superseded by the framework of *spectral triples*, devised axiomatically by Connes in [10], [11].

A choice of spectral triple comes down to fixing an unbounded operator on a representation space for a C^*-algebra which corresponds to a Dirac-type operator and is thus (in the unital case) supposed to be a self-adjoint operator of compact resolvent (see Definition 2.1).

Lie group actions and spectral triples thus provide two different approaches to "smoothness" for noncommutative spaces — for example, the boundedness of the commutators $[D, a]$ is a measure of regularity for a. The validity of this approach was confirmed to some extent by Connes' *reconstruction theorem* [12]: under a slight strengthening of the axioms of [11], spectral triples on commutative C^*-algebras arise from smooth manifolds.

If we think of C^*-algebras as sets of continuous functions on NC spaces, to study a "smooth noncommutative manifold" requires an analog of smooth functions on this "manifold". In other words, we need a "smooth subalgebra" $\mathscr{A} \subseteq A$. Such smooth subalgebras can be obtained in several ways (see e.g. [3], [4]). Here, we follow the familiar construction (see Proposition 2.4 below) obtaining $\mathscr{A}$ from an action of Lie group G on A. A natural question arises from this construction: if we consider a Lie subgroup $G_0 \subseteq G$, how do the two associated smooth subalgebras $\mathscr{A}_0$ and $\mathscr{A}$ differ?

In our case, this difficulty is solved by ergodicity: if the action of G is ergodic, we can construct a summable spectral triple for $\mathscr{A}$ — which need not be possible for G_0, if its action is not ergodic (see Example 8.2 below).

Our paper is inspired by Rieffel's article [33] in which he considers a Lie group G acting by α on a C^*-algebra A. He introduces unbounded Dirac-type operators of the form (see p. 226 of [33])

$$D = \sum \partial_{E_j} \otimes c(e_j), \tag{1.1}$$

where (E_j) and (e_j) are a basis for $\mathfrak{g}$ and its dual basis in $\mathfrak{g}^*$, respectively. They act by the differential ∂ of α and by a Clifford action c, respectively. His paper also puts special emphasis on ergodic actions. However, Rieffel does not study the formal properties of this operator — such as selfadjointness and compact resolvent. The expression (1.1) appeared in [7] too, where it is presented as a "general principle of construction of spectral triple".

In the present document we determine conditions under which this operator yields a spectral triple in the sense of Connes (see Definition 2.1 below) and study its properties including summability and Diximier trace. As [33] seems to suggest, we obtain a finitely summable spectral triple for ergodic actions of compact Lie groups. More precisely (see Theorem 5.5 below for the exact forms of the spectral triples):

THEOREM 1.1. *If G is a compact Lie group of dimension n acting ergodically on a unital C^*-algebra A, then*

(i) *there is a canonical n^+-summable spectral triple on A, which is even when n is even, has a real structure and satisfies the first order condition;*

(ii) *if we are given a covariant representation of (A, G) on $\mathscr{H}_0$ which satisfies an additional finiteness condition, then we obtain an n^+-summable spectral triple from it.*

The above theorem is relevant on two counts:

(1) it recovers for instance the usual spectral triples for noncommutative tori of any dimension (see Section 8);

(2) it links algebraic or "geometric" properties — namely the existence of a covariant representation — with analytic properties i.e. the selfadjointness and finite summability of D.

For covariant representations of non-compact groups of dimension n, we obtain a symmetric operator with bounded commutators (see Proposition 2.12), which is graded when n is even. Furthermore, if the Hilbert space of the triple comes from a G-invariant trace via the GNS construction, an associated real structure is available (see Proposition 3.4), thereby refining the "general principle" mentioned in [7].

A general mean of obtaining spectral triples is given in [13]. This construction is similar to ours in the sense that it assumes a certain symmetry on the initial space — a Riemannian manifold whose isometry group has rank at least 2 in [13], an ergodic action of compact Lie group for us — and estimates the summability of the resulting spectral triple. On the one hand, they rely on a deformation of a (commutative) geometric situation, while we have purely "noncommutative" assumptions, but on the other hand, their result yields orientability and Poincaré duality besides the summability, real structure and first order properties (see [11] for the definitions of these axioms).

Our results more closely ressemble the general construction presented in [37]. Nevertheless, the aforementioned article focuses on semifinite spectral triples and their index properties in the setting of general actions of compact Lie groups, while we put emphasis on "ordinary" spectral triples and their summability properties in the case of ergodic actions.

The notion of "ergodic action" — which plays a crucial role in our results — is well-studied, but our argument depends only on the seminal work [22] of Høegh-Krohn, Landstad and Størmer. Let us just mention that the case of ergodic actions of compact groups on von Neumann algebras was investigated in details by A. Wassermann in a series of articles [38], [39], [40]. We expect the vast literature on this topic to eventually provide new tractable classes of examples. However, many classical examples of spectral triples are already covered by our framework, as we try to indicate in the last section.

Another point of view on our results is that this article (together with the forthcoming [17]) provides some sort of "backward compatibility" of the original article [8] with the more recent framework of spectral triples.

This article starts with a preliminary Section 2, defining precisely the notion of spectral triple that we will use. We prove that, given a covariant representation of A and G on a Hilbert space $\mathscr{H}_0$, a symmetric unbounded operator D with bounded commutators arises naturally. In the next Section 3, we proceed with the particular case when $\mathscr{H}_0$ arises from the GNS construction and show that a real structure (implying the existence of a selfadjoint extension of D) exists in this case. Going back to general covariant representations, we establish carefully in Section 4 that if G is compact, D is essentially selfadjoint. These two threads of results are finally combined in Section 5, where the main theorem is established. Finally, the last Section 8 relates our results to prior work, by examining remarks, examples and counterexamples.

2. SPECTRAL TRIPLES AND COVARIANT REPRESENTATIONS

In this article, we almost exclusively consider *unital* C^*-algebras and *nondegenerated representations* $\pi : A \to B(\mathscr{H})$ of C*-algebras, i.e. $\pi(1_A) = \mathrm{id}_{\mathscr{H}}$.

The expression "spectral triple" has been used to denote several slightly different notions — thus we quickly remind the reader of the definition most suitable for us.

DEFINITION 2.1. Let A be a C^*-algebra. An odd *spectral triple*, also called odd *unbounded Fredholm module*, is a triple $(\pi, \mathscr{H}, D)$ where:

(i) $\mathscr{H}$ is a Hilbert space and $\pi : A \to B(\mathscr{H})$ a $*$-representation of A as bounded operators on $\mathscr{H}$;

(ii) a selfadjoint unbounded operator D — which we will call the *Dirac operator* — with such that:

(a) $\pi(a)(1 + D^2)^{-1}$ is compact for all $a \in A$,

(b) the subalgebra $\mathscr{A}$ of all $a \in A$ such that $\pi(a)(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$ and $[D, \pi(a)]$ extends to a bounded map on $\mathscr{H}$ is dense in A.

An *even spectral triple* is given by the same data, but we further require that a grading γ be given on $\mathscr{H}$ such that (1) A acts by even operators, (2) D is odd.

REMARK 2.2. In the above definition, we do not require that the representation π be faithful. However, many references (including Connes' articles) on this topic include this additional constraint.

The notion of spectral triple is refined by the different additional properties which one can require for them. The most complete collection of such "axioms" is surely the one proposed by Connes in his article [11] (later amended in [12] to prove his reconstruction theorem). Here, we will only need a few of these. In (essential) accordance with the nomenclature of [11] and [12], we call them *reality*, *order one*, *finite summability* and *finiteness*. References for them will be given later on where needed.

ASSUMPTION 2.3. For the rest of this article, G denotes a Lie group of finite dimension n which acts on a C^*-algebra A via a strongly continuous action α. We denote by $\mathfrak{g}$ the Lie algebra of G. We assume that we are given an Ad-invariant positive-definite scalar product μ on $\mathfrak{g}$ (e.g. in the case of a compact G, a negative scalar multiple of the Killing form κ).

Such a G-action defines a "smooth version" $\mathscr{A}$ of A (see for instance Proposition 3.45 p. 138 of [20] for a proof).

PROPOSITION 2.4. *The subalgebra $\mathscr{A}$ of G-smooth elements:*

$$\mathscr{A} := \{a \in A : g \mapsto \alpha_g(a) \text{ is in } C^\infty(G, A)\}$$

is a dense sub-$*$-algebra of A, with a natural Fréchet structure, which is stable under holomorphic functional calculus.

DEFINITION 2.5. Under Assumption 2.3, a *covariant representation of A and G on a Hilbert space $\mathscr{H}_0$* is a representation π of A together with a unitary and strongly continuous representation U of G on $\mathscr{H}_0$ which satisfy the following compatibility condition:

$$\pi(\alpha_g(a))\xi = U_g \pi(a) U_g^* \xi, \tag{2.1}$$

for all $a \in A$, $g \in G$ and $\xi \in \mathscr{H}_0$. In the rest of the present article, to simplify notations we will suppress the representation when unambiguous, so that for instance equation (2.1) reads $\alpha_g(a) = U_g a U_g^*$.

Given a covariant representation as above, we define a smooth domain $\mathscr{H}_0^\infty \subseteq \mathscr{H}_0$ by

$$\mathscr{H}_0^\infty := \{\xi \in \mathscr{H}_0 : g \mapsto U_g \xi \text{ is in } C^\infty(G, \mathscr{H}_0)\}. \tag{2.2}$$

One sees easily that $\mathscr{A}\mathscr{H}_0^\infty \subseteq \mathscr{H}_0^\infty$. Moreover, $\mathscr{H}_0^\infty$ is dense in $\mathscr{H}_0$, since it contains the dense "Gårding's domain" described in p. 306 of [29], and initially introduced in [19].

NOTATION 2.6. Given any vector $X \in \mathfrak{g}$, we denote by:

(i) $\partial_X^{\mathscr{A}}$ the associated infinitesimal generator of the action of G on A, i.e.

$$\partial_X^{\mathscr{A}}(a) := \lim_{t \to 0} \frac{\alpha_{\exp(tX)}(a) - a}{t}$$

for any $a \in \mathscr{A}$.

(ii) ∂_X the associated generator of the action of G on $\mathscr{H}_0$ defined in the same way. Then ∂_X satisfies the relation

$$\langle \partial_X \xi, \eta \rangle + \langle \xi, \partial_X \eta \rangle = 0. \tag{2.3}$$

For all $\xi \in \mathscr{H}_0^\infty$ and all $a \in \mathscr{A}$, taking the derivative of (2.1) we clearly have:

$$\partial_X^{\mathscr{A}}(a)\xi = \partial_X(a\xi) - a\partial_X \xi.$$

In other words, for all $a \in \mathscr{A}$ and all $X \in \mathfrak{g}$,

$$[\partial_X, a] = \partial_X^{\mathscr{A}}(a). \tag{2.4}$$

We refer to Section 5 p. 171 of [20], for facts about Clifford algebras. We just remind the reader of the definition.

DEFINITION 2.7. Given a real vector space (V, μ) with a positive definite scalar product, we can define its *complex Clifford algebra* $\mathbb{C}l(V, \mu)$, i.e. the universal unital C^*-algebra generated by $v \in V$ under the relations $v^* = -v$ and $\forall v, w \in V$,

$$v\,w + w\,v = -2\mu(v, w)1. \tag{2.5}$$

There is a natural $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathbb{C}l(V, \mu)$ induced by the automorphism h defined by $h(v) = -v$.

With this definition, up to isomorphism, $\mathbb{C}l(V, \mu)$ only depends on the dimension $n \in \mathbb{N}$ of V. We denote by $\mathbb{C}l(n)$ this unique algebra. We can specify an isomorphism $\mathbb{C}l(V, \mu) \simeq \mathbb{C}l(n)$ by choosing an orthonormal basis of V.

We can further identify $\mathbb{C}l(n)$. Indeed, the following isomorphisms are well known, see for example Lemma 5.5 p. 178 of [20]: for $N = 2^m$

$$\mathbb{C}l(2m) = M_N(\mathbb{C}), \quad \mathbb{C}l(2m+1) = M_N(\mathbb{C}) \oplus M_N(\mathbb{C}). \tag{2.6}$$

It follows from the above identifications that, up to unitary equivalence, there is a unique representation of $\mathbb{C}l(n)$ for even n, and that there are two inequivalent representations for odd n. A *chirality element* γ can be defined (compare Definition, p. 179 of [20]), which induces another grading operator. Indeed, it satisfies $\gamma^* = \gamma$ and $\gamma^2 = 1$ and moreover $\gamma v \gamma = v$ (in the odd case) or $\gamma v \gamma = -v$ (in the even case). In other words, in this later case the grading is inner. In the odd case, γ is in the center of $\mathbb{C}l(n)$. For irreducible representations, γ has to be sent to either 1 or -1. This distinguishes between the two possible irreducible representations of $\mathbb{C}l(n)$ for odd n. This also justifies that the chirality element does not appear in irreducible representations of odd Clifford algebras.

Section 2 of the article [15] provides an explicit set of generators of the irreducible representations of $\mathbb{C}l(n)$ for all n, together with a concrete involution J_S and (in the even case) a grading operator γ_S. We summarise these results in the following.

PROPOSITION 2.8. *Consider a positive integer n and an irreducible representation c of $\mathbb{C}l(n)$ on a complex vector space S. Up to unitary equivalence, it is determined by n matrices F_j such that*

$$F_j^* = -F_j, \quad F_j F_k + F_k F_j = -2\delta_{jk}. \tag{2.7}$$

If n is even, a grading operator γ_S is available which satisfies $\gamma_S^* = \gamma_S$, $\gamma_S^2 = 1$ and $\gamma_S F_j = -F_j \gamma_S$ for all j. There is an explicit anti-linear map J_S such that for all $j = 1, \dots, n$ and $s, s' \in S$,

$$\langle J_S s, J_S s' \rangle = \langle s', s \rangle, \quad J_S^2 = \varepsilon_J, \quad J_S F_j = \varepsilon_D F_j J_S, \quad J_S\,\gamma_S = \varepsilon_\gamma \gamma_S\,J_S,$$

where

(i) *the last equality (and therefore ε_γ) only appears in the even cases,*

(ii) *$\varepsilon_J, \varepsilon_D$ and ε_γ are all -1 or 1, the proper sign depending on n modulo 8:*

n	0	2	4	6	1	3	5	7
ε_J	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
ε_D	$+$	$+$	$+$	$+$	$-$	$+$	$-$	$+$
ε_γ	$+$	$-$	$+$	$-$				

REMARK 2.9. In the original article, the matrices F_j associated to $\mathbb{C}l(n)$ are denoted $\gamma^j_{(n)}$ (for n even) and $\gamma^j_{(n),\pm}$ (for n odd, the sign corresponding to the two irreducible representations) in [15]. They correspond to a choice of orthonormal basis (v_j) of V via $F_j := c(v_j)$. In the even case, the article [15] actually isolates two possible antilinear maps denoted $J_\pm$ (see Section 2.3 p. 1836).

REMARK 2.10. The article [15] has the advantage of providing an explicit description of γ_S and J_S. However, at a more conceptual level, we can also say that S comes with a *spinor representation* and as such it admits a $\mathbb{Z}/2\mathbb{Z}$-grading (see Definition 5.14 and Proposition 5.15 p. 36 of [25]). The operator J_S then identifies with a charge conjugation operator.

In the rest of this article, following Assumption 2.3, we consider the Clifford algebra $\mathbb{C}l(\mathfrak{g}^*, \mu^*)$ (also denoted simply $\mathbb{C}l(\mathfrak{g}^*)$), generated on the dual of the Lie algebra $\mathfrak{g}$ with the induced scalar product μ^*. We denote by S a fixed finite dimensional Hilbert space equipped with an irreducible representation c of the Clifford algebra $\mathbb{C}l(\mathfrak{g}^*)$. The map c restricts to a linear map $\mathfrak{g}^* \to B(S)$.

The tensor product $\mathfrak{g} \otimes \mathfrak{g}^*$ identifies naturally with the linear endomorphisms of $\mathfrak{g}$. Under this natural identification, the identity map $\mathrm{id}_\mathfrak{g}$ corresponds to $\mathcal{I} := \sum_j E_j \otimes e_j$ where (E_j) and (e_j) are respectively a basis for $\mathfrak{g}$ and its dual basis in $\mathfrak{g}^*$. Of course, $\mathcal{I}$ does not depend on the choice of basis (E_j) for $\mathfrak{g}$.

Following Notation 2.6, the differential ∂ of the action α on $\mathscr{H}_0$ defines a linear map from $\mathfrak{g}$ to linear maps on $\mathscr{H}_0^\infty$. Thus the image D of $\mathcal{I}$ under $\partial \otimes c$ defines a linear map on $\mathscr{H}_0^\infty \otimes S$, which does not depend on the choice of (E_j).

However, to express D more easily, we choose an orthonormal basis (E_j) for $(\mathfrak{g}, \mu)$. Its dual basis (e_j) is thus an orthonormal basis for $(\mathfrak{g}^*, \mu^*)$. For this choice of basis, we set $\partial_j := \partial_{E_j}$ and $F_j := c(e_j)$ and we can write:

$$D := (\partial \otimes c)(\mathcal{I}) = \sum_{j=1}^n \partial_{E_j} \otimes c(e_j) = \sum_{j=1}^n \partial_j \otimes F_j, \tag{2.8}$$

where the operators $F_j := c(e_j)$ satisfy the relations of Proposition 2.8 and D is a linear map defined on $\mathscr{H}_0^\infty \otimes S$. Of course, we can define similarly $\partial_j^\mathscr{A} := \partial_{E_j}^\mathscr{A}$. To summarise our previous discussion:

REMARK 2.11. The operator D defined in (2.8) is independent of the choice of basis (E_j) for $\mathfrak{g}$.

We next investigate the properties of the operator D.

PROPOSITION 2.12. *Let G be a Lie group of finite dimension n which acts on a C^*-algebra A via a strongly continuous action α. Suppose we are given a covariant representation of (A, G) on $\mathscr{H}_0$. The operator D of* (2.8) *is a symmetric unbounded operator D on $\mathscr{H} := \mathscr{H}_0 \otimes_{\mathbb{C}} S$ with domain* $\mathrm{Dom}(D) = \mathscr{H}_0^\infty \otimes S$. *Moreover,*

(i) *for any $a \in \mathscr{A}$, the commutators with D are bounded; more precisely:*

$$a(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D), \quad [D, a] = \sum \partial_j^{\mathscr{A}}(a) \otimes F_j;$$

(i) *if n is even, there is a selfadjoint grading operator γ such that for all $a \in A$,*

$$\gamma^2 = 1, \quad \gamma a = a\gamma, \quad \gamma(\mathrm{Dom}\, D) \subseteq \mathrm{Dom}(D), \quad \gamma D = -D\gamma.$$

DEFINITION 2.13. Under Assumption 2.3, an unbounded operator D obtained from a covariant representation of (A, G) on a Hilbert space $\mathscr{H}_0$ by formula (2.8) is called a *Lie–Dirac operator*. We also use this name for selfadjoint extensions of D when they exist (see e.g. Propositions 3.4 and 4.1).

REMARK 2.14. We note that:

(i) The above proposition, together with Proposition 2.4, already proves the requirement (ii) of Definition 2.1. Condition (i) of the same definition will be examined in Theorem 5.5.

(ii) When n is even, the operator γ determines a grading for an *even* spectral triple.

(iii) If we replace S with any finite (possibly reducible) representation S' of $\mathbb{C}l(n)$, equation (2.8) defines the operator D on $\mathscr{H} := \mathscr{H}_0 \otimes S'$ and the proof below applies verbatim.

Proof of Proposition 2.12. It is clear from the definition of $\mathscr{H}_0^\infty$ that D is defined on $\mathscr{H}_0^\infty \otimes S$. Let us first prove that D is symmetric on this domain: take $\xi \otimes s$ and $\xi' \otimes s'$ in $\mathscr{H}_0^\infty \otimes S$. By (2.3), we get:

$$\begin{aligned}\langle \xi \otimes s, D(\xi' \otimes s')\rangle &= \Big\langle \xi \otimes s, \sum \partial_j \xi' \otimes F_j s' \Big\rangle \\ &= \sum \langle \xi, \partial_j \xi'\rangle \langle s, F_j s'\rangle = \sum \langle -\partial_j \xi, \xi'\rangle \langle -F_j s, s'\rangle \\ &= \Big\langle \sum \partial_j \xi \otimes E_j s, \xi' \otimes s' \Big\rangle = \langle D(\xi \otimes s), \xi' \otimes s'\rangle.\end{aligned}$$

Any $a \in \mathscr{A}$ sends $\mathscr{H}_0^\infty \otimes S$ to itself and $[D, a]$ extends to a bounded operator: this is obvious from the definitions of $\mathscr{A}$ and $\mathscr{H}_0^\infty$ together with equation (2.4).

Let γ be the grading operator for the graded tensor product of the trivially graded space $\mathscr{H}_0$ and S graded by γ_S, or in other words, define the grading on $\mathscr{H}_0 \otimes S$ by the operator $\gamma := 1 \otimes \gamma_S$ in the notations of Proposition 2.8. We clearly get $\gamma^2 = 1$ and $\gamma^* = \gamma$. γ clearly commutes with the action of A. Moreover, $\mathrm{Dom}(D)$ is evidently mapped to itself by γ and the anticommutation relation with D is then easily checked using the properties of γ_S. ∎

PROPOSITION 2.15. *Assume that the adjoint action* Ad *of G on $\mathfrak{g}$ preserves the metric μ (see Assumption* 2.3*) on $\mathfrak{g}$. We say that a projective representation Φ of G on S*

implements the action $\beta := \mathrm{Ad}^*$ *on* $\mathfrak{g}^*$ *if* $\forall T \in \mathbb{C}l(\mathfrak{g}^*, \mu^*), \forall g \in G$,

$$\Phi_g c(T) \Phi_g^* = c(\beta_g(T)). \tag{2.9}$$

(i) *If n is even, then this spatial implementation always exists.*

(ii) *If n is odd, such* Φ *exist if and only if* Ad *preserves the orientation on* $\mathfrak{g}$.

(iii) *For n odd, replacing S by any finite representation* c' *on* S' *(as in Remark* 2.14*), if the orientation is not preserved, then* Φ *exists if and only if* $c'(\gamma)$ *and* $c'(-\gamma)$ *are equivalent in* $B(S')$.

In any case, if Φ *exists, then* $U \otimes \Phi$ *is a unitary projective representation of G on* $\mathscr{H}_0 \otimes S$ *under which D is invariant:* $\forall g \in G$,

$$(U_g \otimes \Phi_g) D (U_g^* \otimes \Phi_g^*) = D. \tag{2.10}$$

REMARK 2.16. A few points:

(i) If G is a connected semi-simple Lie group and $\mu = \kappa$ is the opposite of the Killing form, then both orientation and metric are preserved.

(ii) In general, we cannot require that Φ be an ordinary representation — but projective representations are enough for our purposes (see Proposition 6.1 below).

(iii) This covariance property (and its use of the contragredient representation $\beta := \mathrm{Ad}^*$) further justifies the identification of the Clifford algebra with $\mathbb{C}l(\mathfrak{g}^*)$.

(iv) We can check the ("unbounded") commutation relation (2.10) on finite dimensional spaces given by $E \otimes S$, where E is an irreducible component in the Peter–Weyl decomposition. Indeed, all $E \otimes S$ are globally invariant under both $U \otimes \Phi$ and D.

Proof of Proposition 2.15. We first notice that since the ∂_j are the infinitesimal generators associated to the action implemented by U_g, we have:

$$U_g \partial_j U_g^* = \sum \mathrm{Ad}_{jk}(g) \partial_k,$$

where $\mathrm{Ad}_{jk}(g)$ is the adjoint representation of G on $\mathfrak{g}$. If we can find Φ which implements the contragredient representation $\beta := \mathrm{Ad}^*$ of Ad on the vector space generated by the X_j, checking that D is invariant under $U \otimes \Phi$ is straightforward.

At this point, we want to find a unitary projective action Φ of G on S, which implements β in the sense of (2.9). To find such Φ, we can extend the orthogonal action β of $\mathfrak{g}^*$ into a Bogolyubov automorphism (see [20]) of $\mathbb{C}l(\mathfrak{g}^*, \mu^*)$ — which we also denote β.

In the even case, the identification (2.6) together with the fact that all automorphisms of $M_N(\mathbb{C})$ are inner ensure that for any Bogolyubov automorphism β, we can find a unitary $\Phi : S \to S$ such that $\forall T \in M_N(\mathbb{C})$, $\beta(T) = \Phi \circ T \circ \Phi^*$.

In the odd case, $\mathbb{C}l(2m+1)$ has two inequivalent irreducible representations S_+ and S_- with the same dimension $N = 2^m$ characterised by $\pi_{S_\pm}(\gamma) = \pm 1$. If

β is orientation preserving, then γ is invariant under the Bogolyubov automorphism, since γ does not depend on the choice of oriented orthonormal basis used to defined it — see Definition 5.2 p. 179 of [20].

Now consider a representation S'. We can decompose it into a direct sum of n_+ copies of S_+ and n_- copies of S_-. We have $\mathrm{Tr}(\pi_{S'}(\gamma)) = (n_+ - n_-)N$. If $\pi_{S'}(\gamma)$ is equivalent to $\pi_{S'}(-\gamma)$, then this trace vanishes, i.e. $n_+ = n_-$. Consequently, for all $g \in G$, $\pi_{S'} \circ \beta_g$ is equivalent to $\pi_{S'}$ and we can find a unitary Φ_g such that $\forall T \in \mathbb{C}l(n)$, $\pi_{S'} \circ \beta_g(T) = \Phi_g \pi_{S'}(T) \Phi_g^*$.

So in both even and odd cases, provided β_g preserves the orientation, we can "extract" an irreducible representation S of $\mathbb{C}l(n)$ and get for all $g \in G$ and all $T \in \mathbb{C}l(n)$, $\pi_S \circ \beta_g(T) = \Phi_g T \Phi_g^*$. We can compose these equations for $g_0, g_1 \in G$ using $\beta_{g_0 g_1} \circ \beta_{g_1^{-1}} \circ \beta_{g_0^{-1}} = \mathrm{id}$ and get, for all $T \in \mathbb{C}l(n)$:

$$\Phi_{g_0 g_1} \Phi_{g_1}^* \Phi_{g_0}^* \pi_S(T) \Phi_{g_0} \Phi_{g_1} \Phi_{g_0 g_1}^* = \pi_S(T).$$

Since the representation S is irreducible, the intertwiner $\Phi_{g_0 g_1} \Phi_{g_1}^* \Phi_{g_0}^*$ has to be a scalar — which in turn has to be in $U(1)$ by composition of unitaries. This proves that we get a projective representation.

Conversely, if β_g does not preserve the orientation, then $\beta_g(\gamma) = -\gamma$ and the operator Φ_g satisfies $\pi_{S'}(-\gamma) = \pi_{S'} \circ \beta_g(\gamma) = \Phi_g \pi_{S'}(\gamma) \Phi_g^*$, i.e. $\pi_{S'}(\gamma)$ and $\pi_{S'}(-\gamma)$ are equivalent in $B(S')$. ∎

REMARK 2.17. As a concrete example of a situation where the lifting in S fails, consider $U(1) \rtimes \mathbb{Z}/2\mathbb{Z}$ realised inside $U(1) \times \{\pm 1\}$ by the product $(z, \varepsilon) \cdot (z', \varepsilon') = (z(z')^\varepsilon, \varepsilon\varepsilon')$. If the Clifford algebra $\mathbb{C}l(1)$ is generated by 1 and F, then the adjoint representation gives $\mathrm{Ad}_{(0,-1)}(F) = -F$, which can not be implemented inside a one dimensional irreducible representation of $\mathbb{C}l(1)$.

3. GNS REPRESENTATION

LEMMA 3.1. *Under Assumption* 2.3, *given a G-invariant trace τ on A, the Hilbert space $\mathscr{H}_0$ obtained by the* GNS *construction from (A, τ) is equipped with a natural covariant representation of A and G.*

Proof. From the definition of $\mathscr{H}_0 = \mathrm{GNS}(A, \tau)$, the image $H := \{[a], a \in A\}$ of A in $\mathscr{H}_0$ is dense. We define the representations of A and G on this subset by:

$$\pi(a)([a']) = [aa'], \quad U_g([a]) = [\alpha_g(a)].$$

It is readily checked from these expressions that (2.1) is satisfied. Let us now prove that U_g is unitary:

$$\langle U_g([a]), U_g([a'])\rangle = \tau(\alpha_g(a)^* \alpha_g(a')) = \tau(a^* a') = \langle [a], [a'] \rangle$$

since τ is G-invariant. As α is strongly continuous on A, it is clear that U is strongly continuous on H. It then follows from a standard density argument that U is strongly continuous on $\mathscr{H}_0$. ∎

We need the following slight generalisation of Theorem X.3 of [30].

LEMMA 3.2. *If D is a densely defined unbounded symmetric operator on a Hilbert space $\mathscr{H}$ whose domain is $\mathrm{Dom}(D) \subset \mathscr{H}$ and J is an antilinear map on $\mathscr{H}$ such that for all*

$$\langle J\xi, J\eta\rangle = \langle \eta, \xi\rangle, \quad J^2 = \varepsilon_J, \quad J(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D), \quad JD = \varepsilon_D DJ, \tag{3.1}$$

where ε_J and ε_D in $\{-1, 1\}$, then D admits a selfadjoint extension.

REMARK 3.3. An antilinear operator J which satisfies the equation (3.1) for all $\xi, \eta \in \mathscr{H}$ is called *norm-preserving*.

Proof of Lemma 3.2. We first consider the cases $\varepsilon_D = 1$. If $\varepsilon_J = 1$, we are back to the hypotheses of Theorem X.3 p. 143 of [30], thus the conclusion holds.

If $\varepsilon_J = -1$, we reduce the situation to the previous case by an easy computation using tensor products: define an antilinear map C on $\mathbb{C}^2$ by $C\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\overline{x}_2 \\ \overline{x}_1 \end{pmatrix}$ and then set

$$D_2 = D \otimes 1, \quad J_2 = J \otimes C,$$

where the domain of D_2 is $\mathrm{Dom}(D) \oplus \mathrm{Dom}(D)$. Clearly, the tensor product (over $\mathbb{C}$) of two antilinear maps is well-defined and J_2 is antilinear. The following facts are easily checked using tensor products: D_2 is symmetric, $J_2^2 = 1$, J_2 preserves the norm of $\mathscr{H} \otimes \mathbb{C}^2$ and the domain of D_2. Finally, J_2 and D_2 commute. Hence, we are back to the previous hypotheses and D_2 admits a selfadjoint extension, denoted $\widetilde{D}_2$. It is written as a diagonal block matrix, thus the upper left entry denoted $\widetilde{D}$ is already a selfadjoint operator, which extends D.

It remains to treat the cases of $\varepsilon_D = -1$. We use the same kind of argument: introduce on $\mathbb{C}^2$ the antilinear operator $C'\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \overline{x}_2 \\ \overline{x}_1 \end{pmatrix}$ and then set

$$D_2 = D \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_2 = J \otimes C,$$

the domain of D_2 being $\mathrm{Dom}(D) \oplus \mathrm{Dom}(D)$. Clearly, we have $J_2^2 = J^2 \otimes 1_{\mathbb{C}^2} = \varepsilon_J$, J_2 preserves the norm of $\mathscr{H} \otimes \mathbb{C}^2$ and sends $\mathrm{Dom}(D_2)$ to itself.

Moreover, J_2 and D_2 commute algebraically, as a simple computation using tensor products shows.

We are then back to the previous cases of $\varepsilon_D = 1$. Hence, D_2 admits a selfadjoint extension and we can apply the same argument as in the previous case to extract a selfadjoint extension of D. ∎

Under the same hypotheses as in Lemma 3.1, the Dirac operator satisfies additional properties:

PROPOSITION 3.4. *If* $\mathscr{H}_0 = \mathrm{GNS}(A, \tau)$ *for a G-invariant trace and D is an associated Lie–Dirac operator (see Definition* 2.13*) then:*

(i) *the operator D admits a real structure, i.e. there is a norm-preserving antilinear map* $J : \mathscr{H} \to \mathscr{H}$ *such that for all* $a, b \in A$

$$[a, Jb^* J^{-1}] = 0, \quad J(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D), \quad JD = \varepsilon_D DJ,$$

and $J^2 = \varepsilon_J$, *where* ε_J *and* ε_D *are given by the table of Proposition* 2.8*; in the even case, there is an additional equality:* $J\gamma = \varepsilon_\gamma \gamma J$ *and the value of* ε_γ *is also given by the previous table;*

(ii) *D and J satisfy the first order condition, i.e. for all* $a, b \in \mathscr{A}$,

$$[[D, a], Jb^* J^{-1}] = 0;$$

(iii) *D admits a selfadjoint extension* $\widetilde{D}$.

REMARK 3.5. The above proposition relates to the axioms of [11], [12] in the following way:

(i) Point (i) precisely shows the *reality* condition of [11] (see Axiom (7′) p. 163).

(ii) Point (ii) is the first order condition denoted Axiom (2′) p. 164 of [11].

Of course, it still remains to take care of the compactness of $(1 + D^2)^{-1/2}$.

Proof of Proposition 3.4. We continue to use the notation of the proof of Lemma 3.1. We define an antilinear operator J_0 on H by $J_0([a]) = [a^*]$. This operator preserves the norm on H:

$$\langle J_0([a]), J_0([b]) \rangle = \langle [a^*], [b^*] \rangle = \tau(ab^*) = \tau(b^* a) = \langle [b], [a] \rangle$$

since τ is a trace. It thus extends to an antilinear operator on all of $\mathscr{H}_0$. We then set $J := J_0 \otimes J_S$ — which is well-defined since both J and J_S are antilinear. Moreover, both J_0 and J_S are norm-preserving which implies that J is also norm-preserving.

As both a and $Jb^* J^{-1}$ are bounded (linear) operators on $\mathscr{H}$, it suffices to check the commutation condition on the subset $H \otimes S \subseteq \mathscr{H}_0 \otimes S$, i.e.

$$\begin{aligned} a\, Jb^* J^{-1}([c] \otimes s) &= aJ([b^* c^*] \otimes J_S^{-1} s) = [acb] \otimes s \\ &= J([b^* c^* a^*] \otimes J_S^{-1} s) = Jb^* J([ac] \otimes s) = Jb^* J\, a([c] \otimes s). \end{aligned}$$

Regarding the first order condition, since both $[D, a]$ and $Jb^* J^{-1}$ are bounded linear operators, it suffices to prove the commutation relation on $[c] \otimes s$ for $c \in A$ and $s \in S$. We then have:

$$\begin{aligned} [D, a]\, Jb^* J^{-1}([c] \otimes s) &= \Big(\sum \partial_j^{\mathscr{A}}(a) \otimes F_j \Big)([cb] \otimes s) = \sum [\partial_j^{\mathscr{A}}(a) cb] \otimes F_j s \\ &= Jb^* J^{-1} \Big(\sum [\partial_j^{\mathscr{A}}(a) c] \otimes F_j s \Big) = Jb^* J^{-1}\, [D, a]([c] \otimes s). \end{aligned}$$

Finally, the existence of a selfadjoint extension $\widetilde{D}$ of D follows immediately from the existence of the real structure J and Lemma 3.2. ∎

4. COVARIANT REPRESENTATIONS OF COMPACT LIE GROUPS

Going back to general covariant representations, in the case of a *compact* Lie group, the operator D is essentially selfadjoint, so we do not need to *choose* a selfadjoint extension.

PROPOSITION 4.1. *Let $\mathscr{H}_0$ be a Hilbert space endowed with a covariant representation of (A, G). If G is compact, then the associated Lie–Dirac operator D is essentially selfadjoint.*

REMARK 4.2. In full generality, whenever the Lie–Dirac operator D is essentially selfadjoint, then properties (i)–(ii) of Proposition 2.12 and (i)–(ii) of Proposition 3.4 (if applicable) also hold for $\overline{D}$ — see for instance Proposition 2 in the appendix of [27].

Proof of Proposition 4.1. Peter–Weyl's decomposition theorem enables us to write $\mathscr{H}_0$ as a direct sum of G-representations:

$$\mathscr{H}_0 = \bigoplus_{\ell} E_\ell \otimes \mathbb{C}^{m_\ell} \tag{4.1}$$

where

(i) ℓ is a multi-index labelling the *highest weight* of a representation of G,

(ii) E_ℓ is a finite dimensional Hilbert space, endowed with the representation π_ℓ of G of highest weight ℓ,

(iii) m_ℓ is the multiplicity of E_ℓ inside $\mathscr{H}_0$.

For each ℓ, we can fix m_ℓ subspaces $E_{\ell,k} \subseteq \mathscr{H}_0$ for $k = 1, \ldots, m_\ell$ which are pairwise orthogonal, unitarily equivalent to E_ℓ and exhaust the E_ℓ component of $\mathscr{H}_0$. Denoting $P_{\ell,k}$ the associated projections on $\mathscr{H}_0$ and $Q_{\ell,k} := P_{\ell,k} \otimes 1_S$, we have $Q_{\ell,k} \operatorname{Dom}(D) \subseteq E_{\ell,k} \otimes S \subseteq \operatorname{Dom}(D)$ and $Q_{\ell,k}$ commutes with D. To prove that D is essentially selfadjoint, it suffices to prove that $\operatorname{Ran}(D + \mathrm{i})$ and $\operatorname{Ran}(D - \mathrm{i})$ are dense (see Corollary p. 257 of [29]). Using the decomposition (4.1) and the commutation of $Q_{\ell,k}$ with D, it suffices to prove that for each ℓ, k, $E_{\ell,k} \otimes S = \operatorname{Ran}(Q_{\ell,k} D \pm \mathrm{i})$. Since $Q_{\ell,k} D$ is a symmetric operator on the finite dimensional space $E_{\ell,k} \otimes S$, it induces a basis of eigenvectors whose eigenvalues are real. This implies that both $Q_{\ell,k} D + \mathrm{i}$ and $Q_{\ell,k} D - \mathrm{i}$ are surjective and completes the proof. ∎

5. ERGODIC ACTIONS

In the particular case of *ergodic actions* of compact Lie groups, we can even estimate the summability of the closure $\overline{D}$ of D. To lighten notation, we sometimes write D instead of $\overline{D}$.

DEFINITION 5.1. The action α of G on A is *ergodic* if the only G-invariants elements are the scalars, i.e. if $\forall g \in G, \alpha_g(a) = a$, then $a \in \mathbb{C}1_A$.

REMARK 5.2. If G is compact and the action is ergodic, then a theorem of Høegh-Krohn, Landstad and Størmer (see Theorem 4.1 p. 82 of [22]) proves that the unique G-invariant state is actually a trace τ. Hence the existence of a G-invariant trace τ is automatic!

We now need a brief reminder regarding *Dixmier trace ideals*, for which we follow Chapter IV of [9]. For more details concerning symmetrically normed operator ideals and singular traces we refer the reader to [26] and [36].

DEFINITION 5.3. For $p > 1$, the ideal $\mathcal{L}^{p^+}$ (also denoted $\mathcal{L}^{(p,\infty)}$ in [9] and $\mathscr{I}_{p,\omega}$ in p. 21 of [36]) is given by the compact operators T on $\mathscr{H}$ such that

$$\|T\|_{p+} := \sup_k \frac{\sigma_k(T)}{k^{(p-1)/p}} < \infty,$$

where $\sigma_k(T)$ is defined as the supremum of the trace norms of TE, when E is an orthonormal projection of dimension k, i.e.

$$\sigma_k(T) := \sup\{\|TE\|_1, \dim E = k\}.$$

Equivalently, $\sigma_k(T)$ is the sum of the k largest eigenvalues (counted with their multiplicities) of the positive compact operator $|T| := T^*T$. We extend the definition to the case of $p = 1$: $\mathcal{L}^{1+}$ is the ideal of compact operators T such that

$$\|T\|_{1+} := \sup_k \frac{\sigma_k(T)}{\log k} < \infty. \tag{5.1}$$

The elements of $\mathcal{L}^{p^+}$ are called *p^+-summable* (or (p,∞)-summable — see Section IV.2 α p. 299 and following, of [9]).

The following notion is then the analogue of dimension for spectral triples.

DEFINITION 5.4. A spectral triple is *p^+-summable* if $(1 + D^2)^{-1/2} \in \mathcal{L}^{p^+}$.

Examples of n^+-summable spectral triples include spin manifolds of dimension n equipped with their Dirac operators — see Theorem 11.1 p. 488 and Theorem 7.18 p. 293 of [20]. This last property is related to Weyl's theorem. More material on this topic is available in [34], especially Chapter 8 therein. We are now ready to state our main

THEOREM 5.5. *Let G be a compact Lie group of dimension n acting ergodically on a unital C^*-algebra A, then*

(i) *using the unique G-invariant trace τ of Remmark* 5.2, $\mathscr{H}_0 := \mathrm{GNS}(A, \tau)$ *is endowed with a covariant representation of (A, G) and the Lie–Dirac operator*

$$D = \sum_{j=1}^{n} \partial_j \otimes F_j, \tag{5.2}$$

is essentially selfadjoint; its closure $\overline{D}$ defines an n^+-summable spectral triple $(A, \mathscr{H}_0 \otimes S, \overline{D})$ on A, which is even when n is even, carries a real structure and satisfies the first order condition (as in Proposition 3.4*);*

(ii) *given a covariant representation of (A, G) on $\mathscr{H}_0$ such that the space of G-smooth elements $\mathscr{H}_0^\infty$ is a finitely generated projective module on $\mathscr{A}$, the spectral triple $(A, \mathscr{H}_0 \otimes S, \overline{D})$ obtained from* (5.2) *is n^+-summable and even if n is even.*

REMARK 5.6. This theorem finally proves condition (i) of Definition 2.1! The other properties of the spectral triple are immediate consequences of Proposition 2.12 and 3.4.

In condition (ii), the notation $\mathscr{H}_0^\infty$ is the same as in (2.2). The condition on $\mathscr{H}_0^\infty$ as a $\mathscr{A}$-module mimicks the *finiteness* Axiom (5) found in p. 160 of [11].

REMARK 5.7. The crucial step of the proof below is to control the multiplicities appearing in the Peter–Weyl decomposition Theorem (4.1). Here, we rely on ergodicity and an estimate provided by [22]. However, other means of controlling these multiplicities should lead to analogs of Theorem 5.5 for more general settings.

Proof of Theorem 5.5. We begin with point (i) of the theorem. We are going to prove that D has compact resolvent and is finitely summable by comparing it to the Lie–Dirac operator D_{ref} associated to the C^*-algebra $C(G)$ and its covariant representation on $L^2(G)$ — equipped with the left regular representation V_g of G. Consequently, D_{ref} acts on the Hilbert space $\mathscr{H}_{\mathrm{ref}} := L^2(G) \otimes S$.

Since the tangent space TG is trivial, D_{ref} is actually a Dirac operator on the compact manifold G — equipped with its trivial spin structure. In particular, D_{ref} has compact resolvent and is n^+-summable.

We can now decompose $\mathscr{H}_{\mathrm{ref}}$ for the unitary representation $V_g \otimes 1$ using Peter–Weyl's theorem. The result is a sum of Hilbert G-spaces:

$$\mathscr{H}_{\mathrm{ref}} = \bigoplus_{\ell} E_\ell \otimes \mathbb{C}^{d_\ell} \otimes S \tag{5.3}$$

where

(a) ℓ is a multi-index labelling the *highest weight* of a representation of G,

(b) E_ℓ is a finite dimensional Hilbert space, endowed with the representation π_ℓ of G of highest weight ℓ,

(c) d_ℓ is the dimension of E_ℓ, which is also the multiplicity of E_ℓ inside $L^2(G)$.

We need to perform the same decomposition for $\mathscr{H}_0$. We first decompose A into its *spectral subspaces* (also called *isotypical components*): given a highest weight ℓ, we interpret the associated representation π_ℓ on E_ℓ as a $d_\ell \times d_\ell$-matrix and write $\chi_\ell(g) = d_\ell \operatorname{Tr}(\pi_\ell(g^{-1}))$ for its normalised character. The associated spectral subspace A_ℓ is the norm closed subspace defined by:

$$A_\ell := \overline{\left\{ \int_G \chi_\ell(g)\alpha_g(a)\mathrm{d}g : a \in A \right\}} \subseteq A.$$

Relation (2.2.40) on p. 45 of [5] proves that the (algebraic) direct sum $A^{\mathrm{alg}} := \bigoplus^{\mathrm{alg}} A_\ell$ is norm dense in A. From p. 76 of [22], we get that A_ℓ decomposes into a direct sum of irreducible components, each equivalent to E_ℓ. Moreover, Proposition 2.1 in the same article ensures that the multiplicity m_ℓ of E_ℓ inside A_ℓ satisfies $m_\ell \leqslant d_\ell$ — thereby proving that the dimension of A_ℓ is bounded by d_ℓ^2.

Relying on the dense subset A^{alg} of A, it is easy to prove that the unique G-invariant trace τ is faithful and therefore $\mathscr{H}_0$ decomposes as a sum of Hilbert spaces:

$$\mathscr{H}_0 = \bigoplus E_\ell \otimes \mathbb{C}^{m_\ell}.$$

Comparing the above expression with (5.3), it is clear that there is an inclusion $\iota_0 : \mathscr{H}_0 \to L^2(G)$ which commutes with the action of G. This inclusion extends to an inclusion $\iota : \mathscr{H} \to \mathscr{H}_{\mathrm{ref}}$.

Since D_{ref} has compact resolvent, $\mathscr{H}_{\mathrm{ref}}$ admits a Hilbertian basis of eigenvectors. If for each ℓ, we choose a decomposition $\bigoplus_{k=1}^{d_\ell} E_{\ell,k}$ of the term $E_\ell \otimes \mathbb{C}^{d_\ell}$ in (5.3), it is easily checked that the associated projections $P_{\ell,k}$ intertwine the action of G on $\mathscr{H}_{\mathrm{ref}}$ and thus commute with D_{ref}. Moreover, we can pick the spaces $E_{\ell,k}$ so that if $k \leqslant m_\ell$, then $E_{\ell,k} \in \iota(\mathscr{H})$ and if $k > m_\ell$, then $E_{\ell,k}$ is orthogonal to $\iota(\mathscr{H})$.

Hence, we can choose a basis of $\mathscr{H}_{\mathrm{ref}}$ of eigenvectors of D_{ref}. Since $P_{\ell,k}$ commutes with D_{ref}, we can even take a basis which is compatible with the decomposition $1_{\mathscr{H}_{\mathrm{ref}}} = \bigoplus P_{\ell,k}$. It is then clear that D_{ref} and D coincide on the relevant building blocks $E_{\ell,k}$, therefore providing a basis of $\mathscr{H}$ of eigenvectors for D. The associated eigenvalues are the same for D_{ref} and D. In particular, since D_{ref} has compact resolvent, so does D.

To prove that D is n^+-summable, note first that as a consequence of the previous discussion the eigenvalues $(\mu_k)_{k\in\mathbb{N}}$ of $|D|$ are the same as those $(\lambda_k)_{k\in\mathbb{N}}$ of $|D_{\mathrm{ref}}|$, but they have lower (possibly vanishing) multiplicities. This implies that if the eigenvalues (μ_k) and (λ_k) (repeated with their multiplicities) are in increasing order, then for all $k \in \mathbb{N}$, $\lambda_k \leqslant \mu_k$.

Formally, we want to prove that $(1 + D^2)^{-1/2}$ is in the ideal $\mathcal{L}^{n+}$ and we already know that $(1 + D_{\mathrm{ref}}^2)^{-1/2} \in \mathcal{L}^{n+}$.

Since the function $f(x) = (1 + x^2)^{-1/2}$ is decreasing on $\mathbb{R}^+$, we see that if (λ_k) is the sequence of eigenvalues of $|D_{\mathrm{ref}}|$ in *increasing* order, then $(\lambda'_n) :=$

$(f(\lambda_k))$ is the sequence of eigenvalues of $(1 + D_{\mathrm{ref}}^2)^{-1/2}$ in *decreasing* order. Moreover, as $\lambda_k \leqslant \mu_k$, we have $\lambda'_k = f(\lambda_k) \geqslant \mu'_k := f(\mu_k)$. In the notations of Definition 5.3, we have:

$$\sigma_{k,\mathrm{ref}} := \sigma_k((1 + D_{\mathrm{ref}}^2)^{-1/2}) = \sum_{p=0}^{k-1} \lambda'_p\,, \quad \sigma_k := \sigma_k((1 + D^2)^{-1/2}) = \sum_{p=0}^{k-1} \mu'_p\,,$$

and the monotony of f implies $\sigma_{k,\mathrm{ref}} \geqslant \sigma_k$. That $(1 + D_{\mathrm{ref}}^2)^{-1/2} \in \mathcal{L}^{n+}$ means

$$\|(1 + D_{\mathrm{ref}}^2)^{-1/2}\|_{n^+} = \sup_k \frac{\sigma_{k,\mathrm{ref}}}{k^{(n-1)/n}} < \infty.$$

This in turn implies

$$\|(1 + D^2)^{-1/2}\|_{n^+} = \sup_k \frac{\sigma_k}{k^{(n-1)/n}} \leqslant \sup_k \frac{\sigma_{k,\mathrm{ref}}}{k^{(n-1)/n}} < \infty$$

and completes the proof that D is n^+-summable.

We now turn our attention to point (ii). The proof is essentially the same, but we need to compare D with $D_{\mathrm{ref},k} := D_{\mathrm{ref}} \otimes 1_k$ acting on $L^2(G) \otimes S \otimes \mathbb{C}^k$ instead of D_{ref}. The finiteness condition on $\mathscr{H}_0^\infty$ can be written $\mathscr{H}_0^\infty \simeq p\mathscr{A}^k$, thus yielding a covariant inclusion $\mathscr{H}_0 \subseteq \mathrm{GNS}(A \otimes \mathbb{C}^k, \underline{\tau})$ where $\underline{\tau}$ is defined on $A \otimes \mathbb{C}^k$ by:

$$\underline{\tau}(a_1, \ldots, a_k) = \frac{1}{k} \sum_{j=1}^{k} \tau(a_j).$$

This is indeed a state since it is the restriction to diagonal matrices of the state $\tau \otimes \mathrm{Tr}$ defined on $A \otimes M_k(\mathbb{C})$.

$D_{\mathrm{ref},k}$ is n^+-summable since functional calculus leads to

$$(1 + D_{\mathrm{ref},k}^2)^{-1/2} = (1 + D_{\mathrm{ref}}^2)^{-1/2} \otimes 1_k,$$

and this is a finite sum of operators $\sum_j (1 + D_{\mathrm{ref}}^2)^{-1/2} \otimes e_{jj}$ which are all in the ideal $\mathcal{L}^{n+}$. Another way to prove this property would be to use the "scale invariance" considered in IV.2.β p. 305 of [9].

In any case, we conclude as in point (i): using a Peter–Weyl decomposition, we see that $D_{\mathrm{ref},k}$ and D coincide on the relevant irreducible components and a comparison of multiplicities leads to the n^+-summability of D. ∎

REMARK 5.8. The above proof is based on methods from noncommutative geometry but another proof using only representation theory should be possible. It would involve a comparison of D^2 with Δ (see equation (6.2) below). Δ is the Casimir operator and so can be treated by representation theoretic methods. Together with the estimate of Proposition 2.1 in [22], it would follow that $(1 + \Delta)^{-1}$ is n^+-summable.

6. EVALUATION OF DIXMIER TRACE

The n^+-summability of D established in the previous section enables us to use the Dixmier trace Tr_ω (see IV.2 of [9]):

PROPOSITION 6.1. *Let G be a compact Lie group of dimension n acting ergodically on a unital C^*-algebra A, consider a state ω of B_∞ as in Definition* 7.16 *p.* 288 *of* [20] *and a spectral triple $(A, \mathscr{H}, D)$ as in Theorem* 5.5.

Given a positive real number n_0 such that $(1+D^2)^{-n_0/2} \in \mathcal{L}^{1+}$, for any $T \in B(\mathscr{H})$ we can evaluate the Dixmier trace Tr_ω in the expression $\phi(T) := \mathrm{Tr}_\omega(T(1+D^2)^{-n_0/2})$. In particular, there is a scalar λ such that for any $a \in A$,

$$\mathrm{Tr}_\omega(a(1+D^2)^{-n_0/2}) = \lambda\tau(a). \tag{6.1}$$

REMARK 6.2. Theorem 5.5 ensures the existence of real numbers $n_0 \geqslant n$ such that $(1+D^2)^{-n_0/2} \in \mathcal{L}^{1+}$: according to Lemma 7.37 p. 316 of [20], since $(1+D^2)^{-1/2}$ is positive and in $\mathcal{L}^{n+}$, $(1+D^2)^{-n/2} \in \mathcal{L}^{1+}$.

Proof of Proposition 6.1. It follows directly from the hypothesis $(1+D^2)^{-n_0/2} \in \mathcal{L}^{1+}$ that for any $T \in B(\mathscr{H})$, $T(1+D^2)^{-n_0/2}$ is in the ideal $\mathcal{L}^{1+}$ of $B(\mathscr{H})$ and we can evaluate $\phi(T)$. The precise value of $\phi(T)$ may depend on the choice of ω, however.

A consequence of Proposition 2.15 above is that for any $g \in G$, we have

$$(U_g \otimes \Phi_g)(1+D^2)^{-n_0/2}(U_g \otimes \Phi_g)^* = (1+D^2)^{-n_0/2}.$$

Since Tr_ω is a trace (see Proposition 3 (b) p. 306 of [9]) we have:

$$\begin{aligned}\phi(\alpha_{g^{-1}}(a)) &= \mathrm{Tr}_\omega((U_g \otimes \Phi_g)^*(a \otimes \mathrm{id}_S)(U_g \otimes \Phi_g)(1+D^2)^{-n_0/2}) \\ &= \mathrm{Tr}_\omega((a \otimes \mathrm{id}_S)(U_g \otimes \Phi_g)(1+D^2)^{-n_0/2}(U_g \otimes \Phi_g)^*) \\ &= \mathrm{Tr}_\omega((a \otimes \mathrm{id}_S)(1+D^2)^{-n_0/2}) = \phi(a).\end{aligned}$$

Theorem 4.1 p. 82 of [22] proves that all G-invariant linear forms on A are proportional to the unique normalised G-invariant trace τ, so (6.1) is proved. ∎

Under a slightly different condition, we can refine the result. To this end, we follow the conventions of [9] (see p. 303) and write $\mathcal{L}_0^{1+}$ for the closure of the finite rank operators under the norm $\|\cdot\|_{1+}$ defined in (5.1). It follows from p. 288 of [20] that $\mathcal{L}_0^{1+}$ lies inside the common kernel of all Dixmier traces.

THEOREM 6.3. *Under the same hypotheses as Proposition* 6.1 *above,*

(i) *the positive unbounded operator $\Delta := -\sum_j \partial_j^2 \otimes 1$ with domain* $\mathrm{Dom}(\Delta) = \mathscr{H}_0^\infty \otimes_{\mathbb{C}} S$ *is essentially selfadjoint and $(1+\Delta)^{-n/2} \in \mathcal{L}^{1+}$.*

Given a positive real number n_0 such that $(1+\Delta)^{-n_0/2} \in \mathcal{L}^{1+}$,

(ii) *the operator $(1+D^2)^{-n_0/2}$ is in $\mathcal{L}^{1+}$ and*

$$(1+D^2)^{-n_0/2} - (1+\Delta)^{-n_0/2} \in \mathcal{L}_0^{1+}; \tag{6.2}$$

(iii) *the form* $\phi(T) := \mathrm{Tr}_\omega(T(1+D^2)^{-n_0/2})$ *extends to a trace on* $A \otimes B(S)$ *and*

$$\mathrm{Tr}_\omega(T(1+D^2)^{-n_0/2}) = \lambda(\tau \otimes \mathrm{Tr}_S)(T), \tag{6.3}$$

where the trace Tr_S *on the finite dimensional algebra* $B(S)$ *satisfies* $\mathrm{Tr}_S(1) = 1$ *and* λ *is the same as in equation* (6.1) *above.*

REMARK 6.4. A few comments on the above theorem, using the involutive algebra Ω generated by A and the commutators $\{[D,a], a \in \mathscr{A}\}$.

(i) It is clear from the expression Proposition 2.12 above that $\Omega \subseteq A \otimes B(S)$, and therefore the equation (6.3) evaluates the Dixmier trace on Ω.

(ii) We remind the reader that a spectral triple is *tame* in the sense of p. 465 in [20], if the functional $T \mapsto \mathrm{Tr}_\omega(T(1+D^2)^{-n_0/2})$ is a trace on Ω. It follows from the above Theorem 6.3 that the spectral triple of Theorem 5.5 is tame.

(iii) The condition on $(1+\Delta)^{-n_0/2}$ is actually easier to evaluate in concrete cases than the original condition on $(1+D^2)^{-n_0/2}$ — see Example 7.1 below.

Proof of Theorem 6.3. Regarding item (i), it is clear that Δ is positive as sum of the positive operators $-\partial_j^2 \otimes 1$. Proving that Δ is essentially selfadjoint proceeds as in Theorem 5.5 above. We also follow the proof of this theorem to show that this operator is n^+-summable.

Only this time, we compare Δ to the Casimir operator Δ_{ref} acting on the manifold G. In this situation, the expression $(1+\Delta_{\mathrm{ref}})^{-n/2}$ represents an elliptic pseudodifferential operator of order $-n$ on compact Riemannian manifold of dimension n, which is therefore in $\mathcal{L}^{1+}$ by Theorem 7.18 p. 293 of [20].

To compare D^2 and Δ in item (ii), notice first that if we can prove (6.2), then it follows that $(1+D^2)^{-n_0/2} \in \mathcal{L}^{1+}$. An easy computation yields:

$$D^2 = -\sum_j \partial_j^2 \otimes 1 + \sum_{j<k} [\partial_j, \partial_k] \otimes F_j F_k.$$

Selecting a couple $j < k$, we denote $\lambda_{j,k}$ the (absolute value of the) norm of $[\partial_j, \partial_k]$ measured using the metric μ and $T_{j,k} := [\partial_j, \partial_k] \otimes F_j F_k$.

For any ξ in a finite dimensional Hilbert space involved in the Peter–Weyl decomposition (4.1), we have:

$$\langle \xi, T_{j,k}^* T_{j,k} \xi \rangle \leqslant \lambda_{j,k}^2 \langle \xi, (1+\Delta)\xi \rangle.$$

Indeed, we know that the expression of the Casimir operator actually does not depend on the choice of orthonormal basis in $\mathfrak{g}$, so up to renormalisation, $[\partial_j, \partial_k]$ appears in the sum defining Δ. From this relation, we get

$$\langle \eta, (1+\Delta)^{-1/2} T_{j,k}^* T_{j,k} (1+\Delta)^{-1/2} \eta \rangle \leqslant \lambda_{j,k}^2 \langle \eta, \eta \rangle,$$

which in turn proves that $T_{j,k}(1+\Delta)^{-1/2}$ extends to a bounded operator on $\mathscr{H}$. At this point, remark that $(1+\Delta)^{-1/2}$ commutes with $T_{j,k}$, as can be easily checked on the finite dimensional spaces obtained from the Peter–Weyl decomposition.

If we write $T := \sum_{j<k} T_{j,k} = D^2 - \Delta$, the above shows that $T(1+\Delta)^{-1/2}$ is a bounded operator. Moreover, the previous commutation relation proves that $T(1+\Delta)^{-1/2} = (1+\Delta)^{-1/4}T(1+\Delta)^{-1/4}$ is selfadjoint. Using this commutation relation, we want to write:

$$(1+D^2)^{-n_0/2} - (1+\Delta)^{-n_0/2} = ((1+T(1+\Delta)^{-1})^{-n_0/2} - 1)(1+\Delta)^{-n_0/2}. \tag{6.4}$$

Of course, this expression only makes sense if $1+T(1+\Delta)^{-1}$ is invertible, which may not be satisfied. However, it appears from the previous discussion that $T(1+\Delta)^{-1} = T(1+\Delta)^{-1/2}(1+\Delta)^{-1/2}$ is the product of the bounded operator $T(1+\Delta)^{-1/2}$ with the compact operator $(1+\Delta)^{-1/2}$ and is therefore compact in its own right. Consequently, the projection P on the eigenspaces associated to the eigenvalues λ with $|\lambda| > 1/2$ is a finite dimensional projection. We then split $\mathscr{H} = P\mathscr{H} \oplus P^\perp\mathscr{H}$ and the commutation of $(1+\Delta)^{-1/2}$ with $T(1+\Delta)^{-1/2}$ ensure that $(1+D^2)^{-n/2} - (1+\Delta)^{-n/2}$ splits in the sum of two endomorphisms, on $P\mathscr{H}$ and $P^\perp\mathscr{H}$.

Since $P\mathscr{H}$ is finite dimensional, this part clearly lives in $\mathcal{L}_0^{1+}$. On $P^\perp\mathscr{H}$, by construction, $1+T(1+\Delta)^{-1}$ is invertible and therefore the expression (6.4) has a meaning. Consider the scalar function:

$$f(t) := (1+t)^{-n_0/2} - 1.$$

The image of the compact operator $T(1+\Delta)^{-1}$ under this continuous function with $f(0) = 0$ is a compact operator. The difference $(1+D^2)^{-n_0/2} - (1+\Delta)^{-n_0/2}$ thus identifies with the product of the compact operator $K = ((1+T(1+\Delta)^{-1})^{-n_0/2} - 1)$ and the element of $\mathcal{L}^{1+}$ given by $T' = (1+\Delta)^{-n_0/2}$. The discussion of Dixmier traces in p. 288 of [20] proves that

(a) the norm $\|\cdot\|_{1+}$ is symmetric thus $\|KT'\|_{1+} \leqslant \|K\|\,\|T'\|_{1+}$;
(b) Dixmier traces are continuous for $\|\cdot\|_{1+}$.

Since Dixmier traces vanish on finite-rank operators, they also vanish on the $\|\cdot\|_{1+}$-closure of finite-rank operators, i.e. $\mathcal{L}_0^{1+}$. The operator K is compact, so it can be approximated (in operator norm) by a sequence of finite-rank operators K_n. Since $\|\cdot\|_{1+}$ is a symmetric norm, we have $\|KT' - K_nT'\|_{1+} \leqslant \|K - K_n\|\,\|T'\|_{1+}$, which proves that the sequence K_nT' of finite rank operators converges to KT' for the $\|\cdot\|_{1+}$-norm. In other words, $(1+D^2)^{-n_0/2} - (1+\Delta)^{-n_0/2} \in \mathcal{L}_0^{1+}$.

Item (iii) follows from (6.2): for any Dixmier trace Tr_ω and any $T \in A \otimes B(S)$,

$$\mathrm{Tr}_\omega(T(1+D^2)^{-n_0/2}) = \mathrm{Tr}_\omega(T(1+\Delta)^{-n_0/2}).$$

Since the operator $-\sum_j \partial_j^2$ is the Casimir operator on the compact Lie group G, $(1+\Delta)^{-n_0/2}$ commutes with the unitary $U_g \otimes 1$. Similarly, for any unitary $U' \in \mathcal{U}(S)$ of $B(S)$, $1 \otimes U'$ commutes with $(1+\Delta)^{-n_0/2}$. It is easy to see that the action of $G \times \mathcal{U}(S)$ on $A \otimes B(S)$ implemented by $U_g \otimes U'$ is ergodic.

The same computation as in Proposition 6.1 proves that the linear form $\phi(T)$ on $A \otimes B(S)$ is invariant under the ergodic action of $G \times \mathcal{U}(S)$. It is therefore proportional $\tau \otimes \mathrm{Tr}$. Since this is an extension of the linear form of point (i), the scalar λ has to be the same in both cases. ∎

7. SUMMABILITY: THE CASE OF THE 2-SPHERE S^2

In this section, we discuss the questions of summability in relation with the following example.

EXAMPLE 7.1. We can apply our construction to the 2-sphere on which the group $G = SU(2)$ is acting.

Let us start by simple observations on our main result.

REMARK 7.2. Theorem 5.5 only gives an upper bound on summability, i.e. the scalar λ can take the value $\lambda = 0$. Assuming ergodicity, in general we cannot do better: if G acts ergodically on A, then letting K act trivially on A, we obtain an ergodic $G \times K$-action. However:

(i) The degree of freedom that we gain by using the real number n_0 in Proposition 6.1 and Theorem 6.3 can compensate this problem, as we will see in this section.

(ii) If there is a n_0 such that λ is finite and nonzero, then this "dimension" is unique: indeed, for all $\varepsilon > 0$, $(1+D^2)^{-(n_0+\varepsilon)/2} = (1+D^2)^{-n_0/2}(1+D^2)^{-\varepsilon/2}$ is the product of $(1+D^2)^{-n_0/2} \in \mathcal{L}^{1+}$ and the compact operator $(1+D^2)^{-\varepsilon/2}$ and its Dixmier trace therefore vanishes (see proof of Theorem 6.3).

(iii) We can consider an ergodic action with *full multiplicity* — i.e. in the decomposition (4.1) $m_\ell = d_\ell$, dimension of the representation E_ℓ (compare with Theorem A, p. 1482 of [38]). In this case, the eigenvalues of D are the same as those of D_{ref} acting on $\mathscr{H}_{\mathrm{ref}}$ (see proof of Theorem 5.5), therefore the techniques available for manifolds apply, and following for instance the lines of Proposition 7.7 p. 269 of [20], it appears that for $n_0 = n$, λ is nonvanishing.

PROPOSITION 7.3. *If we consider $A = C(S^2)$ equipped with its natural $SU(2)$ action,*

(i) *the action of $SU(2)$ on A is ergodic.*

Using the spectral triple constructed as in Theorem 5.5,

(ii) *the operator $(1+\Delta)^{-3/2}$ is traceclass and thus*

$$\mathrm{Tr}_\omega((1+D^2)^{-3/2}) = 0;$$

(iii) *the operator $(1+\Delta)^{-1}$ is in $\mathcal{L}^{1+}$ and its Dixmier trace does not vanish; more precisely:*

$$\mathrm{Tr}_\omega((1+D^2)^{-1}) = 2. \tag{7.1}$$

REMARK 7.4. The estimate (7.1) together with Theorem 6.3 proves that all $T \in A \otimes B(S)$ are measurable, i.e. $\mathrm{Tr}_\omega(T(1+D^2)^{-1})$ does not depend on the choice of ω (see e.g. Definition 7.16 p. 288 of [20]).

Proof of Proposition 7.3. It is well-known that the spherical harmonics provide a decomposition of the Hilbert space $\mathscr{H}_0 = L^2(S^2)$ into an orthogonal Hilbert sum of eigenspaces of $\Delta_0 = -(\partial_1^2 + \partial_2^2 + \partial_3^2)$. Formally,

$$\mathscr{H}_0 = \bigoplus_{\ell \geqslant 0} E_{2\ell},$$

where $E_{2\ell}$ is the (Hilbert space of the) irreducible representation of $SU(2)$ of dimension $2\ell+1$. It follows that the eigenvalues of Δ_0 are $\ell(\ell+1)$ (for $\ell \in \mathbb{N}$) and they have multiplicity $2\ell+1$. A consequence is that the only G-invariant functions in $\mathscr{H}_0$ are the constants. Since $A \subseteq \mathscr{H}_0$, it appears that the action of G on A is ergodic and thus item (i) is proven.

To proceed in the direction of items (ii) and (iii), notice that the Lie group $G = SU(2)$ has dimension 3, so following (2.6), the Hilbert space we consider for the Dirac operator of Theorem 5.5 is $\mathscr{H} = \mathscr{H}_0 \otimes \mathbb{C}^2$, meaning the operator Δ as defined in Theorem 6.3 has eigenvalues $\ell(\ell+1)$ with multiplicities $2(2\ell+1)$.

Next step is to evaluate $\|(1+\Delta)^{-n_0/2}\|_{1+}$ and prove by direct examination that it is finite for $n_0 = 3$ and $n_0 = 2$. Of course, the case $n_0 = 3$ is also a consequence of Theorem 6.3 item (i) and we complete the computation as an illustration only. Consider the sum $\sigma_k((1+\Delta)^{-n_0/2})$ (see Definition 5.3), which we simply denote σ_k. We can evaluate it explicitly for $k(\ell_0) = \sum_{\ell=0}^{\ell_0} 2(2\ell+1) = 2(\ell_0+1)^2$ corresponding to the eigenvalues of $\bigoplus_{0 \leqslant \ell \leqslant \ell_0} E_{2\ell}$. In this case, we get:

$$\sigma_{k(\ell_0)} = \sum_{\ell=0}^{\ell_0} 2(2\ell+1)(\ell(\ell+1))^{-n_0/2}.$$

For $n_0 = 3$, we get the upper bound:

$$\frac{(2\ell+1)}{(\ell(\ell+1))^{3/2}} \leqslant \frac{2}{\ell\sqrt{\ell(\ell+1)}} = O(\ell^{-2}),$$

where we used Landau's notation. This shows that for $n_0 = 3$, the series σ_k of positive terms actually converges. It follows that the operator $(1+\Delta)^{-3/2}$ is traceclass and thus its image under the Dixmier trace is 0 (see p. 304 of [9]). As a consequence of Theorem 6.3 item (ii), this also proves that $\mathrm{Tr}_\omega((1+D^2)^{-3/2}) = 0$, i.e. $\lambda = 0$ in equation (6.1). This completes the discussion of item (ii).

Taking now $n_0 = 2$, we can estimate the general term of the series $\sigma_{k(\ell_0)}$:

$$\frac{2(2\ell+1)}{\ell(\ell+1)} \sim 4\ell^{-1},$$

and since all the terms in the series are positive, this series itself is equivalent to $4\log(\ell_0)$. More generally, for all $k \in [\![k(\ell_0), k(\ell_0+1)]\!]$ we get the estimate:

$$0 \leqslant \frac{\sigma_{k(\ell_0)}}{\log(k(\ell_0+1))} \leqslant \frac{\sigma_k}{\log(k)} \leqslant \frac{\sigma_{k(\ell_0+1)}}{\log(k(\ell_0))}. \tag{7.2}$$

Using the equivalences $\sigma_{k(\ell_0)} \sim 4\log(\ell_0)$ and $\log(k(\ell_0)) = \log(2(\ell_0+1)^2) \sim 2\log(\ell_0)$, it appears that both sides of the inequality (7.2) converge to 2 for $\ell_0 \to 0$. Following Definition 2 p. 305 of [9], this proves that the value of the Dixmier trace does not depend on ω and actually:

$$\mathrm{Tr}_\omega((1+D^2)^{-1}) = 2.$$

In other words, the scalar λ in (6.1) is equal to 2. ∎

8. EXAMPLES AND COUNTEREXAMPLES

We begin this section with a few comments on Theorem 5.5.

(i) For an n^+-summable operator D, we know that a Hochschild cocycle φ on $\mathscr{A}$ can be defined by:

$$\varphi(a^0, a^1, \dots, a^n) = \lambda_n \,\mathrm{Tr}_\omega(\gamma a^0 [D, a^1] \cdots [D, a^n] |D|^{-n}), \tag{8.1}$$

as stated in IV.2 Theorem 8 p. 308 of [9] — which was later improved in [14]. In the case of noncommutative tori (see Example 8.1 below), the equation (8.1) leads (up to normalisation) to the cyclic cocycle:

$$\varphi(a^0, a^1, a^2) = \tau(a^0(\partial_1(a^1)\partial_2(a^2) - \partial_2(a^1)\partial_1(a^2))).$$

Would it be possible in the setting of Theorem 5.5(i) to relate (8.1) to cyclic cocycles as constructed in III.6 example 12 c p. 256 of [9])?

(ii) Another interesting improvement would be to find sufficient conditions for the Poincaré duality (Axiom (6′) of [11]) to hold. However, this property would really depend on the algebraic structure on A, and not just on the multiplicity of its spectral subspaces.

(iii) Examples of covariant representations of (A, G) on $\mathscr{H}_0$ satisfying the hypotheses of the second point of the theorem can be constructed from finite dimensional representations $\beta : G \to B(\mathscr{K})$ by setting: $\mathscr{H}_0 := \mathrm{GNS}(A, \tau) \otimes \mathscr{K}$ on which G acts by $U \otimes \beta$, where U is defined by Lemma 3.1. Sub-covariant representations of such $\mathscr{H}_0$ are also suitable.

To illustrate our results and hypotheses we give examples showing that frequently spectral triples arise from Lie group actions, and we hint at some possible interesting generalizations.

EXAMPLE 8.1. The spectral triples on noncommutative tori were among the first examples considered by Connes. Indeed, they already appear in his article [11] (see p. 166). His original example was in dimension 2 but the construction was later extended to include noncommutative tori of any dimension (see Section 12.3 and especially p. 545 of [20]). We illustrate the notion in the two-dimensional case which was studied as early as in [31]: the C^*-noncommutative torus A_θ is the universal C^*-algebra generated by two unitaries U and V subject to the relation $UV = e^{2\pi i\theta} VU$ for $\theta \in \mathbb{R}$.

It is well known that the actions of T^n on the noncommutative tori (even of dimension n) are ergodic (see p. 537 of [20]). Hence our construction fully applies to these algebras. The unbounded operator D obtained from our Theorem 5.5 is exactly the same as the operator defined in (12.24) p. 545 of [20]. The n^+-summability of D corresponds to Proposition 12.14 p. 545 of [20] and is a sharp estimate of summability in this case.

EXAMPLE 8.2. It is easy to see that some hypothesis on the fixed-point algebra is unavoidable. For example, already the action of S^1 on $\mathbb{T}^2$ given by $\lambda \cdot (z_1, z_2) := (\lambda z_1, z_2)$ yields an unbounded operator on the Hilbert space $L^2(\mathbb{T}^2)$ which certainly does not have compact resolvent because it has an infinite dimensional kernel.

However, note that this operator has "compact (even summable) resolvent with coefficients in $C(\mathbb{T}^2)$", i.e. $(1 + D^2)^{-1} \otimes 1 \in \mathcal{L}^p \otimes_\pi C(\mathbb{T})$. In this sense, it yields an (unbounded) $\mathcal{LC}$-Kasparov module from $C(\mathbb{T}^2)$ to $\mathcal{L}^p \otimes_\pi C(\mathbb{T})$ in the sense of [21].

EXAMPLE 8.3. Quantum Heisenberg manifolds (QHM) provide another illustration of our results. This family of algebras was introduced in [32] as an example of Rieffel's strict quantization-deformation and largely studied since, e.g. [1], [2], [7], [16], [18] to name just a few elements of the available literature. In particular, QHM admit an ergodic action of the (non-compact) Heisenberg group G and a unique G-invariant trace τ — yielding properties very similar to those assumed in Theorem 5.5. In the article [7], Chakraborty and Sinha constructed a family of spectral triples on QHM whose Hilbert space is obtained from τ by the GNS construction. The expression of the operator is given by (2.8) (compare Proposition 9 of [7]) and they went further, proving summability in Theorem 10 p. 431.

When applied to quantum Heisenberg manifolds, our results recover a real structure and thus complement [7]. However, the non-compactness of the Heisenberg group prevents us from reproducing summability. We will encounter a similar phenomenon in Example 8.5 below. To bypass this problem, we elaborate on our construction in another article [17], where we give a setting to treat the case of QHM.

EXAMPLE 8.4. Another, related, example is given by Kasparov's Dirac element (see for example [23] or [24] for the Dirac-dual Dirac method at work). It is easy to generalize our results in the following two ways: first we can treat more general modules than the uniquely determined spinor module we used in our construction of the Dirac-type operator. In fact, it suffices that the module S used in the representation be a complex module over $\mathbb{C}l(n)$ (which carries a real structure if one wants to recover Proposition 2.12). In the Real case, this means that the module is equipped with a Real structure which is compatible with the canonical Real structure on $\mathbb{C}l(n)$. Secondly, one may include a Real structure on the C^*-algebra A which is preserved by the action of G. "Real structures" in this sense were already used by Kasparov in his very first definition of KK-theory (see, for example, [35] for an overview).

In order to carry over the summability results for the associated spectral triple it suffices to decompose the "spinor module" into irreducible representations.

With this slight generalization, it is possible to include Kasparov's Dirac element into our framework. It is given by the Hilbert space $\Omega_{\mathbb{C}}(\mathbb{R}^n)$ of L^2-forms on $\mathbb{R}^n$ on which $C(\mathbb{R}^n)$ acts naturally, together with the Hodge–Dirac operator. Our Proposition 3.4 is in the spirit of Wolf's theorem ([25], Theorem 5.7). However, the operator one obtains has continuous spectrum and therefore does not have compact resolvent as in the example of the Heisenberg manifold. This shows that some hypothesis is necessary on the relative size of the Lie group compared with the algebra in order to apply our techniques. Note however that for every function $f \in C(\mathbb{R}^n)$ with compact support the operator $f(\mathrm{i}+D)^{-1}$ is in fact compact and the mapping $f \mapsto f(\mathrm{i}+D)^{-1}$ is continuous (this also holds for Schwartz functions). In this sense, it should be possible to generalize our results to non-compact Lie groups by passing to nonunital spectral triples.

EXAMPLE 8.5. A less simple-minded example is provided by the harmonic oscillator $d + d^* + c(x)$ acting as in the last example on $\Omega_{\mathbb{C}}(\mathbb{R}^n)$, where $c(x)$ denotes Clifford multiplication by x and $d + d^*$ the Hodge–Dirac operator. This operator has recently come to our attention in [41] and was also used heavily in [21]. And it again has a group-theoretic interpretation: it can be obtained as the operator associated to an action of the Heisenberg group of $\mathbb{R}^n$ on $\mathbb{R}^n$. The fact that it has summable resolvent (which is classical and seen by using Hermite polynomials) shows that when the Lie group is non-compact, it may induce a selfadjoint operator in the above way which has nevertheless summable resolvent.

This example is closely related to the spectral triple on $C(\mathbb{T})$ obtained by our techniques: the class of the spectral triple on $C(\mathbb{R})$ defined by our techniques is in a sense a unitalization of the one defined in this example.

EXAMPLE 8.6. Further examples can be obtained from *(Cuntz–)Pimsner algebras* as defined in [28]. Indeed, consider a C^*-correspondence E over a C^*-algebra A, i.e. a (right) Hilbert module E with a (faithful) left A-action $\phi : A \to \mathcal{L}(E)$.

Such objects are called "Hilbert bimodules" in [28]. These data E and A define a Pimsner algebra $\mathcal{O}_E$.

Assume moreover that a Lie group G acts on A and that there is a compatible G-action on E, in the sense of Remark 4.10-(2) in [28]. The latter then provides us with a G-action on the Pimsner algebra, which commutes with the canonical gauge action, thereby defining an action of $G \times S^1$ on $\mathcal{O}_E$ — to which our theory may apply. In general, even if G acts ergodically on A, the action of $G \times S^1$ on $\mathcal{O}_E$ need not be ergodic, as illustrated by the Cuntz algebra $\mathcal{O}_2$ generated by $A = \mathbb{C}$ (on which the trivial group $G = \{1\}$ acts ergodically) and $E = \mathbb{C}^2$: the S^1-action induced on $\mathcal{O}_2$ is clearly not ergodic!

Yet, if we consider *Hilbert bimodules* (E with a left- *and* a right-Hilbert module structure) instead of C^*-correspondences, the resulting Pimsner algebras are in fact *generalised crossed products* (see [2]) and better results are available. For instance, if G acts ergodically on A, then the induced $G \times S^1$-action on $\mathcal{O}_E$ is ergodic, and our theory applies to its fullest extent (namely Theorem 5.5). This sort of results form the core of our forthcoming article [17].

Let us conclude with a nonexample showing that the previous theory cannot be extended in general to noncompact groups. In particular, the G-invariant trace may not be available:

EXAMPLE 8.7. There is an ergodic action of $\mathbb{R}$ on the infinite Cuntz algebra $\mathcal{O}_\infty$. Since this algebra is purely infinite and simple, it has no trace.

To prove it, identify $\mathcal{O}_\infty$ with the Cuntz algebra generated by the vector space $L^2(\mathbb{R})$ and consider the natural unitary $\mathbb{R}$-action by translation on $L^2(\mathbb{R})$. This action has no finite dimensional invariant subspaces, and following Theorem 5.4 and Corollary 5.5 of [6], this suffices to conclude that the action of $\mathbb{R}$ on $\mathcal{O}_\infty$ is ergodic.

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