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ON THE SYMMETRIZATION OF GENERAL WIENER–HOPF OPERATORS

A. BÖTTCHER and F.-O. SPECK

For Florian-Horia Vasilescu on his 75th birthday

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ABSTRACT. This article focuses on general Wiener–Hopf operators given as $W = P_2A|_{P_1X}$ where X, Y are Banach spaces, $P_1 \in \mathcal{L}(X), P_2 \in \mathcal{L}(Y)$ are any projectors and $A \in \mathcal{L}(X, Y)$ is boundedly invertible. It presents conditions for W to be equivalently reducible to a Wiener–Hopf operator in a symmetric space setting where $X = Y$ and $P_1 = P_2$. The results and methods are related to the so-called Wiener–Hopf factorization through an intermediate space and the construction of generalized inverses of W in terms of factorizations of A.

KEYWORDS: *Wiener–Hopf operator, symmetrization, factorization, generalized inverse.*

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1. INTRODUCTION AND MAIN RESULTS

We investigate operators of the form

$$W = P_2A|_{P_1X} : P_1X \to P_2Y \tag{1.1}$$

under the basic assumptions that X and Y are Banach spaces, $P_1 \in \mathcal{L}(X)$ and $P_2 \in \mathcal{L}(Y)$ are projectors, and $A \in \mathcal{L}(X, Y)$. Such operators are called *general Wiener–Hopf operators* (WHO) [10], [19] or simply truncations or compressions of an operator [14]. In the *symmetric setting*, where $X = Y, P_1 = P_2 = P$, the operator W is commonly written in the form $W = T_P(A) = PA|_{PX} : PX \to PX$. Following [22], the setting in which $X \neq Y$ or $P_1 \neq P_2$ is referred to as the *asymmetric space setting*. Throughout this paper we assume that the so-called *underlying operator* A is invertible, i.e., that A is a linear homeomorphism of X onto Y. For brevity and by tradition, thinking of the group $G\mathcal{L}(X)$ of the invertible elements in the Banach algebra $\mathcal{L}(X)$, we denote the set of invertible operators in $\mathcal{L}(X, Y)$ by $G\mathcal{L}(X, Y)$. The requirement that A be in $G\mathcal{L}(X, Y)$ is satisfied in many important applications. This requirement is in fact no limitation of generality, since if

W is any operator given by (1.1), we may suitably change X, Y, P_1, P_2 so that W is of the form (1.1) with an invertible operator A; see, e.g., Proposition 5.1 of [24].

A fundamental idea to solve equations $Wf = g$ consists in the construction of certain factorizations of the underlying operator A that yield explicit formulas for generalized inverses of W. Various "general" factorization theorems are known in the existing literature. See, for instance, [8], [10], [16], [21], [22], [25]. We will here not embark on the constructive factorization of scalar or matrix functions; for an overview on that wide research area see [1], [4], [9], [15], [20] and the literature cited therein.

In 1968, Čebotarev [8] considered so-called abstract Wiener–Hopf equations in a purely algebraic context: given a unital ring $\mathcal{R}$, an element $p \in \mathcal{R}$ satisfying $p^2 = p$, and an invertible element $a \in \mathcal{R}$, he studied, in terms of certain factorizations of a, the one-sided invertibility of $w = pap$ in the subring $\mathcal{R}_p = \{t \in \mathcal{R} : t = ptp\}$. In 1969, Devinatz and Shinbrot [10] introduced the notion of a general Wiener–Hopf operator. They worked in the symmetric setting with separable Hilbert spaces $X = Y$ and with orthogonal projectors $P_1 = P_2 = P$, which implied that they had to deal with both topological and algebraical questions, and they proved a criterion for the invertibility of $T_P(A)$ in terms of a factorization of A. We should mention that the original idea of operator factorization was proposed before by Shinbrot [19] in 1964 in the context of one-dimensional singular integral operators. For more details of the pre-history we refer the reader to [22].

General asymmetric Wiener–Hopf operators were first investigated in [21]. A strong motivation to study the operator (1.1) in an asymmetric space setting is given by the theory of pseudo-differential operators, which naturally act between Sobolev-like spaces of different orders; see Eskin's book [13]. Their symmetrization (lifting) by generalized Bessel potential operators is considered in [11]. Furthermore, Toeplitz operators with singular symbols are another source of motivation for considering symmetrization. We will briefly touch these two concrete applications in the examples later in Section 2.

In [22], the second author introduced the notion of a *cross factorization* and proved that the generalized invertibility of W is equivalent to the existence of a cross factorization of A. In the recent paper [24], two further kinds of operator factorizations were studied, the *Wiener–Hopf factorization of A through an intermediate space* and the *full range factorization $W = LR$* where L is left invertible and R is right invertible. The main theorem of [24] states the equivalence between all three factorizations, partly under the restrictive condition that the two projectors P_1 and P_2 are equivalent. Unfortunately, one proof in [24] contains a gap. This gap, which will be filled in Section 4 of the present paper, actually motivated us to look after the matter again. Our efforts resulted in a symmetrization criterion (Theorem 1.1 below) and a new proof of a basic theorem of [24] (Theorem 1.2 below).

Our first topic here is the symmetrization of asymmetric WHOs. To be precise, we call the setting X, Y, P_1, P_2 *symmetrizable* if there are a Banach space Z, operators $M_+ \in G\mathcal{L}(X,Z)$, $M_- \in G\mathcal{L}(Z,Y)$, and a projector $P \in \mathcal{L}(Z)$ such that

$$M_+(P_1X) = PZ, \quad M_-(QZ) = Q_2Y, \tag{1.2}$$

where $Q = I_Z - P$ and $Q_2 = I_Y - P_2$. Note that the invertibility of M_+ and M_- in conjunction with (1.2) implies that

$$U_+ := M_+|_{P_1X} : P_1X \to PZ, \quad V_- := M_-|_{QZ} : QZ \to Q_2Y, \tag{1.3}$$

are invertible. As will be made explicit below in (2.2), the invertibility of V_- yields (and is in fact equivalent to) the invertibility of

$$V_+ := (PM_-^{-1}|_{P_2Y})^{-1} : PZ \to P_2Y.$$

If the setting X, Y, P_1, P_2 is symmetrizable, then asymmetric WHOs may also be symmetrized: given an operator of the form (1.1), there is an operator $\widetilde{A} \in \mathcal{L}(Z)$ such that $A = M_-\widetilde{A}M_+$ and $W = V_+\widetilde{W}U_+ = V_+T_P(\widetilde{A})U_+$. Indeed, we have $\widetilde{A} = M_-^{-1}AM_+^{-1}$, and since $PM_-^{-1} = PM_-^{-1}P_2$ and $PM_+P_1 = M_+P_1$, we get

$$\begin{aligned} V_+\widetilde{W}U_+ &= (PM_-^{-1}|_{P_2Y})^{-1}PM_-^{-1}AM_+^{-1}|_{PZ}(PM_+|_{P_1X}) \\ &= (PM_-^{-1}|_{P_2Y})^{-1}PM_-^{-1}P_2AM_+^{-1}M_+|_{P_1X} = P_2A|_{P_1X} = W. \end{aligned}$$

As usual, we call two operators T and S *equivalent*, written $T \sim S$, if there exist linear homeomorphisms E and F such that $T = FSE$. Thus, in the case of a symmetrizable setting, $W \sim T_P(\widetilde{A})$.

Here is our first main result. Given two Banach spaces Z_1 and Z_2, we write $Z_1 \cong Z_2$ if the two spaces are isomorphic, that is, if there exists an operator A in $G\mathcal{L}(Z_1, Z_2)$. We also put $Q_1 = I_X - P_1$, $Q_2 = I_Y - P_2$.

THEOREM 1.1. *The following are equivalent:*
(i) *the setting X, Y, P_1, P_2 is symmetrizable;*
(ii) *$P_1X \cong P_2Y$ and $Q_1X \cong Q_2Y$;*
(iii) *$P_1 \sim P_2$.*

The theorem implies in particular that every setting given by two separable Hilbert spaces X, Y and two infinite-dimensional bounded projectors P_1, P_2 is symmetrizable.

In Section 3, we will recall two types of factorizations of the underlying operator A, the cross factorization (CFn) and the Wiener–Hopf factorization through an intermediate space (FIS). Note that the existence of a CFn for A is equivalent to the generalized invertiblity of W in the sense that there exists an operator $W^- \in \mathcal{L}(P_2Y, P_1X)$ such that $WW^-W = W$. Herewith our second main result.

THEOREM 1.2. *The following are equivalent:*
(i) *A has a* CFn *and $P_1 \sim P_2$;*
(ii) *A has a* FIS.

Theorem 1.2 is already in [24], and it is the theorem whose proof in that paper contains a gap. We here give another, more straightforward proof. In addition, in Section 4, we will repair the gap of the proof in [24], thus saving also the original proof. Theorems 1.1 and 1.2 will be proved in Sections 2 and 3, respectively. There we will also present concrete examples.

2. SYMMETRIZATION

In this section we prove Theorem 1.1. In what follows we repeatedly use the following well known fact whose proof can be found, for example, on p. 332 of [3], pp. 16–17 of [18], or pp. 21–22 of [22].

LEMMA 2.1. *If B is in $G\mathcal{L}(Z_1, Z_2)$ and $R_1, S_1 \in \mathcal{L}(Z_1)$, $R_2, S_2 \in \mathcal{L}(Z_2)$ are projectors such that $R_1 + S_1 = I_{Z_1}$ and $R_2 + S_2 = I_{Z_2}$, then*

$$(2.1)\qquad \begin{aligned} &R_2 B|_{\operatorname{im} R_1} : \operatorname{im} R_1 \to \operatorname{im} R_2 \text{ is invertible} \\ &\Leftrightarrow S_1 B^{-1}|_{\operatorname{im} S_2} : \operatorname{im} S_2 \to \operatorname{im} S_1 \text{ is invertible,} \end{aligned}$$

and in the case of invertibility, the Schur complement identity

$$(2.2)\qquad (S_1 B^{-1}|_{S_2 Z_2})^{-1} = S_2 B|_{S_1 Z_1} - S_2 B R_1 (R_2 B|_{R_1 Z_1})^{-1} R_2 B|_{S_1 Z_1}$$

holds.

Proof of Theorem 1.1. (i) ⇒ (ii) Assume that the setting X, Y, P_1, P_2 is symmetrizable and let M_+, M_-, P be the corresponding operators defined in (1.3). Combining (2.1) and the invertibility of the operators (1.3), we obtain that the operators

$$U_- := Q_1 M_+^{-1}|_{QZ} : QZ \to Q_1 X, \quad V_+ := P M_-^{-1}|_{P_2 Y} : P_2 Y \to PZ$$

are invertible. Hence $V_+^{-1} U_+ : P_1 X \to P_2 Y$ and $V_- U_-^{-1} : Q_1 X \to Q_2 Y$ are isomorphisms.

(ii) ⇒ (iii) Let $U : P_1 X \to P_2 Y$ and $V : Q_1 X \to Q_2 Y$ be isomorphisms. Consider the operators E and F defined by

$$E := U P_1 + V Q_1 : X \to Y, \quad F := U^{-1} P_2 + V^{-1} Q_2 : Y \to X.$$

It can be straightforwardly verified that $EF = I_Y$ and $FE = I_X$, that is, these two operators are invertible. Moreover, we have

$$\begin{aligned} E P_1 F &= E P_1 (U^{-1} P_2 + V^{-1} Q_2) = E P_1 U^{-1} P_2 + E P_1 V^{-1} Q_2 \\ &= E U^{-1} P_2 + E P_1 Q_1 V^{-1} P_2 = E U^{-1} P_2 = (U P_1 + V Q_1) U^{-1} P_2 \\ &= U P_1 U^{-1} P_2 + V Q_1 U^{-1} P_2 = U U^{-1} P_2 + V Q_1 P_1 U^{-1} P_2 = P_2, \end{aligned}$$

which shows that $P_1 \sim P_2$.

(iii) $\Rightarrow$ (i) Suppose $P_2 = EP_1F$ with $E \in G\mathcal{L}(X,Y)$ and $F \in G\mathcal{L}(Y,X)$. The operator $U := E|_{P_1X} : P_1X \to P_2Y$ is clearly injective together with E, and since $P_2y = EP_1Fy$, it is also surjective. Consider the linear map

$$\Phi : X/P_1X \to Y/P_2Y, \quad \Phi(x + P_1X) = Ex + P_2Y.$$

This map is well-defined, since if $x_1 + P_1X = x_2 + P_1X$, then $x_1 - x_2 \in P_1X$ and hence $Ex_1 - Ex_2 = E(x_1 - x_2) \in P_2Y$. The map Φ is injective, because if $Ex = P_2y \in P_2Y$, then $Ex = EP_1Fy$, whence $x = P_1Fy \in P_1X$. Finally, Φ is surjective because so is E. It follows that Φ is an isomorphism. Consequently, $X/P_1X \cong Y/P_2Y$, and since $X/P_1X \cong Q_1X$ and $Y/P_2Y \cong Q_2Y$, we arrive at the conclusion that $Q_1X \cong Q_2Y$. Let $V : Q_1X \to Q_2Y$ be any isomorphism.

We claim that X, Y, P_1, P_2 is symmetrized by $Z := P_1X \times Q_2Y$, the projector $P : Z \to Z$ defined by $P(z_1, z_2) := z_1$, and the operators $M_\pm$ given by

$$M_+ : X \to Z, x \mapsto (P_1x, VQ_1x), \quad M_- : Z \to Y, (z_1, z_2) \mapsto Uz_1 + z_2.$$

The operator M_+ is obviously injective and surjective. The injectivity of M_- can be seen as follows: if $Uz_1 + z_2 = 0$ with $z_1 = P_1x \in P_1X$ and $z_2 = Q_2y \in Q_2Y$, then $UP_1x + Q_2y = 0$, which implies that $Uz_1 = UP_1x \in P_2Y \cap Q_2Y$, whence $Uz_1 = 0$ and thus $z_2 = 0$, and since $U : P_1X \to P_2Y$ is invertible, we conclude that $z_1 = 0$. The surjectivity of M_- is again obvious: if $y \in Y$, then $y = P_2y + Q_2y = UP_1x + Q_2y$ with $P_1x \in P_1X$. Finally, we have

$$\begin{aligned} M_+(P_1X) &= \{(P_1x, 0) : x \in X\} = PZ, \\ M_-(QZ) &= M_-(\{(0, Q_2y) : y \in Y\}) = \{Q_2y : y \in Y\} = Q_2Y, \end{aligned}$$

which shows that (1.2) is satisfied. The proof of Theorem 1.1 is complete. ∎

REMARK 2.2. From Theorem 1.1 we see that if $P_1 \sim P_2$, then

$$\begin{aligned} P_1X \times Q_2Y &\cong P_1X \times Q_1X \cong P_1X \oplus Q_1X = X, \\ P_1X \times Q_2Y &\cong P_2Y \times Q_2Y \cong P_2Y \oplus Q_2Y = Y, \end{aligned}$$

and hence

$$X \cong P_1X \times Q_2Y \cong Y. \tag{2.3}$$

However, (2.3) does not imply that $P_1 \sim P_2$. A counterexample is provided by the setting $X = Y = \ell^2(\mathbb{Z})$,

$$\begin{aligned} P_1 &: (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) \mapsto (\ldots, 0, 0, 0, x_1, x_2, \ldots), \\ P_2 &: (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) \mapsto (\ldots, 0, 0, x_0, 0, 0, \ldots). \end{aligned}$$

Indeed, condition (2.3) holds because X, Y, P_1X, Q_2Y are infinite-dimensional separable Hilbert spaces, but P_1 and P_2 are clearly not equivalent.

EXAMPLE 2.3. A concrete case where symmetrization was used (without calling it symmetrization) occurs in the proof of the Fisher–Hartwig conjecture in [2]; see also pp. 281–283 of [3]. A Fisher–Hartwig symbol is a function on the complex unit circle $\mathbb{T}$ of the form

$$a(t) = b(t) \prod_{j=1}^{N} |t - t_j|^{2\alpha_j}, \quad t \in \mathbb{T},$$

where b is a piecewise continuous function on $\mathbb{T}$ that is invertible in L^∞, $t_1, \ldots, t_N$ are distinct points on $\mathbb{T}$, and $\alpha_1, \ldots, \alpha_N$ are complex numbers whose real parts lie in the interval $(-1/2, 1/2)$. The Toeplitz operator generated by a is an operator of the form $T(a) = P_2 M(a)|_{\operatorname{im} P_1}$, where $M(a)$ acts on certain Lebesgue spaces over $\mathbb{T}$ by the rule $f \mapsto af$ and P_1, P_2 are the Riesz projectors of the Lebesgue spaces onto their Hardy spaces. The operators $M(a)$ and $T(a)$ are in general neither bounded nor invertible on L^p and the corresponding Hardy spaces H^p. However, things can be saved by passing to weighted spaces. Put $\varrho(t) = \prod_{j=1}^{N} |t - t_j|^{\mathrm{Re}\alpha_j}$. For $1 < p < \infty$, let

$$L^p(\varrho^{\pm 1}) = \Big\{ f \in L^1 : \|f\|^p := \int_{\mathbb{T}} |f(t)|^p \varrho(t)^{\pm p} |\mathrm{d}t| < \infty \Big\}.$$

The Riesz projector P, which may be defined as $P = (I + S)/2$ with the Cauchy singular integral operator S given by

$$(Sf)(t) = \lim_{\varepsilon \to 0} \frac{1}{\pi \mathrm{i}} \int_{|\tau - t| > \varepsilon} \frac{f(\tau)}{\tau - t} \mathrm{d}\tau, \quad t \in \mathbb{T},$$

is bounded on the spaces $L^p(\varrho^{\pm 1})$ if $\mathrm{Re}\alpha_j \in (-1/r, 1/r)$ where $r = \max(p, q)$ with $1/p + 1/q = 1$. Thus, assume the real parts $\mathrm{Re}\alpha_j$ are all in $(-1/r, 1/r)$. Finally, consider the setting

$$X = L^p(\varrho), \quad P_1 = P, \quad Y = L^p(\varrho^{-1}), \quad P_2 = P.$$

It turns out that $M(a) \in \mathcal{GL}(X, Y)$ and hence we are in the setting (1.1) with the invertible operator $A = M(a)$. The Toeplitz operator $T(a)$ acts from $PL^p(\varrho)$ to $PL^p(\varrho^{-1})$. Thus, it is a WHO in an asymmetric setting. It can be shown that the setting X, Y, P_1, P_2 is symmetrized by $Z = L^p$, $P =$ Riesz projector, $M_+ := M(\eta)$, $M_- := M(\xi)$, where

$$\eta(t) = \prod_{j=1}^{N} \Big(1 - \frac{t}{t_j}\Big)^{\alpha_j}, \quad \xi(t) = \prod_{j=1}^{N} \Big(1 - \frac{t_j}{t}\Big)^{\alpha_j}.$$

We have $T(a) = V_+ T(b) U_+$ with $T(b) \in \mathcal{L}(L^p, L^p)$, which reduces the study of $T(a)$ to the investigation of the much simpler operator $T(b)$.

EXAMPLE 2.4. Another useful application of symmetrization is the reduction of WHOs and pseudo-differential operators in scales of Sobolev spaces to operators acting in L^p spaces by Bessel potential operators for a half-line, half-space, quarter plane, or Lipschitz domain [12], [13], [17]. The same idea works for Wiener–Hopf plus/minus Hankel operators, convolution type operators with symmetry, and convolutionally equivalent operators [7], and it also works for other scales of spaces such as the Sobolev–Slobodetski spaces $W^{s,p}$ and the Zygmund spaces Z^s, as well as for matrix operators, cf. [5]. To illustrate the strategy, we here confine us to the basic variant of classical WHOs in Bessel potential spaces (one-dimensional, scalar, $p = 2$).

Let $\mathcal{F}$ be the Fourier transformation, $(\mathcal{F}f)(\xi) = \int_{\mathbb{R}} f(x)e^{i\xi x}dx$, and let H^s denote the Sobolev space of all distributions f on $\mathbb{R}$ such that $\lambda^s\mathcal{F}f \in L^2$, where $\lambda(\xi) = (\xi^2 + 1)^{1/2}$. The well-known Bessel potential operators are given by

$$\Lambda^s := A_{\lambda^s} := \mathcal{F}^{-1}\lambda^s \cdot \mathcal{F} : H^r \to H^{r-s},$$
$$\Lambda^s_\pm := A_{\lambda^s_\pm} := \mathcal{F}^{-1}\lambda^s_\pm \cdot \mathcal{F} : H^r \to H^{r-s},$$

where $\lambda_\pm(\xi) = \xi \pm i$; see, for example, [11], [13], [17]. Here r and s are real numbers. Let H^s_+ and H^s_- stand for the subspace of all distributions in H^s that are supported on $[0,\infty)$ and $(-\infty, 0]$, respectively. We then have

$$\Lambda^s_+(H^r_+) = H^{r-s}_+, \quad \Lambda^s_-(H^r_-) = H^{r-s}_-.$$

In terms of operator identities, this may be rephrased as follows. If $P_1^{(s)}$ and $P_2^{(s)}$ are any bounded projectors on H^s such that $\operatorname{im} P_1^{(s)} = H^s_+$ and $\ker P_2^{(s)} = H^s_-$, then

$$\Lambda^s_+ P_1^{(r)} = P_1^{(r-s)}\Lambda^s_+ P_1^{(r)}, \quad P_2^{(r-s)}\Lambda^s_- = P_2^{(r-s)}\Lambda^s_- P_2^{(r)}.$$

In accordance with [13], a *classical Wiener–Hopf operator* is given by

$$T = r_+ A_\Phi|_{H^r_+} : H^r_+ \to H^s(\mathbb{R}_+)$$

where $H^s(\mathbb{R}_+)$ is the common Hilbert space of all restrictions of distributions in H^s to $\mathbb{R}_+ = (0,\infty)$, $r_+ : f \mapsto f|_{\mathbb{R}_+}$ is the restriction operator, and A_Φ is a convolution (or translation invariant) operator of order $r - s$, that is, A_Φ is of the form

$$A_\Phi = \mathcal{F}^{-1}\Phi \cdot \mathcal{F} : H^r \to H^s \quad \text{with } \lambda^{s-r}\Phi \in L^\infty(\mathbb{R}).$$

Obviously, T is equivalent to the general Wiener–Hopf operator W given by

$$W = P_2^{(s)} A_\Phi|_{H^r_+} : P_1^{(r)} H^r \to P_2^{(s)} H^s,$$

where $P_2^{(s)} := \ell^{(s)} r_+ \in \mathcal{L}(H^s)$ and $\ell^{(s)} : H^s(\mathbb{R}_+) \to H^s$ is any bounded extension operator that is left invertible by r_+. The projector $P_1^{(r)}$ may be an arbitrary projector in $\mathcal{L}(H^r)$ such that $\operatorname{im} P_1^{(r)} = H^r_+$. The equivalence between T and W is simply given by $W = \ell^{(s)}T$ and $T = r_+W$. Thus, in the case at hand the setting X, Y, P_1, P_2 is $H^r, H^s, P_1^{(r)}, P_2^{(s)}$. As an interpretation of results in Section 2 of [17], a symmetrization of W is achieved by the so-called lifting to L^2: choosing

$$Z := H^0 = L^2(\mathbb{R}), \quad M_+ := \Lambda^r_+, \quad M_- := \Lambda^{-s}_-, \quad P := \ell_0 r_+,$$

where $\ell_0 : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ denotes the extension by zero, we get, with $\Phi_0 := \lambda_-^s \Phi \lambda_+^{-r}$, $P_1^{(0)} := \ell_0 r_+$, $P_2^{(0)} := \ell_0 r_+$,

$$\begin{aligned} W &= P_2^{(s)} A_\Phi|_{H_+^r} = P_2^{(s)} \Lambda_-^{-s} A_{\Phi_0} \Lambda_+^r|_{H_+^r} \\ &= P_2^{(s)} \Lambda_-^{-s}|_{P_2^{(0)} H^0} P_2^{(0)} A_{\Phi_0}|_{H_+^0} P_1^{(0)} \Lambda_+^r|_{H_+^r} \\ &= P_2^{(s)} \Lambda_-^{-s}|_{L_+^2} P A_{\Phi_0}|_{L_+^2} P \Lambda_+^r|_{H_+^r} =: E W_0 F. \end{aligned}$$

3. FACTORIZATIONS

DEFINITION 3.1. Let X, Y be Banach spaces, let $P_1 \in \mathcal{L}(X), P_2 \in \mathcal{L}(Y)$ be projectors, and let A be an operator in $G\mathcal{L}(X,Y)$. A factorization

$$\begin{array}{cccccccc} A & = & & A_- & & C & & A_+ \\ & : & Y & \leftarrow & Y & \leftarrow & X & \leftarrow X. \end{array}$$

is referred to as a *cross factorization* of A (with respect to X, Y, P_1, P_2), in brief CFn, if the factors $A_\pm$ and C possess the properties

$$(3.1) \qquad A_+ \in G\mathcal{L}(X), \quad A_- \in G\mathcal{L}(Y), \quad A_+(P_1 X) = P_1 X, \quad A_-(Q_2 Y) = Q_2 Y,$$

and $C \in G\mathcal{L}(X,Y)$ splits the spaces X, Y both into four complemented subspaces such that

$$(3.2) \qquad \begin{array}{ccccc} X & = & \overbrace{X_1 \oplus X_0}^{P_1 X} & \oplus & \overbrace{X_2 \oplus X_3}^{Q_1 X} \\ & & \downarrow & C \swarrow\!\!\!\!\searrow & \downarrow \\ Y & = & \underbrace{Y_1 \oplus Y_2}_{P_2 Y} & \oplus & \underbrace{Y_0 \oplus Y_3}_{Q_2 Y}. \end{array}$$

The last property means that C maps each X_j bijectively onto Y_j , $j = 0,1,2,3$, i.e., the complemented subspaces $X_0, X_1, \ldots, Y_3$ are images of the corresponding projectors $p_0, p_1, \ldots, q_3$, namely

$$(3.3) \qquad \begin{array}{ll} X_1 = p_1 X = C^{-1} P_2 C P_1 X, & X_0 = p_0 X = C^{-1} Q_2 C P_1 X, \\ X_2 = p_2 X = C^{-1} P_2 C Q_1 X, & X_3 = p_3 X = C^{-1} Q_2 C Q_1 X, \\ Y_1 = q_1 Y = C P_1 C^{-1} P_2 Y, & Y_2 = q_2 Y = C Q_1 C^{-1} P_2 Y, \\ Y_0 = q_0 Y = C P_1 C^{-1} Q_2 Y, & Y_3 = q_3 Y = C Q_1 C^{-1} Q_2 Y. \end{array}$$

The operators $A_\pm$ are called *strong WH factors* and C is said to be a *cross factor*, since it maps a part of $P_1 X$ onto a part of $Q_2 Y$ ($X_0 \to Y_0$) and a part of $Q_1 X$ onto a part of $P_2 Y$ ($X_2 \to Y_2$), which are all complemented subspaces.

The last two equalities in (3.1) can be formulated in various different ways, for instance as

$$A_+P_1 = P_1A_+P_1, A_+^{-1}P_1 = P_1A_+^{-1}P_1, P_2A_- = P_2A_-P_2, P_2A_-^{-1} = P_2A_-^{-1}P_2.$$

The *cross factorization theorem* tells us that if $A \in G\mathcal{L}(X,Y)$, then W is generalized invertible if and only if a cross factorization of A exists. In that case a formula for a generalized inverse of W is given by

$$W^- = A_+^{-1}P_1C^{-1}P_2A_-^{-1}|_{P_2Y} : P_2Y \to P_1X.$$

A crucial consequence is the equivalence of W and $P_2C|_{P_1X}$, that is, $W \sim P_2C|_{P_1X}$:

$$W = P_2A_-|_{P_2Y}P_2C|_{P_1X}P_1A_+|_{P_1X} = EP_2C|_{P_1X}F$$

where E, F are linear homeomorphisms. We refer to [22] for more details.

In [6], [24] another type of factorization was studied. This factorization is quite different from the previous and more interesting for certain applications.

DEFINITION 3.2. Suppose X, Y are Banach spaces, $P_1 \in \mathcal{L}(X), P_2 \in \mathcal{L}(Y)$ are projectors, and A is an operator in $G\mathcal{L}(X,Y)$. A factorization

$$\begin{array}{cccccccc} A & = & & A_- & & C & & A_+ \\ & : & Y & \leftarrow & Z & \leftarrow & Z & \leftarrow X. \end{array}$$

is called a *Wiener–Hopf factorization through an intermediate space* Z (with respect to the setting X, Y, P_1, P_2), in brief FIS, if Z, $A_\pm$, and C possess the following properties:

(i) Z is a Banach space;

(ii) $A_+ \in G\mathcal{L}(X,Z), C \in G\mathcal{L}(Z), A_- \in G\mathcal{L}(Z,Y)$;

(iii) there exists a projector $P \in \mathcal{L}(Z)$ such that, with $Q := I_Z - P$,

$$A_+(P_1X) = PZ, \quad A_-(QZ) = Q_2Y; \tag{3.4}$$

(iv) C splits the space Z twice into four complemented subspaces such that

$$\begin{array}{ccccc} Z & = & \overbrace{X_1 \oplus X_0}^{PZ} & \oplus & \overbrace{X_2 \oplus X_3}^{QZ} \\ & & \downarrow & C\swarrow\!\!\!\searrow & \downarrow \\ Z & = & \underbrace{Y_1 \oplus Y_2}_{PZ} & \oplus & \underbrace{Y_0 \oplus Y_3}_{QZ}. \end{array} \tag{3.5}$$

Thus, C maps each X_j bijectively onto Y_j, $j = 0,1,2,3$, i.e., the complemented subspaces $X_0, X_1, \ldots, Y_3$ are again images of corresponding projectors $p_0, p_1, \ldots, q_3$, namely $X_0 = p_0Z = C^{-1}QCPZ$, $X_1 = p_1Z = C^{-1}PCPZ$, $\ldots$, $Y_3 = q_3Z = CQC^{-1}QZ$, similarly as in (3.3) (with X and Y replaced by Z).

Again $A_\pm$ are called *strong WH factors* and C is said to be a *cross factor*, now acting from a space Z onto the same space Z. If the factor C in a FIS is the identity, we speak of a *canonical* FIS.

Proof of Theorem 1.2. We begin with the proof of the implication (i) $\Rightarrow$ (ii). Since $P_1 \sim P_2$, we infer from Theorem 1.1 that the setting X, Y, P_1, P_2 is symmetrizable. So let Z, P, $M_\pm$, $U_\pm$, $V_\pm$ be as in Section 1. Since A has a CFn, the operator $W = P_2 A|_{P_1 X}$ is generalized invertible. Consider the operators

$$\widetilde{A} := M_-^{-1} A M_+^{-1} : Z \to Z, \quad \widetilde{W} := P\widetilde{A}|_{PZ}.$$

Since $\widetilde{W} = U_- W U_+^{-1}$, the operator $\widetilde{W}$ is generalized invertible together with W. It follows that $\widetilde{A}$ has a CFn $\widetilde{A} = A_- C A_+$. But as A is an operator from Z to Z, this CFn is automatically a FIS. We so have $A = (M_- A_-) C (A_+ M_+)$, and because

$$A_+ M_+ (P_1 X) = A_+ U_+ (P_1 X) = A_+(PZ) = PZ,$$
$$M_- A_- (QZ) = M_-(QZ) = V_-(QZ) = Q_2 Y,$$

the factorization $A = (M_- A_-) C (A_+ M_+)$ is a FIS. This completes the proof of the implication (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i) Let $A = A_- C A_+$ be a FIS. A straightforward computation along the lines of pp. 27–29 in [22] shows that

$$W^- := A_+^{-1} P C^{-1} P A_-^{-1}|_{P_2 Y} : P_2 Y \to P_1 X \tag{3.6}$$

is a generalized inverse of W. Hence A has a CFn. From (3.4) we see that

$$A_+|_{P_1 X} = PA_+|_{P_1 X} : P_1 X \to PZ, \quad A_-|_{QZ} = Q_2 A_-|_{QZ} : QZ \to Q_2 Y$$

are invertible operators. Now (2.1) implies that

$$Q_1 A_+^{-1}|_{QZ} : QZ \to Q_1 X, \quad PA_-^{-1}|_{P_2 Y} : P_2 Y \to PZ$$

are also invertible. Consequently,

$$P_1 X \cong PZ \cong P_2 Y, \quad Q_1 X \cong QZ \cong Q_2 Y,$$

which, by Theorem 1.1, completes the proof. ∎

REMARK 3.3. We want to repeat outside the proof that (3.6) is a generalized inverse of the operator W. We also remark that a FIS of A implies the equivalence

$$W = P_2 A_-|_{PZ} PC|_{PZ} PA_+|_{P_1 X} = EPC|_{PZ} F.$$

This factorization of W is nicer than the corresponding factorization resulting from a CFn because the middle factor C in a FIS sorts the spaces as in (3.5), which is of higher quality than (3.2).

EXAMPLE 3.4. Consider the situation of Example 2.3. For simplicity, suppose b is identically 1. Then $T(b) = I_{L^p}$ is invertible and hence so also is $T(a) = V_+ U_+ : PL^p(\varrho) \to PL^p(\varrho^{-1})$. From the cross factorization theorem we deduce that $M(a)$ has a CFn

$$\begin{aligned} M(a) \quad &= \qquad A_- \qquad\quad C \qquad\quad A_+ \\ &: \quad L^p(\varrho^{-1}) \leftarrow L^p(\varrho^{-1}) \leftarrow L^p(\varrho) \leftarrow L^p(\varrho). \end{aligned}$$

Let P be the Riesz projector and put $Q = I - P$; the underlying space is suppressed in this notation. Furthermore, let us simply write c for the operator $M(c)$ of multiplication by c. The operator $Pa|_{\operatorname{im} P} : PL^p(\varrho) \to PL^p(\varrho^{-1})$ may be identified with the invertible Toeplitz operator $T(a)$. From (2.1) we deduce that $Qa^{-1}|_{\operatorname{im} Q} : QL^p(\varrho^{-1}) \to QL^p(\varrho)$ is also invertible. The last operator may be identified with $JT(\widetilde{a}^{-1})J : QL^p(\varrho^{-1}) \to QL^p(\varrho)$, where $J : \operatorname{im} Q \to \operatorname{im} P$ is the flip operator and $\widetilde{a}$ is defined by $\widetilde{a}(t) := a(1/t)$. One can show that

$$T(a) = T(\xi)T(\eta), \quad [T(a)]^{-1} = T(\eta^{-1})T(\xi^{-1}),$$
$$JT(\widetilde{a}^{-1})J = JT(\widetilde{\eta}^{-1})T(\widetilde{\xi}^{-1})J, \quad [JT(\widetilde{a}^{-1})J]^{-1} = JT(\widetilde{\xi})T(\widetilde{\eta})J.$$

A concrete CFn is given by

$$A_- = I + QaP(PaP)^{-1}P, \quad C = PaP + (Qa^{-1}Q)^{-1}Q, \quad A_+ = I + (PaP)^{-1}PaQ;$$

here and in the following we use the abbreviations

$$(Pa|_{\operatorname{im} P})^{-1} =: (PaP)^{-1}, \quad (Qa^{-1}|_{\operatorname{im} Q})^{-1} =: (Qa^{-1}Q)^{-1}.$$

The inverses of A_-, C, and A_+ are

$$A_-^{-1} = I - QaP(PaP)^{-1}P, \quad C = (PaP)^{-1} + Qa^{-1}Q, \quad A_+^{-1} = I - (PaP)^{-1}PaQ.$$

To verify the equality $A = A_-CA_+$ note that

$$A_-C = PaP + (Qa^{-1}Q)^{-1}Q + QaP,$$

whence

$$\begin{aligned} A_-CA_+ &= (PaP + (Qa^{-1}Q)^{-1}Q + QaP)(I + (PaP)^{-1}PaQ) \\ &= PaP + PaQ + (Qa^{-1}Q)^{-1}Q + QaP + QaP(PaP)^{-1}PaQ, \end{aligned}$$

and since

$$(Qa^{-1}Q)^{-1}Q = QaQ - QaP(PaP)^{-1}PaQ$$

due to (2.2), we obtain that

$$A_-CA_+ = PaP + PaQ + QaP + QaQ = a = A.$$

Finally, the last two equalities in (3.1) are obvious in the case at hand. The splitting (3.2) now takes the form

$$\begin{array}{ccccc} L^p(\varrho) & = & \overbrace{PL^p(\varrho) \quad \oplus \quad \{0\}}^{PL^p(\varrho)} & \oplus & \overbrace{\{0\} \quad \oplus \quad QL^p(\varrho)}^{QL^p(\varrho)} \\ & & \downarrow T(\xi)T(\eta) & I \swarrow\!\!\!\searrow & JT(\widetilde{\xi})T(\widetilde{\eta})J \downarrow \\ L^p(\varrho^{-1}) & = & \underbrace{PL^p(\varrho^{-1}) \quad \oplus \quad \{0\}}_{P_2Y} & \oplus & \underbrace{\{0\} \quad \oplus \quad QL^p(\varrho^{-1})}_{QL^p(\varrho^{-1})}. \end{array}$$

Since the setting X, Y, P_1, P_2 is symmetrizable, Theorems 1.1 and 1.2 imply that $M(a)$ also admits a FIS

$$\begin{array}{rcl} M(a) & = & B_- \quad D \quad B_+ \\ & : & L^p(\varrho^{-1}) \leftarrow L^p \leftarrow L^p \leftarrow L^p(\varrho). \end{array}$$

It is easily seen that a concrete FIS is given by $B_+ = M(\eta)$, $B_- = M(\xi)$, $D = I_{L^p}$. Thus, in this case (3.5) becomes

$$\begin{array}{ccccc} L^p & = & \overbrace{PL^p \oplus \{0\}}^{PL^p} & \oplus & \overbrace{\{0\} \oplus QL^p}^{QL^p} \\ & & \downarrow I & I \diagup\!\!\!\!\diagdown & I \downarrow \\ L^p & = & \underbrace{PL^p \oplus \{0\}}_{PL^p} & \oplus & \underbrace{\{0\} \oplus QL^p}_{QL^p}. \end{array}$$

It is obvious that the FIS is much simpler than the CFn.

EXAMPLE 3.5. The great advantage of the "equivalent reduction" of W to W_0 performed in Example 2.4 is that the generalized inversion of W is reduced to a factorization of Φ_0. In applications we often have $\Phi_0 \in GC^\mu(\ddot{\mathbb{R}})$, i.e., Φ is Hölder continuous on the two-point compactification $\ddot{\mathbb{R}} = [-\infty, +\infty]$ of the real line with Hölder conditions at $\pm\infty$. The factorization problem for this class of functions is completely solved; see [17]. Instead of embarking on the subtleties of factorization of functions in $GC^\mu(\ddot{\mathbb{R}})$, we move from the real line to unit circle $\mathbb{T}$ and to the Wiener–Hopf factorization of functions defined on $\mathbb{T}$. In this context things are a little easier.

The "middle factor" of a Wiener–Hopf factorization is a function of the form $c(t) = t^n$ ($t \in \mathbb{T}$) with $n \in \mathbb{Z}$. The multiplication operator $M(c) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is unitarily equivalent to $U^n : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, where U the shift operator defined by $(Ux)_j = x_{j-1}$. So let us work in $\ell^2(\mathbb{Z})$. Given a set $E \subset \mathbb{Z}$, we denote by $\ell^2(E)$ the subspace of the sequences in $\ell^2(\mathbb{Z})$ which are supported in E. After the symmetrization described in Example 2.4 and after passing from $\mathbb{R}$ to $\mathbb{Z}$, we are in the context where $X = Y = Z = \ell^2(\mathbb{Z})$, P is the orthogonal projection of $\ell^2(\mathbb{Z})$ onto $\ell^2(\{0,1,2,\ldots\})$, and $Q = I - P$. Let further $\mathbb{Z}_+ = \{0,1,2,\ldots\}$, $\mathbb{Z}_- = \{-1,-2,-3,\ldots\}$, $B_n^+ = \{0,\ldots,n-1\}$, $A_n^- = \{-1,\ldots,-n\}$. For $C = U^n$ with $n \geqslant 0$, the splittings (3.2) and (3.5) are

$$\begin{array}{ccccc} \ell^2(\mathbb{Z}) & = & \overbrace{\ell^2(\mathbb{Z}_+) \oplus \{0\}}^{\ell^2(\mathbb{Z}_+)} & \oplus & \overbrace{\ell^2(A_n^-) \oplus \ell^2(\mathbb{Z}_- \backslash A_n^-)}^{\ell^2(\mathbb{Z}_-)} \\ & & \downarrow & U^n \diagup\!\!\!\!\diagdown & \downarrow \\ \ell^2(\mathbb{Z}) & = & \underbrace{\ell^2(\mathbb{Z}_+ \backslash B_n^+) \oplus \ell^2(B_n^+)}_{\ell^2(\mathbb{Z}_+)} & \oplus & \underbrace{\{0\} \oplus \ell^2(\mathbb{Z}_-)}_{\ell^2(\mathbb{Z}_-)}, \end{array}$$

while for $C = U^{-n}$ with $n \geqslant 0$, we obtain

$$\begin{array}{ccccc}
\ell^2(\mathbb{Z}) & = & \overbrace{\ell^2(\mathbb{Z}_+\backslash B_n^+) \oplus \ell^2(B_n^+)}^{\ell^2(\mathbb{Z}_+)} & \oplus & \overbrace{\{0\} \quad \oplus \quad \ell^2(\mathbb{Z}_-)}^{\ell^2(\mathbb{Z}_-)} \\
 & & \downarrow \qquad\qquad\qquad U^{-n} & \swarrow\!\!\!\searrow & \qquad\qquad \downarrow \\
\ell^2(\mathbb{Z}) & = & \underbrace{\ell^2(\mathbb{Z}_+) \quad \oplus \quad \{0\}}_{\ell^2(\mathbb{Z}_+)} & \oplus & \underbrace{\ell^2(A_n^-) \oplus \ell^2(\mathbb{Z}_-\backslash A_n^-)}_{\ell^2(\mathbb{Z}_-)}.
\end{array}$$

Note that the cross factors arising in the Wiener–Hopf factorization of matrix functions are of the form $\mathrm{diag}(U^{n_1}, \ldots, U^{n_k})$ with $n_1, \ldots, n_k \in \mathbb{Z}$; see [9], [15].

4. ADDENDUM TO [24]

One of the main results of [24] is Theorem 2.2 on page 399, which is identical to Theorem 1.2 of the present paper. In the first part of the proof of Theorem 2.2 (page 405, lines 22–23) we find the conclusion "The factor properties of A_+ imply $P_1 \sim P$ and the factor properties of A_- imply $P_2 \sim P$, therefore $P_1 \sim P_2$ is necessarily satisfied", which is not correct. One can here directly conclude that $\mathrm{im}\, P_1 \cong \mathrm{im}\, P$ and $\ker P_2 \cong \ker P$ but not that $P_1 \sim P_2$.

To fix the gap we now replace that text "..." by the following: "The factor properties of A_+ and A_- imply that the definition of $\widetilde{A} = A_-A_+$ contains a canonical FIS (with respect to X, Y, P_1, P_2) through the same intermediate space Z that appeared in the (non-canonical) FIS of $A = A_-CA_+$. Hence $\widetilde{W} = P_2\widetilde{A}|_{P_1X}$ is invertible and $\mathrm{im}\, P_2 \cong \mathrm{im}\, P_1$. By a symmetry argument we prove that $\ker P_2 \cong \ker P_1$ holds, as well, exchanging the roles of X and Y, of P_1 and Q_2, and of P_2 and Q_1. Namely $A^{-1} = A_+^{-1}C^{-1}A_-^{-1}$ can be seen as a FIS of A^{-1} with respect to Y, X, Q_2, Q_1. Therefore $\widetilde{A}^{-1} = A_+^{-1}A_-^{-1}$ is a canonical FIS with respect to Y, X, Q_2, Q_1 through the same space Z as before, $\widetilde{W}_* = Q_1A_+^{-1}A_-^{-1}|_{Q_2Y}$ is invertible and $\mathrm{im}\, Q_1 \cong \mathrm{im}\, Q_2$, i.e., $\ker P_2 \cong \ker P_1$. Together with $\mathrm{im}\, P_2 \cong \mathrm{im}\, P_1$ we arrive at $P_2 \sim P_1$". The rest of the proof of Theorem 2.2 is then correct in the existing form.

Beside of this change two misprints have to be corrected.

(i) The first obvious misprint occurs in formula (1.5) on page 397. Line 4 has to start with $Y_0 = q_0Y = CP_1C^{-1}Q_2Y$.

(ii) The second appears in Corollary 2.5 on page 400 where the assumption $\ker P_2 \cong \ker P_1$ was forgotten. Under the general assumption that A is boundedly invertible, this corollary correctly reads as follows. *If W is generalized invertible **and if** $P_1 \sim P_2$, there exists a* FIS *of A with $Z = X$ and a* FIS *with $Z = Y$ (and a* FIS *with any prescribed intermediate space that is isomorphic to X and Y).*

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ALBRECHT BÖTTCHER, FAKULTÄT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT CHEMNITZ, 09107, GERMANY
E-mail address: aboettch@mathematik.tu-chemnitz.de

FRANK-OLME SPECK, DEPARTAMENTO DE MATEMÁTICA, INSTITUTO SUPERIOR TÉCNICO, UNIVERSIDADE DE LISBOA, 1049-001, PORTUGAL
E-mail address: fspeck@tecnico.pt

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