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MOURRE'S COMMUTATORS METHOD FOR A DISSIPATIVE FORM PERTURBATION

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ABSTRACT. We prove uniform resolvent estimates for an abstract operator given by a dissipative perturbation of a self-adjoint operator in the sense of forms. For this we adapt the commutators method of Mourre. We also obtain the limiting absorption principle and uniform estimates for the derivatives of the resolvent. Finally, we study the absolutely continuous subspace in the sense of Davies. This abstract work is motivated in particular by the Schrödinger and wave equations on a wave guide with dissipation at the boundary.

KEYWORDS: *Mourre's theory, limiting absorption principle, dissipative operators, quadratic forms, relatively smooth operators in the sense of Kato.*

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1. INTRODUCTION

The main purpose of this paper is to prove uniform resolvent estimates and the limiting absorption principle for a dissipative operator obtained by a form-perturbation of a self-adjoint operator. For this we prove a suitable version of the commutators method of Mourre.

Given a self-adjoint operator H_0 on a Hilbert space $\mathcal{H}$, the purpose of the Mourre method (see [24]) is to prove uniform estimates for the weighted resolvent

$$\langle A\rangle^{-\delta}(H_0-z)^{-1}\langle A\rangle^{-\delta} \tag{1.1}$$

when z is close to the real axis. Here $\delta > \frac{1}{2}$ and A is a self-adjoint operator which satisfies in particular

$$\mathbf{1}_J(H_0)[H_0,\mathrm{i}A]\mathbf{1}_J(H_0) \geqslant \alpha\mathbf{1}_J(H_0), \quad \text{for some } \alpha > 0. \tag{1.2}$$

J is an open interval of $\mathbb{R}$ and $\mathbf{1}_J$ is the characteristic function of J. Under some additional assumptions on A and its commutators with H_0, it is proved that the operator (1.1) is uniformly bounded for $\mathrm{Re}(z)$ in a compact subset of J and $\mathrm{Im}(z) \neq 0$. In addition to these resolvent estimates, the method gives the limiting absorption principle: not only the operator (1.1) is uniformly bounded for z close to the real axis, but for $\mathrm{Re}(z) \in J$ it has a limit when $\pm \mathrm{Im}(z) \searrow 0$.

Compared to previous commutators methods (see for instance [20], [21], [22], [31]), the assumption (1.2) on the commutator is spectrally localized with respect to H_0. This proved to be very efficient for difficult problems such as the N-body problem (see for instance [9], [12], [29] and references therein).

There are many generalizations of the original result in various directions. Here we only refer to [2] for a general overview of the subject and focus on dissipative operators.

In [37], we generalized the result of [24] for a dissipative operator $H = H_0 - \mathrm{i}V$, where $V \geqslant 0$ is relatively bounded with respect to H_0. The main difficulty in this setting is that we cannot localize spectrally with respect to the non-selfadjoint operator H. It turns out that we can obtain a similar result using the spectral projections of the self-adjoint part H_0. It is even possible to use the dissipative part to weaken the assumption as follows:

$$\mathbf{1}_J(H_0)([H_0, \mathrm{i}A] + \beta V)\mathbf{1}_J(H_0) \geqslant \alpha \mathbf{1}_J(H_0), \quad \alpha > 0,\ \beta \geqslant 0. \tag{1.3}$$

Notice that for a general maximal dissipative operator we only know that the spectrum is included in the lower half-plane $\{\mathrm{Im}(z) \leqslant 0\}$ and the estimates for the weighted resolvent (1.1) (with H_0 replaced by $H = H_0 - \mathrm{i}V$) are only available for $\mathrm{Im}(z) > 0$.

Then in [5] we adapted to this setting the results of [14], [15] about the derivatives of the resolvent. We also mention [6] for a closely related context.

The present work is motivated by the dissipative wave guide. We consider a domain of the form $\Omega = \mathbb{R}^p \times \omega \subset \mathbb{R}^d$ where $p \in \{1, \dots, d-1\}$ and ω is a smooth open bounded and connected subset of $\mathbb{R}^{d-p}$. Given $a \in W^{1,\infty}(\partial\Omega)$, we consider on $L^2(\Omega)$ the operator

$$H_a = -\Delta \tag{1.4}$$

with domain

$$\mathcal{D}(H_a) = \{u \in H^2(\Omega), \partial_\nu u = \mathrm{i}au \text{ on } \partial\Omega\}. \tag{1.5}$$

This operator appears in the spectral analysis of the wave equation

$$\begin{cases} \partial_t^2 w - \Delta w = 0 & \text{on } \mathbb{R}_+ \times \Omega, \\ \partial_\nu w + a\partial_t w = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ (w, \partial_t w)|_{t=0} = (w_0, w_1) & \text{on } \Omega, \end{cases} \tag{1.6}$$

or the Schrödinger equation

$$\begin{cases} -\mathrm{i}\partial_t u - \Delta u = 0 & \text{on } \mathbb{R}_+ \times \Omega, \\ \partial_\nu u = \mathrm{i} a u & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ u|_{t=0} = u_0 & \text{on } \Omega. \end{cases} \tag{1.7}$$

The function a is the absorption index on the boundary. We assume that a takes non-negative values. We could also consider a (dissipative) perturbation of the free laplacian in (1.4).

The wave equation with dissipation at the boundary has already been studied on a compact domain (we refer for instance to the stabilisation results of [4] and [23]) or on exterior domains (see [1] about the local energy decay for (1.6) and (1.7)). But little is known for these equations on a domain which is neither bounded nor a compact (or asymptotically vanishing) perturbation of the Euclidean space.

In [38] we have studied (1.7) on a wave guide in the particular case where $\dim(\omega) = 1$ and a is greater than a positive constant at least on one side of the boundary. In this situation it is possible to compute almost explicitely some spectral properties of H_a. In particular, its spectrum $\sigma(H_a)$ is included in the half-plane $\{z \in \mathbb{C} : \mathrm{Im}(z) < -\gamma\}$ for some $\gamma > 0$ with a uniform estimate for the resolvent on the real axis. This gives in particular exponential decay for the solution of (1.7).

In this paper, we are interested in uniform estimates for the resolvent of H_a when the spectral parameter z with $\mathrm{Im}(z) > 0$ goes to some $E > 0$. The question is irrelevant on a compact domain, since the spectrum is given by a sequence of eigenvalues with negative imaginary parts. In [1], the authors study the local energy decay with a compactly supported weight, so the corresponding resolvent has a meromorphic continuation through the real axis. And, as mentioned above, the damping is so strong in [38] that there is a spectral gap around the real axis. When the absorption index a is not that strong, for instance if it is compactly supported on $\partial\Omega$, the essential spectrum is included in the real axis. And with a polynomially decaying weight, there is no meromorphic continuation of the resolvent through this essential spectrum.

The dissipative operator H_a cannot be written as $H_0 - \mathrm{i}V$ for self-adjoint operators H_0 and $V \geqslant 0$, so we cannot apply the results of [37] to obtain the limiting absorption principle in this setting. However the quadratic form q_a associated with H_a (see (3.2) below) can be written as $q_a = q_0 - \mathrm{i}q_\Theta$ where q_0 is the quadratic form corresponding to a self-adjoint operator and q_Θ is a non-negative quadratic form relatively bounded with respect to q_0. Inspired by the self-adjoint results of [3], we generalize the Mourre method for this kind of operators. This will give uniform estimates for the resolvent, the limiting absorption principle, and estimates for the derivatives of the resolvent (or, more generally, for powers of the resolvent with inserted operators). For this we will have to show uniform estimates for operators of the form

$$\langle A\rangle^{\delta}\mathbf{1}_{\mathbb{R}_-}(A)(H-z)^{-1}\mathbf{1}_{\mathbb{R}_+}(A)\langle A\rangle^{\delta}$$

or

$$\langle A\rangle^{-\delta}(H-z)^{-1}\mathbf{1}_{\mathbb{R}_+}(A)\langle A\rangle^{\delta-1}$$

as in [14]. Notice that the Mourre method has already been used for wave guides in a self-adjoint context. See for instance [19], [33] and references therein.

The original motivation for the Mourre theory was to prove the absence of singular spectrum for the self-adjoint operator H_0 in the interval J which appears in (1.2). This is an important question in scattering theory.

The absolutely continuous spectrum and the corresponding absolutely continuous subspace are *a priori* only defined for self-adjoint operators, so this is not the main motivation for generalizing the abstract theory to the dissipative setting. However, an absolutely continuous subspace corresponding to a maximal dissipative operator H on $\mathcal{H}$ has been defined in [7] as the closure in $\mathcal{H}$ of

$$\left\{\varphi\in\mathcal{H}\,:\,\exists C_\varphi\geqslant 0,\forall\psi\in\mathcal{H},\int_0^{+\infty}|\langle e^{-itH}\varphi,\psi\rangle|^2\,dt\leqslant C_\varphi\|\psi\|_{\mathcal{H}}^2\right\}.$$

This definition coincide with the usual one for a self-adjoint operator. Notice that there are other generalizations for the notion of absolutely continuous subspace in the litterature (see for instance [26], [34], [35], [39], [40], [41]). We prove in this paper that the uniform resolvent estimates given by the Mourre theory give results on the absolutely continuous subspace in the sense of Davies. For this we will use the dissipative generalization of the theory of relatively smooth operators in the sense of Kato.

This paper is organized as follows. In Section 2 we give precise definitions for the dissipative operator H which we consider and the corresponding conjugate operator A. Then in Section 3 we describe the applications which motivated this abstract work: the Schrödinger operator on a wave-guide or on a half-space with dissipation at the boundary, and then the Schrödinger operator on $\mathbb{R}^d$ whose absorption index becomes singular for low frequencies. In Section 4 we state and prove the main theorem of this paper about uniform estimates and the limiting absorption principle. Finally we discuss the resolvent estimates for the derivatives of the resolvent in Section 5 and the absolutely continuous subspace in Section 6.

We close this introduction with some general notation. We set

$$\mathbb{C}_+ := \{z\in\mathbb{C}\,:\,\mathrm{Im}(z)>0\},$$

and for $I\subset\mathbb{R}$:

$$\mathbb{C}_{I,+} := \{z\in\mathbb{C}\,:\,\mathrm{Re}(z)\in I,\mathrm{Im}(z)>0\}.$$

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, we denote by $\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ the space of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$.

2. DISSIPATIVE OPERATORS AND ASSOCIATED CONJUGATE OPERATORS

In this section we recall some basic facts about dissipative operators given by form perturbations of self-adjoint operators, and we introduce the corresponding conjugate operators. Let $\mathcal{H}$ be a complex Hilbert space.

DEFINITION 2.1. We say that an operator T with domain $\mathcal{D}(T)$ on the Hilbert space $\mathcal{H}$ is dissipative (respectively accretive) if

$$\forall \varphi \in \mathcal{D}(T), \quad \operatorname{Im}\langle T\varphi, \varphi\rangle_{\mathcal{H}} \leqslant 0, \quad (\text{respectively } \operatorname{Re}\langle T\varphi, \varphi\rangle_{\mathcal{H}} \geqslant 0).$$

Moreover T is said to be maximal dissipative (maximal accretive) if it has no other dissipative (accretive) extension on $\mathcal{H}$ than itself.

With these conventions, an operator T is (maximal) dissipative if and only if $\mathrm{i}T$ is (maximal) accretive. Moreover we recall that a dissipative operator T is maximal dissipative if and only if $(T - z)$ has a bounded inverse on $\mathcal{H}$ for some (and hence any) $z \in \mathbb{C}$ with $\operatorname{Im}(z) > 0$.

Let q_0 be a quadratic form closed, densely defined, symmetric and bounded from below. We denote by $\mathcal{K}$ its domain. The space $\mathcal{K}$ is endowed with the norm

$$\|\varphi\|_{\mathcal{K}}^2 = q_0(\varphi) + (\gamma + 1)\|\varphi\|_{\mathcal{H}}^2, \tag{2.1}$$

where $\gamma \geqslant 0$ is such that $q_0(\varphi) \geqslant -\gamma\|\varphi\|_{\mathcal{H}}^2$ for all $\varphi \in \mathcal{K}$. We identify $\mathcal{H}$ with its dual and denote by $\mathcal{K}^*$ the dual of $\mathcal{K}$. Then the form q_0 naturally defines an operator $\widetilde{H}_0$ in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ by

$$\forall \varphi, \psi \in \mathcal{K}, \quad \langle \widetilde{H}_0\varphi, \psi\rangle_{\mathcal{K}^*,\mathcal{K}} = q_0(\varphi, \psi)$$

(given a quadratic form, we always use the same notation for the corresponding sesquilinear form). We denote by H_0 the self-adjoint operator on $\mathcal{H}$ (with domain $\mathcal{D}(H_0)$) associated with q_0 by the representation theorem (see Theorem VI.2.6 of [17]).

Let q_Θ be a symmetric and non-negative form on $\mathcal{H}$. We assume that q_Θ is q_0-bounded. This means that its domain contains $\mathcal{K}$ and that there exists $C_\Theta \geqslant 0$ such that for all $\varphi \in \mathcal{K}$ we have

$$|q_\Theta(\varphi)| \leqslant C_\Theta\|\varphi\|_{\mathcal{K}}^2. \tag{2.2}$$

We set $q = q_0 - \mathrm{i}q_\Theta$ and denote by $\widetilde{H}$ the corresponding operator in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$. We also denote by $\Theta \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ the operator corresponding to q_Θ. Thus in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ we have

$$\widetilde{H} = \widetilde{H}_0 - \mathrm{i}\Theta. \tag{2.3}$$

We recall from [38] the following lemma:

LEMMA 2.2. *Let q_{R} be a non-negative, densely defined, closed form on a Hilbert space $\mathcal{H}$. Let q_{I} be a symmetric form relatively bounded with respect to q_{R}. Then the form $q_{\mathrm{R}} - \mathrm{i}q_{\mathrm{I}}$ (defined on the same domain as q_{R}) is sectorial and closed.*

We apply this lemma to $q_{\mathrm{R}} = q_0 + \gamma$ (we recall that γ was defined in (2.1)) and $q_{\mathrm{I}} = q_\Theta$. This proves that $q + \gamma$ is sectorial and closed. We denote by H_γ the maximal accretive operator associated with $q + \gamma$ by the representation theorem. Since q_Θ is non-negative, this is a dissipative and hence maximal dissipative operator. Thus $H := H_\gamma - \gamma$ is a maximal dissipative operator and it is associated with q in the sense of the representation theorem. This means in particular that its domain $\mathcal{D}(H)$ is the set of $u \in \mathcal{K}$ such that

$$\exists f \in \mathcal{H}, \forall \phi \in \mathcal{K}, \quad q(u,\phi) = \langle f, \phi \rangle_{\mathcal{H}},$$

and that for $\varphi \in \mathcal{D}(H)$ we have

$$\langle H\varphi, \varphi \rangle_{\mathcal{H}} = q(\varphi).$$

It is important to note that the form q_Θ is not assumed to be closable, so it is not *a priori* associated with any operator on $\mathcal{H}$. This means that we cannot write an equality analogous to (2.3) involving the operators H_0 and H. In particular we cannot write a resolvent identity between $(H-z)^{-1}$ and $(H_0 - z)^{-1}$. This identity was important in [37] to obtain results on $(H-z)^{-1}$ by using spectral localizations with respect to H_0.

We can use (2.3) to write a resolvent identity in the sense of forms. According to the Lax–Milgram theorem the operators $(\widetilde{H}_0 - z)^{-1}$ and $(\widetilde{H} - z)$ have bounded inverses in $\mathcal{L}(\mathcal{K}^*, \mathcal{K})$ for all $z \in \mathbb{C}_+$. Moreover, in $\mathcal{L}(\mathcal{K}^*, \mathcal{K})$ we have the resolvent identities

$$\begin{aligned}(\widetilde{H} - z)^{-1} &= (\widetilde{H}_0 - z)^{-1} + \mathrm{i}(\widetilde{H}_0 - z)^{-1}\Theta(\widetilde{H} - z)^{-1} \\ &= (\widetilde{H}_0 - z)^{-1} + \mathrm{i}(\widetilde{H} - z)^{-1}\Theta(\widetilde{H}_0 - z)^{-1}.\end{aligned} \tag{2.4}$$

Notice that for $\varphi \in \mathcal{H} \subset \mathcal{K}^*$ we have

$$(H-z)^{-1}\varphi = (\widetilde{H} - z)^{-1}\varphi,$$

so we can study $(\widetilde{H} - z)^{-1}$ to obtain information on $(H-z)^{-1}$.

We now introduce the conjugate operator A for H. Before the definition we recall from [3] (see Lemmas 1.1.3 and 1.1.4) the following result.

LEMMA 2.3. *Let A be a self-adjoint operator on $\mathcal{H}$. Assume that $\mathcal{K}$ is left invariant by $\mathrm{e}^{-\mathrm{i}tA}$ for all $t \in \mathbb{R}$. Then by restriction the family of operators $(\mathrm{e}^{-\mathrm{i}tA})_{t\in\mathbb{R}}$ defines a continuous semigroup on $\mathcal{K}$. Moreover the domain of the generator of this semigroup is*

$$\{\varphi \in \mathcal{D}(A) \cap \mathcal{K} \,:\, A\varphi \in \mathcal{K}\}.$$

Given $t \in \mathbb{R}$, we remark that under the assumption of Lemma 2.3 we can extend by duality the operator $\mathrm{e}^{-\mathrm{i}tA}$ to a bounded operator on $\mathcal{K}^*$, which is also left invariant.

DEFINITION 2.4. Let A be a self-adjoint operator on $\mathcal{H}$. We say that A is a *conjugate operator (in the sense of forms)* to H on the interval J if the following conditions are satisfied:

(i) The form domain $\mathcal{K}$ of H is left invariant by e^{-itA} for all $t \in \mathbb{R}$. We denote by $\mathcal{E}$ the domain of the generator of $e^{-itA}|_{\mathcal{K}}$.

(ii) There exists $C \geqslant 0$ such that for $\varphi, \psi \in \mathcal{E}$ we have

$$|\langle \widetilde{H}_0 A\varphi, \psi\rangle_{\mathcal{K}^*,\mathcal{K}} - \langle \widetilde{H}_0 \varphi, A\psi\rangle_{\mathcal{K}^*,\mathcal{K}}| \leqslant C\|\varphi\|_{\mathcal{K}}\|\psi\|_{\mathcal{K}} \tag{2.5}$$

and

$$|\langle \widetilde{H} A\varphi, \psi\rangle_{\mathcal{K}^*,\mathcal{K}} - \langle \widetilde{H} \varphi, A\psi\rangle_{\mathcal{K}^*,\mathcal{K}}| \leqslant C\|\varphi\|_{\mathcal{K}}\|\psi\|_{\mathcal{K}}.$$

Thus the commutators $[\widetilde{H}_0, iA]$ and $[\widetilde{H}, iA]$ extend to operators in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ which we denote by B_0 and B, respectively.

(iii) There exists $C \geqslant 0$ such that for $\varphi, \psi \in \mathcal{E}$ we have

$$|\langle BA\varphi, \psi\rangle_{\mathcal{K}^*,\mathcal{K}} - \langle B\varphi, A\psi\rangle_{\mathcal{K}^*,\mathcal{K}}| \leqslant C\|\varphi\|_{\mathcal{K}}\|\psi\|_{\mathcal{K}}.$$

(iv) There exist $\alpha \in]0,1]$ and $\beta \geqslant 0$ such that

$$\mathbf{1}_J(H_0)(B_0 + \beta\Theta)\mathbf{1}_J(H_0) \geqslant \alpha \mathbf{1}_J(H_0). \tag{2.6}$$

REMARK 2.5. If $H = H_0 - iV$ and A is a conjugate operator for H on J in the sense of Definition 2.3 in [37] then H can be seen as a perturbation of H_0 in the sense of forms and A is a conjugate operator for H on J in the sense of Definition 2.4.

The domain $\mathcal{E}$ of the generator of $e^{-itA}|_{\mathcal{K}}$ is endowed with the graph norm of A on $\mathcal{K}$:

$$\|\varphi\|_{\mathcal{E}} = \|A\varphi\|_{\mathcal{K}} + \|\varphi\|_{\mathcal{K}}.$$

When dealing with a family of operators indexed by a parameter λ, it may be important to track the dependency in λ of all the quantities which appear in this definition. For Schrödinger operators, this is for instance the case for high-frequency estimates (then λ is the semiclassical parameter) or low-frequency estimates (see for instance the last example of Section 3). In this case we will refer to the following refined version of Definition 2.4:

DEFINITION 2.6. We say that A is a *conjugate operator (in the sense of forms) to H on J and with bounds $(\alpha, \beta, \Upsilon) \in]0,1] \times \mathbb{R}_+ \times \mathbb{R}_+$* if all the assumptions of Definition 2.4 are satisfied (in particular α and β are the constants which appears in (2.6)) and moreover

$$\|B\| \leqslant \sqrt{\alpha}\Upsilon, \quad \|B + \beta\Theta\|\|B_0\| \leqslant \alpha\Upsilon \quad \text{and} \quad \|[B, A]\| + \beta\|[\Theta, A]\| \leqslant \alpha\Upsilon,$$

where all the norms are in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$.

These definitions include the assumptions needed to prove a uniform estimate and the limiting absorption principle for the resolvent of H. However it is known that in order to estimate the derivatives of the resolvent we have to control more commutators of H with the conjugate operator A.

DEFINITION 2.7. Let $N \in \mathbb{N}^*$. We set $B_1 = B$. We say that the self-adjoint operator A is a *conjugate operator for H on J up to order N* if it is a conjugate operator in the sense of Definition 2.4 and if for all $n \in \{1, \ldots, N\}$ the operator $[B_n, \mathrm{i}A]$ defined (inductively) in $\mathcal{L}(\mathcal{E}, \mathcal{E}^*)$ extends to an operator in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$, which we denote by B_{n+1}.

Again, for a family of operators it may be useful to control the size of these multiple commutators.

DEFINITION 2.8. We say that A is a *conjugate operator for H on J with bounds $(\alpha, \beta, \Upsilon_N)$ up to order N* if there exists $\Upsilon \geqslant 0$ such that it is a conjugate operator for H on J with bounds $(\alpha, \beta, \Upsilon)$ in the sense of Definition 2.6, it is a conjugate operator up to order N in the sense of Definition 2.7, and

$$\Upsilon + \frac{1}{\alpha} \sum_{n=2}^{N+1} \|B_n\|_{\mathcal{L}(\mathcal{K}, \mathcal{K}^*)} \leqslant \Upsilon_N.$$

3. THE DISSIPATIVE WAVE GUIDE AND OTHER APPLICATIONS

Before going further, we give some applications to illustrate the definitions of Section 2 and to motivate the upcoming abstract theorems.

We first recall that for the free laplacian $-\Delta$ on $\mathbb{R}^d$ an example of conjugate operator is given by the generator of dilations

$$A = -\frac{\mathrm{i}}{2}(x \cdot \nabla + \nabla \cdot x) = -\mathrm{i}(x \cdot \nabla) - \frac{\mathrm{i}d}{2}. \tag{3.1}$$

Indeed for all $t \in \mathbb{R}$ the dilation $\mathrm{e}^{-\mathrm{i}tA}$ maps u to $\mathrm{e}^{-\mathrm{i}tA}u : x \mapsto \mathrm{e}^{-\mathrm{d}t/2}u(\mathrm{e}^{-t}x)$. In particular it leaves invariant the form domain $H^1(\mathbb{R}^d)$. Moreover a straightforward computation gives $[-\Delta, \mathrm{i}A] = -2\Delta$, so A is conjugate to $-\Delta$ on any interval $J \Subset \mathbb{R}_+^*$ with bound $\alpha = 2\inf(J) > 0$. The study of more general Schrödinger operators is usually inspired by this model case.

As mentioned in introduction, this work is in particular motivated by the dissipative wave guide with damping at the boundary. We recall that we considered a domain Ω of the form $\mathbb{R}^p \times \omega$ where $p \in \{1, \ldots, d-1\}$ and ω is a smooth open bounded and connected subset of $\mathbb{R}^{d-p}$, and $a \in W^{1,\infty}(\partial\Omega, \mathbb{R}_+)$.

For boundary value problems, we have to consider the restrictions on the boundary of functions defined on the domain. The trace theorems are well known when Ω is a half-space or a bounded domain in $\mathbb{R}^d$ (see for instance Section 1.5 of [11]). We easily obtain an analogous result on a wave guide. Given $\phi \in C_0^\infty(\overline{\Omega})$, we denote by $T_{\partial\Omega}\phi$ the restriction of ϕ on $\partial\Omega$. We will only use the following version of the trace theorem.

LEMMA 3.1. *The map $\phi \mapsto T_{\partial\Omega}\phi$ extends to a bounded operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$.*

Proof. Let $\phi \in C_0^\infty(\overline{\Omega})$. Let $x \in \mathbb{R}^d$. By the trace theorem on ω we have

$$\|\phi(x,\cdot)\|_{L^2(\partial\omega)}^2 \lesssim \|\phi(x,\cdot)\|_{H^1(\omega)}^2.$$

The result follows after integration over $x \in \mathbb{R}^d$. ∎

The quadratic forms q_0, q_Θ and q_a are defined on $\mathcal{K} = H^1(\Omega)$ by

$$q_0(\varphi) = \int_\Omega |\nabla\varphi|^2, \quad q_\Theta(\varphi) = \int_{\partial\Omega} a|\varphi|^2 \quad \text{and} \quad q_a(\varphi) = q_0 - \mathrm{i}q_\Theta. \tag{3.2}$$

The self-adjoint part q_0 is a closed, densely defined, symmetric and non-negative quadratic form on $L^2(\Omega)$. The imaginary part q_Θ is non-negative, and according to Lemma 3.1 it is relatively bounded with respect to q_0. Thus we are in the setting of Section 2, and in particular there is a maximal dissipative operator associated with q_a by the representation theorem. As in [38] we can prove that this is the operator H_a defined by (1.4)–(1.5).

The form q_0 is associated with the operator H_0 (defined as H_a with the Neumann boundary condition $\partial_\nu u = 0$ on $\partial\Omega$), but the imaginary part q_Θ cannot be associated with an operator on $L^2(\Omega)$. Indeed, assume by contradiction that there exists an operator Θ on $L^2(\Omega)$ with domain $\mathcal{D}(\Theta)$ dense in $H^1(\Omega)$ and such that

$$\forall u \in \mathcal{D}(\Theta), \forall v \in H^1(\Omega), \quad \langle \Theta u, v\rangle_{L^2(\Omega)} = q_\Theta(u,v).$$

Let $u \in \mathcal{D}(\Theta)$. We have $\langle \Theta u, v\rangle = 0$ for any $v \in C_0^\infty(\Omega)$, so $\Theta u = 0$ in $L^2(\Omega)$. Then $q_\Theta(u,v) = 0$ for all $v \in H^1(\Omega)$, which means that $au = 0$ on $\partial\Omega$. Unless a vanishes almost everywhere on $\partial\Omega$, this gives a contradiction with the fact that $\mathcal{D}(\Theta)$ is dense in $H^1(\Omega)$.

Moreover, since $\mathcal{D}(H_a) \neq \mathcal{D}(H_a^*) = \mathcal{D}(H_{-a})$ there is no hope to write H_a as $H_{\text{s.a.}} - \mathrm{i}V$ for some self-adjoint operator $H_{\text{s.a.}}$ and some non-negative self-adjoint operator V relatively bounded with respect to $H_{\text{s.a.}}$ with relative bound less than 1 as is required in [5], [37].

We define $\widetilde{H}_0$ and $\widetilde{H}_a$ in $\mathcal{L}(\mathcal{K},\mathcal{K}^*)$ as in Section 2. Let L_ω be the Laplacian with Neumann boundary condition on the compact domain ω. L_ω is self-adjoint on $L^2(\omega)$ with compact resolvent. We denote by $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ its eigenvalues and by $(\varphi_n)_{n\in\mathbb{N}}$ a corresponding sequence of orthonormal eigenfunctions. The spectrum of H_0 is given by $\bigcup_{n\in\mathbb{N}} \lambda_n + \mathbb{R}_+ = \mathbb{R}_+$, and the eigenvalues of L_ω are the thresholds in the spectrum of H_0. We denote by $\mathcal{T}$ the set of these thresholds. Assume that $u \in \mathcal{D}(H_0)$ and $\lambda \in \mathbb{R}$ are such that $H_0 u = \lambda u$. We denote by (x,y) with $x \in \mathbb{R}^p$ and $y \in \omega$ a general point in Ω. Let

$$\widehat{u} : (\xi, y) \in \mathbb{R}^p \times \omega \mapsto \int_{x\in\mathbb{R}^p} \mathrm{e}^{-\mathrm{i}\langle x,\xi\rangle} u(x,y)\,\mathrm{d}x$$

be the partial Fourier transform of u with respect to x. Then for almost all $\xi \in \mathbb{R}^p$ we have

$$(L_\omega + |\xi|^2 - \lambda)\widehat{u}(\xi, \cdot) = 0.$$

Since L_ω has a discrete set of eigenvalues, $\widehat{u}(\xi, \cdot)$ vanishes for ξ outside a set of measure 0 in $\mathbb{R}^p$. This proves that $u = 0$, and hence H_0 has no eigenvalue.

We denote by ∇_x the gradient with respect to the first p variables on Ω. Then we consider the generator A_x of dilations in the first p variables, defined by

$$(A_x u)(x, y) = -\mathrm{i}x \cdot \nabla_x u(x, y) - \frac{\mathrm{i}p}{2}.$$

Then for $u \in L^2(\Omega)$, $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^p \times \omega$ we have

$$\mathrm{e}^{-\mathrm{i}tA_x} u(x, y) = \mathrm{e}^{-tp/2} u(\mathrm{e}^{-t}x, y). \tag{3.3}$$

PROPOSITION 3.2. *Let $J \subset \mathbb{R}_+^* \setminus \mathcal{T}$ be a compact interval. Let $N \in \mathbb{N}^*$. Assume that for $\gamma \in \mathbb{N}^p$ with $|\gamma| \leqslant N + 1$ we have*

$$|\partial_x^\gamma a(x, y)| \leqslant c_\gamma \langle x \rangle^{-|\gamma|}. \tag{3.4}$$

Then A_x is conjugate to H_a on J up to order N.

This proposition implies that all the abstract results of the following sections will apply to the operator H_a and its conjugate operator A_x.

Proof. Step 1. According to (3.3) the form domain $\mathcal{K} = H^1(\Omega)$ is left invariant by $\mathrm{e}^{-\mathrm{i}tA_x}$ for any $t \in \mathbb{R}$. Let φ, ψ in $C_0^\infty(\overline{\Omega})$ (the set of restrictions of functions in $C_0^\infty(\mathbb{R}^d)$ on Ω). If $-\Delta_x$ denotes the Laplacian in the first p directions we have on Ω

$$[-\Delta, \mathrm{i}A_x] = [-\Delta_x, \mathrm{i}A_x] = -2\Delta_x,$$

so

$$\langle [\widetilde{H}_0, \mathrm{i}A_x]\varphi, \psi \rangle = 2\langle \nabla_x \varphi, \nabla_x \psi \rangle_{L^2(\Omega)}.$$

Since A_x only acts on the x directions, it can be seen as an operator on $L^2(\partial\Omega)$. Then on $\partial\Omega$ we have

$$[a, \mathrm{i}A_x] = -(x \cdot \nabla_x)a,$$

and hence

$$\langle [-\mathrm{i}\Theta_a, \mathrm{i}A_x]\varphi, \psi \rangle = \mathrm{i} \int_{\partial\Omega} a(A_x\varphi\overline{\psi} - \varphi\overline{A_x\psi}) = \mathrm{i} \int_{\partial\Omega} (x \cdot \nabla_x a)\varphi\overline{\psi}.$$

Similarly we obtain for any $n \in \{0, \dots, N\}$

$$\langle \mathrm{ad}_{\mathrm{i}A_x}^n(\widetilde{H}_a)\varphi, \psi \rangle = 2^n \langle \nabla_x \varphi, \nabla_x \psi \rangle_{L^2(\Omega)} - \mathrm{i} \int_{\partial\Omega} ((-x \cdot \nabla_x)^n a)\varphi\,\overline{\psi}.$$

By Lemma 3.1, this implies in particular that the forms $\mathrm{ad}_{\mathrm{i}A_x}(\widetilde{H}_0)$ and $\mathrm{ad}_{\mathrm{i}A_x}^n(\widetilde{H}_a)$ for $n \in \{1, \dots, N\}$ extend to forms on $H^1(\Omega)$. It remains to check the last assumption of Definition 2.4.

Step 2. Let $u \in L^2(\Omega)$. For almost all $x \in \mathbb{R}^p$ we have $u(x,\cdot) \in L^2(\omega)$ so we can find a sequence $(u_n(x))_{n\in\mathbb{N}}$ in $\mathbb{C}^{\mathbb{N}}$ such that

$$u(x,\cdot) = \sum_{n\in\mathbb{N}} u_n(x)\varphi_n \quad \text{and in particular} \quad \sum_{n\in\mathbb{N}} |u_n(x)|^2 = \|u(x,\cdot)\|^2_{L^2(\omega)}.$$

This defines a sequence $(u_n)_{n\in\mathbb{N}}$ of functions in $L^2(\mathbb{R}^p)$ with

$$\sum_{n\in\mathbb{N}} \|u_n\|^2_{L^2(\mathbb{R}^p)} = \|u\|^2_{L^2(\Omega)}.$$

With the same proof as for Proposition 4.3 in [38] we can check that for $z \in \mathbb{C}\setminus\mathbb{R}_+$ we have

$$(H_0 - z)^{-1}u = \sum_{n\in\mathbb{N}} (-\Delta_x + \lambda_n - z)^{-1} u_n \otimes \varphi_n.$$

Moreover if $u \in \mathcal{D}(H_0)$ then $u_n \in H^2(\mathbb{R}^p)$ for all $n \in \mathbb{N}$ and we have

$$H_0 u = \sum_{n\in\mathbb{N}} (-\Delta_x + \lambda_n) u_n \otimes \varphi_n.$$

For a bounded operator T we set $\operatorname{Im}(T) = (T - T^*)/(2\mathrm{i})$. Since H_0 and $-\Delta_x$ do not have any eigenvalue we can write for any $n \in \mathbb{N}$

$$\begin{aligned}\mathbf{1}_J(H_0)(u_n \otimes \varphi_n) &= \frac{1}{\pi}\lim_{\mu\to 0}\int_J \operatorname{Im}(H_0 - (\tau + \mathrm{i}\mu))^{-1}(u_n\otimes\varphi_n)\,\mathrm{d}\tau \\ &= \frac{1}{\pi}\lim_{\mu\to 0}\int_J \operatorname{Im}(-\Delta_x + \lambda_n - (\tau+\mathrm{i}\mu))^{-1}(u_n)\otimes\varphi_n\,\mathrm{d}\tau \\ &= \mathbf{1}_J(-\Delta_x + \lambda_n)(u_n)\otimes\varphi_n.\end{aligned}$$

There exist $m \in \mathbb{N}$ and $\varepsilon > 0$ such that $J \subset [\lambda_m + \varepsilon, \lambda_{m+1} - \varepsilon]$. Then we have $\mathbf{1}_J(-\Delta_x + \lambda_n) = 0$ for $n \geqslant m+1$ and therefore

$$\mathbf{1}_J(H_0)u = \sum_{n=0}^{m} \mathbf{1}_J(-\Delta_x + \lambda_n)(u_n)\otimes\varphi_n.$$

This gives

$$\begin{aligned}&\langle[\widetilde{H}_0, \mathrm{i}A_x]\mathbf{1}_J(H_0)u, \mathbf{1}_J(H_0)u\rangle_{L^2(\Omega)} \\ &\quad= \langle -2\Delta_x \mathbf{1}_J(H_0)u, \mathbf{1}_J(H_0)u\rangle_{L^2(\Omega)} \\ &\quad= \sum_{n=0}^{m}\langle -2\Delta_x\mathbf{1}_J(-\Delta_x+\lambda_n)(u_n)\otimes\varphi_n, \mathbf{1}_J(-\Delta_x+\lambda_n)(u_n)\otimes\varphi_n\rangle_{L^2(\Omega)} \\ &\quad\geqslant 2\varepsilon\sum_{n=0}^{m}\|\mathbf{1}_J(-\Delta_x+\lambda_n)(u_n)\otimes\varphi_n\|^2_{L^2(\Omega)} \geqslant 2\varepsilon\|\mathbf{1}_J(H_0)u\|^2_{L^2(\Omega)}.\end{aligned}$$

This proves (2.6) with $\alpha = 2\varepsilon$ and concludes the proof of the proposition. ∎

We will continue this example in Remark 4.9 and Proposition 6.7. We could similarly analyse the same problem on the half-space

$$\Omega = \{(x_1, \dots, x_d) \in \mathbb{R}^d \,:\, x_d > 0\}. \tag{3.5}$$

We also mention the Schrödinger operator on $\mathbb{R}^d$ with dissipation on the hyperplane $\Sigma = \mathbb{R}^{d-1} \times \{0\}$ given by the transmission condition

$$\partial_{x_d} u(x', 0^+) - \partial_{x_d} u(x', 0^-) = -\mathrm{i} a(x') u(x', 0) \quad \text{on } \Sigma. \tag{3.6}$$

Here we have denoted by $x = (x', x_d)$ a general point in $\mathbb{R}^d$, with $x' \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$. When $d = 1$ this corresponds to the second derivative with (dissipative) Dirac potential, usually denoted by

$$u \mapsto -u'' - \mathrm{i} a \delta(x) u.$$

More precisely, given $a \in W^{1,\infty}(\Sigma)$ we consider on $L^2(\mathbb{R}^d)$ the operator $H_a = -\Delta$ with domain

$$\mathcal{D}(H_a) = \{u \in H^1(\mathbb{R}^d) \cap H^2(\mathbb{R}^d \setminus \Sigma) \,:\, u \text{ satisfies (3.6)}\}.$$

Given $u \in \mathcal{D}(H_a)$ we define $H_a u$ as the function in $L^2(\mathbb{R}^d)$ which coincides with the distribution $-\Delta u$ on $\mathbb{R}^d \setminus \Sigma$. The operator H_a is associated with the quadratic form

$$q_a : \varphi \mapsto \int_{\mathbb{R}^d} |\nabla \varphi|^2 \, \mathrm{d}x - \mathrm{i} \int_{\Sigma} a(x') |u(x', 0)|^2 \, \mathrm{d}x',$$

defined on $\mathcal{D}(q_a) = H^1(\Omega)$.

In both cases, we can take the generator of dilations (3.1) as a conjugate operator on any compact interval $J \subset \mathbb{R}_+^*$ if for all $k \in \mathbb{N}$ the function $(x' \cdot \nabla')^k a$ is bounded on $\partial\Omega$ or Σ (we have denoted by ∇' the gradient in the first $(d-1)$ variables).

In the same spirit as the last example, we can also mention the dissipative quantum graphs with some infinite edges and dissipation at the vertices, given by the conditions

$$u_1(0) = \cdots = u_{n_v}(0), \quad \sum_{j=1}^{n_v} u'(0) = -\mathrm{i} a_v u(0), \tag{3.7}$$

where for a vertex v we have $a_v \geqslant 0$ and the integer n_v is the number of edges attached to v. For precise definitions we refer to [27], which deals with the limiting absorption principle for such a quantum graph with self-adjoint boundary conditions at the vertices (in particular (3.7) with $a_v = 0$ for all v). For various non-selfadjoint conditions on quantum graphs we also refer to [13].

We finish this section with the example of the Schrödinger operator with dissipation by a potential in low dimensions and for low frequencies. In this case the dissipative Mourre theory in the sense of operators as given in [5], [37], can be applied, but not uniformly.

We consider on $\mathbb{R}^d$, $d \geqslant 3$, the Schrödinger operator
$$H_\lambda = -\Delta - \frac{\mathrm{i}}{\lambda^2} a\Big(\frac{x}{\lambda}\Big),$$
where $\lambda > 0$ and $a \in C^\infty(\mathbb{R}^d, \mathbb{R}_+)$ is of very short range: for some $\rho > 0$ there exist constants c_α, $\alpha \in \mathbb{N}^d$, such that
$$|\partial^\gamma a(x)| \leqslant c_\gamma \langle x \rangle^{-2-\rho-|\gamma|}.$$
In order to obtain low frequency resolvent estimates for the Schrödinger operator $-\Delta - \mathrm{i}a$ we have to prove uniform resolvent estimates for H_λ close to the spectral parameter 1 uniformly in $\lambda > 0$ (see [5] for the wave equation). Since a is bounded the multiplication by $\frac{1}{\lambda^2} a(\frac{x}{\lambda})$ defines a bounded operator on $L^2(\mathbb{R}^d)$, so for any $\lambda > 0$ we can apply to H_λ the dissipative Mourre theory for perturbations in the sense of operators. However this absorption index becomes singular when λ is close to 0 and it is not clear that this method gives estimates which are uniform in λ. According to Proposition 7.2 in [5] we have for $u \in \mathcal{S}$
$$\|a\Big(\frac{x}{\lambda}\Big)u\|_{H^s} \lesssim \lambda^2 \|u\|_{H^{s+2}}$$
whenever s and $s+2$ belong to $]-\frac{d}{2}, \frac{d}{2}[$. The same applies if we replace a by $(x\cdot\nabla)^k a$ for some $k \in \mathbb{N}$. This proves that the commutator between the dissipative part of H_λ and the generator of dilations A defines an operator in $\mathcal{L}(H^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$ uniformly in $\lambda > 0$ if $d \geqslant 5$. But not if $d \in \{3,4\}$. However, for any $d \geqslant 3$ it defines a uniformly bounded operator in $\mathcal{L}(H^1(\mathbb{R}^d), H^{-1}(\mathbb{R}^d))$, so it is fruitful to see it as a perturbation of the free laplacian in the sense of forms. This idea is used (in a more general setting) in [18].

4. UNIFORM RESOLVENT ESTIMATES AND LIMITING ABSORPTION PRINCIPLE

In this section we prove the uniform resolvent estimates and the limiting absorption principle in the abstract setting:

THEOREM 4.1. *Assume that A is a conjugate operator to H on the interval J with bounds $(\alpha, \beta, \Upsilon)$, in the sense of Definition* 2.6.

(i) *Let $I \subset \mathring{J}$ be a compact interval and $\delta > \frac{1}{2}$. Then there exists $C \geqslant 0$ (which only depends on C_Θ, I, J, δ, β and Υ) such that for all $z \in \mathbb{C}_{I,+}$ we have*

$$\|\langle A\rangle^{-\delta}(H-z)^{-1}\langle A\rangle^{-\delta}\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{C}{\alpha}. \tag{4.1}$$

(ii) *Moreover for all $\lambda \in \mathring{J}$ the limit*
$$\langle A\rangle^{-\delta}(H-(\lambda+\mathrm{i}0))^{-1}\langle A\rangle^{-\delta} = \lim_{\mu\to 0^+} \langle A\rangle^{-\delta}(H-(\lambda+\mathrm{i}\mu))^{-1}\langle A\rangle^{-\delta}$$
exists in $\mathcal{L}(\mathcal{H})$ and defines a continuous function of λ on J (it is Hölder-continuous of index $\frac{2\delta-1}{2\delta+1}$ with a constant of size $\alpha^{-4\delta/(2\delta+1)}$).

REMARK 4.2. Taking the adjoint we obtain the same estimate with $(H - z)^{-1}$ replaced by $(H^* - \overline{z})^{-1}$.

The rest of this section is devoted to the proof of Theorem 4.1. To simplify the notation, the symbol " $\lesssim$ " will be used to replace " $\leqslant C$ " where C is a constant which depends on C_Θ, I, J, δ, β and Υ. The dependency in $\alpha \in]0,1]$, $z \in \mathbb{C}_{I,+}$ and in the parameter ε (which will be introduced in the proof) will always be explicit.

Let $\phi \in C_0^\infty(\mathbb{R}, [0,1])$ be supported in $\mathring{J}$ and equal to 1 on a neighborhood of I (notice that all the estimates below will also depend on the choice of ϕ). We set $\Phi = \phi(H_0)$ and $\Phi^\perp = (1-\phi)(H_0)$. We have

$$\Phi \in \mathcal{L}(\mathcal{K}^*, \mathcal{K}) \quad \text{and} \quad \Phi^\perp \in \mathcal{L}(\mathcal{H}) \cap \mathcal{L}(\mathcal{K}) \cap \mathcal{L}(\mathcal{K}^*).$$

Now let

$$M_0 = \Phi(B_0 + \beta\Theta)\Phi \quad \text{and} \quad M = \Phi(B + \beta\Theta)\Phi.$$

The operators M_0 and M are bounded on $\mathcal{H}$, and M_0 is the self-adjoint part of M. After multiplication by Φ on both sides, assumption (2.6) reads

$$M_0 \geqslant \alpha\Phi^2. \tag{4.2}$$

The proof of the following lemma is postponed to the end of the section.

LEMMA 4.3. *The operator $[M, A]$, a priori defined as an operator in $\mathcal{L}(\mathcal{E}, \mathcal{E}^*)$, extends to an operator in $\mathcal{L}(\mathcal{K}^*, \mathcal{K})$ which we denote by $[M, A]_\mathcal{K}$. Moreover we have*

$$\|[M, A]_\mathcal{K}\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})} \lesssim \alpha.$$

Let $\varepsilon \geqslant 0$. The operator $H - \mathrm{i}\varepsilon M$ is maximal dissipative on $\mathcal{H}$ with domain $\mathcal{D}(H)$, so for $z \in \mathbb{C}_+$ it has a bounded inverse $(H - \mathrm{i}\varepsilon M - z)^{-1}$ in $\mathcal{L}(\mathcal{H}, \mathcal{D}(H))$. As above for $\widetilde{H}$, the operator $(\widetilde{H} - \mathrm{i}\varepsilon M - z) \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ has a bounded inverse

$$G_z(\varepsilon) := (\widetilde{H} - \mathrm{i}\varepsilon M - z)^{-1} \in \mathcal{L}(\mathcal{K}^*, \mathcal{K}).$$

The Mourre method relies on the so-called quadratic estimates (see Proposition II.5 in [24]). Here we will use the following version:

PROPOSITION 4.4. *Let γ_0 be a quadratic form closed, densely defined, symmetric and bounded from below. Let P_0 be the corresponding self-adjoint operator. Let $\mathcal{K}_{\gamma_0}$ be the domain of the form γ_0. Let γ_I be a non-negative and γ_0-bounded form on $\mathcal{H}$. Let P be the maximal dissipative operator associated with the form $\gamma_0 - \mathrm{i}\gamma_I$, and $\widetilde{P}$ the corresponding operator in $\mathcal{L}(\mathcal{K}_{\gamma_0}, \mathcal{K}_{\gamma_0}^*)$. Let γ a non-negative form on $\mathcal{K}_{\gamma_0}$ which satisfies $\gamma \leqslant \gamma_I$. Then for $z \in \mathbb{C}_+$ and $\varphi \in \mathcal{K}_{\gamma_0}^*$ we have*

$$\gamma((\widetilde{P} - z)^{-1}\varphi) \leqslant |\langle (\widetilde{P} - z)^{-1}\varphi, \varphi \rangle_{\mathcal{K}_{\gamma_0}, \mathcal{K}_{\gamma_0}^*}|$$

and

$$\gamma((\widetilde{P}^* - \overline{z})^{-1}\varphi) \leqslant |\langle (\widetilde{P} - z)^{-1}\varphi, \varphi \rangle_{\mathcal{K}_{\gamma_0}, \mathcal{K}_{\gamma_0}^*}|.$$

If $\varphi \in \mathcal{H}$ we can replace $\widetilde{P}$ by P in these estimates.

Proof. For $z \in \mathbb{C}_+$ and $\varphi \in \mathcal{K}_{\gamma_0}^*$ we have

$$\begin{aligned}\gamma((\widetilde{P}-z)^{-1}\varphi) &\leqslant \frac{1}{2\mathrm{i}}\langle((\widetilde{P}^*-\overline{z})-(\widetilde{P}-z))(\widetilde{P}-z)^{-1}\varphi,(\widetilde{P}-z)^{-1}\varphi\rangle_{\mathcal{K}_{\gamma_0}^*,\mathcal{K}_{\gamma_0}}\\ &\leqslant \frac{1}{2\mathrm{i}}\langle(\widetilde{P}-z)^{-1}\varphi,\varphi\rangle_{\mathcal{K}_{\gamma_0},\mathcal{K}_{\gamma_0}^*}-\frac{1}{2\mathrm{i}}\langle\varphi,(\widetilde{P}-z)^{-1}\varphi\rangle_{\mathcal{K}_{\gamma_0}^*,\mathcal{K}_{\gamma_0}}\\ &\leqslant \operatorname{Im}\langle(\widetilde{P}-z)^{-1}\varphi,\varphi\rangle_{\mathcal{K}_{\gamma_0},\mathcal{K}_{\gamma_0}^*}.\end{aligned}$$

The second estimate is proved similarly. ∎

PROPOSITION 4.5. *Let $\mathcal{K}_0$ stand either for $\mathcal{K}$ or $\mathcal{H}$. Then there exists $\varepsilon_0 \in]0,1]$ (which depends on C_Θ, I, J, β and Υ) such that for $Q \in \mathcal{L}(\mathcal{K}_0^*)$, $z \in \mathbb{C}_{I,+}$ and $\varepsilon \in]0,\varepsilon_0]$ we have:*

$$\|\Phi^\perp G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K})} \lesssim \|Q\|_{\mathcal{L}(\mathcal{K}_0^*)} + \|Q^*G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)}^{1/2}, \tag{4.3}$$

$$\|\Phi G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K})} \lesssim \frac{1}{\sqrt{\alpha}\sqrt{\varepsilon}}\|Q^*G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)}^{1/2}, \tag{4.4}$$

$$\|G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K})} \lesssim \|Q\|_{\mathcal{L}(\mathcal{K}_0^*)} + \frac{\|Q^*G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)}^{1/2}}{\sqrt{\alpha}\sqrt{\varepsilon}}, \tag{4.5}$$

and for $\varphi \in \mathcal{K}_0^*$ with $\|\varphi\|_{\mathcal{K}_0^*} \leqslant 1$:

$$q_\Theta(\Phi G_z(\varepsilon)Q\varphi) + q_\Theta(\Phi^\perp G_z(\varepsilon)Q\varphi) \lesssim \|Q\|_{\mathcal{L}(\mathcal{K}_0^*)}^2 + \|Q^*G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)}. \tag{4.6}$$

These estimates also hold if $G_z(\varepsilon)$ is replaced by $G_z(\varepsilon)^*$ on the left-hand sides.

Applied with $Q = \mathrm{Id}_{\mathcal{K}^*}$, (4.5) gives an estimate on $G_z(\varepsilon)$ alone.

COROLLARY 4.6. *For $z \in \mathbb{C}_{I,+}$ and $\varepsilon \in]0,\varepsilon_0]$ we have*

$$\|G_z(\varepsilon)\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})} + \|G_z(\varepsilon)^*\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})} \lesssim \frac{1}{\alpha\varepsilon}.$$

Proof of Proposition 4.5. *Step* 1. Let $z \in \mathbb{C}_{I,+}$. Since $\Phi + \Phi^\perp = 1$, (4.5) is a direct consequence of (4.3) and (4.4). Let $\varphi \in \mathcal{K}_0^*$. According to (4.2) and Proposition 4.4 applied with $Q\varphi \in \mathcal{K}^*$ and the form $\widetilde{q}$ corresponding to $\alpha\varepsilon\Phi^2$ we have

$$\begin{aligned}\|\Phi G_z(\varepsilon)Q\varphi\|_{\mathcal{H}}^2 &= \frac{1}{\alpha\varepsilon}\langle\alpha\varepsilon\Phi^2 G_z(\varepsilon)Q\varphi, G_z(\varepsilon)Q\varphi\rangle_{\mathcal{H}} \leqslant \frac{1}{\alpha\varepsilon}|\langle G_z(\varepsilon)Q\varphi, Q\varphi\rangle_{\mathcal{K},\mathcal{K}^*}|\\ &\leqslant \frac{1}{\alpha\varepsilon}\|Q^*G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)}\|\varphi\|_{\mathcal{K}_0^*}^2.\end{aligned}$$

Since ϕ is compactly supported in J, there exists a constant c which only depends on J such that

$$\|\Phi G_z(\varepsilon)Q\varphi\|_{\mathcal{K}}^2 \leqslant \frac{c}{\alpha\varepsilon}\|Q^*G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)}\|\varphi\|_{\mathcal{K}_0^*}^2.$$

The same holds with $G_z(\varepsilon)$ replaced by $G_z(\varepsilon)^*$, and (4.4) is proved.

Step 2. Since the quadratic form q_Θ is non-negative we can apply the Cauchy–Schwarz inequality: for $\psi_1, \psi_2 \in \mathcal{K}$ we have

$$q_\Theta(\psi_1 + \psi_2) \leqslant q_\Theta(\psi_1) + 2\sqrt{q_\Theta(\psi_1)}\sqrt{q_\Theta(\psi_2)} + q_\Theta(\psi_2) \leqslant 2q_\Theta(\psi_1) + 2q_\Theta(\psi_2).$$

In particular

$$q_\Theta(\Phi G_z(\varepsilon)Q\varphi) \leqslant 2q_\Theta(G_z(\varepsilon)Q\varphi) + 2q_\Theta(\Phi^\perp G_z(\varepsilon)Q\varphi).$$

According to Proposition 4.4 we have

$$q_\Theta(G_z(\varepsilon)Q\varphi) \leqslant |\langle Q^* G_z(\varepsilon)Q\varphi, \varphi\rangle_{\mathcal{K}_0,\mathcal{K}_0^*}| \leqslant \|Q^* G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)} \|\varphi\|^2_{\mathcal{K}_0^*}. \tag{4.7}$$

On the other hand, according to (2.2)

$$q_\Theta(\Phi^\perp G_z(\varepsilon)Q\varphi) \leqslant C_\Theta \|\Phi^\perp G_z(\varepsilon)Q\varphi\|^2_{\mathcal{K}}. \tag{4.8}$$

We obtain

$$q_\Theta(\Phi G_z(\varepsilon)Q\varphi) \leqslant 2\|Q^* G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)} \|\varphi\|^2_{\mathcal{K}_0} + 2C_\Theta \|\Phi^\perp G_z(\varepsilon)Q\varphi\|^2_{\mathcal{K}}. \tag{4.9}$$

Thus (4.6) will be a consequence of (4.3). The proof of (4.3) relies itself on (4.9).

Step 3. According to the resolvent identity (as in (2.4)) we have in $\mathcal{L}(\mathcal{K}_0^*, \mathcal{K})$

$$\Phi^\perp G_z(\varepsilon)Q = \Phi^\perp(\widetilde{H}_0 - z)^{-1}Q + \mathrm{i}\Phi^\perp(\widetilde{H}_0 - z)^{-1}(\Theta + \varepsilon\Phi B\Phi + \varepsilon\beta\Phi\Theta\Phi)G_z(\varepsilon)Q.$$

By functional calculus the operator $\Phi^\perp(\widetilde{H}_0 - z)^{-1}$ belongs to $\mathcal{L}(\mathcal{K}^*, \mathcal{K})$ uniformly in $z \in \mathbb{C}_{I,+}$. Let $\varphi \in \mathcal{K}_0^*$ and $\psi \in \mathcal{K}^*$. According to the Cauchy–Schwarz inequality we have

$$\begin{aligned}|\langle \Phi^\perp(\widetilde{H}_0 - z)^{-1}\Theta G_z(\varepsilon)Q\varphi, \psi\rangle_{\mathcal{K},\mathcal{K}^*}| &= |q_\Theta(G_z(\varepsilon)Q\varphi, \Phi^\perp(\widetilde{H}_0 - \bar{z})^{-1}\psi)| \\ &\leqslant q_\Theta(G_z(\varepsilon)Q\varphi)^{1/2} q_\Theta(\Phi^\perp(\widetilde{H}_0 - \bar{z})^{-1}\psi)^{1/2}.\end{aligned}$$

According to (2.2) we have

$$q_\Theta(\Phi^\perp(\widetilde{H}_0 - \bar{z})^{-1}\psi) \lesssim \|\psi\|^2_{\mathcal{K}^*}.$$

With (4.7) this proves that

$$\|\Phi^\perp(\widetilde{H}_0 - z)^{-1}\Theta G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K})} \lesssim \|Q^* G_z(\varepsilon)Q\|^{1/2}_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)}.$$

Then we have

$$\begin{aligned}\varepsilon\|\Phi^\perp(\widetilde{H}_0 - z)^{-1}\Phi B\Phi G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K})} &\lesssim \sqrt{\alpha}\varepsilon\|\Phi G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K})} \\ &\lesssim \sqrt{\varepsilon}\|Q^* G_z(\varepsilon)Q\|^{1/2}_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)}.\end{aligned}$$

On the other hand, by the Cauchy–Schwarz inequality and (4.9) we have for $\varphi \in \mathcal{K}_0^*$ and $\psi \in \mathcal{K}^*$

$$\begin{aligned}&\varepsilon\beta|\langle \Phi^\perp(\widetilde{H}_0 - z)^{-1}\Phi\Theta\Phi G_z(\varepsilon)Q\varphi, \psi\rangle_{\mathcal{K},\mathcal{K}^*}| \\ &\qquad = \varepsilon\beta|q_\Theta(\Phi G_z(\varepsilon)Q\varphi, \Phi(\widetilde{H}_0 - \bar{z})^{-1}\Phi^\perp\psi)|\end{aligned}$$

$$\leqslant \varepsilon\beta q_\Theta(\Phi G_z(\varepsilon)Q\varphi)^{1/2} q_\Theta(\Phi(\widetilde{H}_0-\bar{z})^{-1}\Phi^\perp\psi)^{1/2}$$
$$\lesssim \varepsilon(\|Q^*G_z(\varepsilon)Q\|^{1/2}_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)} + \|\Phi^\perp G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K})})\|\varphi\|_{\mathcal{K}_0^*}\|\psi\|_{\mathcal{K}^*},$$

hence

$$\varepsilon\beta\|\Phi^\perp(\widetilde{H}_0-z)^{-1}\Phi\Theta\Phi G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K})}$$
$$\lesssim \varepsilon(\|Q^*G_z(\varepsilon)Q\|^{1/2}_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)} + \|\Phi^\perp G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K})}).$$

Finally we obtain

$$\|\Phi^\perp G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K})} \lesssim (\|Q\|_{\mathcal{L}(\mathcal{K}_0^*)} + \|Q^*G_z(\varepsilon)Q\|^{1/2}_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)} + \varepsilon\|\Phi^\perp G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K})}).$$

This gives (4.3) when $\varepsilon > 0$ is small enough. Then (4.9) and (4.8) give (4.6). ∎

LEMMA 4.7. *On $\mathcal{L}(\mathcal{D}(A),\mathcal{D}(A)^*)$ we have*

$$G_z(\varepsilon)BG_z(\varepsilon) = \mathrm{i}AG_z(\varepsilon) - \mathrm{i}G_z(\varepsilon)A - \varepsilon G_z(\varepsilon)[M,A]_{\mathcal{K}}G_z(\varepsilon).$$

Proof. Let $\varphi,\psi \in \mathcal{D}(A)$. Since $\mathcal{E}$ is dense in $\mathcal{K}$ we can consider sequences $(\varphi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $\varphi_n \to G_z(\varepsilon)\varphi$ and $\psi_n \to G_z(\varepsilon)^*\psi$ in $\mathcal{K}$. Since $B \in \mathcal{L}(\mathcal{K},\mathcal{K}^*)$ we have

$$\langle B\varphi_n,\psi_m\rangle \xrightarrow[n,m\to\infty]{} \langle G_z(\varepsilon)BG_z(\varepsilon)\varphi,\psi\rangle.$$

On the other hand, since $\varphi_n,\psi_m \in \mathcal{D}(A)$ and $A\varphi_n, A\psi_m \in \mathcal{K}$ we can write

$$\langle B\varphi_n,\psi_m\rangle = \langle[\widetilde{H},\mathrm{i}A]\varphi_n,\psi_m\rangle = \langle[\widetilde{H}-\mathrm{i}\varepsilon M - z,\mathrm{i}A]\varphi_n,\psi_m\rangle - \varepsilon\langle[M,A]\varphi_n,\psi_m\rangle.$$

According to Lemma 4.3 we have

$$\langle[M,A]\varphi_n,\psi_m\rangle \xrightarrow[n,m\to\infty]{} \langle G_z(\varepsilon)[M,A]_{\mathcal{K}}G_z(\varepsilon)\varphi,\psi\rangle.$$

And finally

$$\begin{aligned}
&\lim_{n,m\to\infty}\langle[\widetilde{H}-\mathrm{i}\varepsilon M - z,\mathrm{i}A]\varphi_n,\psi_m\rangle\\
&\quad= \mathrm{i}\lim_{n\to\infty}\lim_{m\to\infty}\langle A\varphi_n,(\widetilde{H}-\mathrm{i}\varepsilon M - z)^*\psi_m\rangle_{\mathcal{K},\mathcal{K}^*}\\
&\qquad - \mathrm{i}\lim_{m\to\infty}\lim_{n\to\infty}\langle(\widetilde{H}-\mathrm{i}\varepsilon M - z)\varphi_n, A\psi_m\rangle_{\mathcal{K}^*,\mathcal{K}}\\
&\quad= \mathrm{i}\lim_{n\to\infty}\langle A\varphi_n,\psi\rangle_{\mathcal{K},\mathcal{K}^*} - \mathrm{i}\lim_{m\to\infty}\langle\varphi, A\psi_m\rangle_{\mathcal{K}^*,\mathcal{K}}\\
&\quad= \mathrm{i}\lim_{n\to\infty}\langle A\varphi_n,\psi\rangle_{\mathcal{H}} - \mathrm{i}\lim_{m\to\infty}\langle\varphi, A\psi_m\rangle_{\mathcal{H}}\\
&\quad= \mathrm{i}\lim_{n\to\infty}\langle\varphi_n, A\psi\rangle_{\mathcal{H}} - \mathrm{i}\lim_{m\to\infty}\langle A\varphi,\psi_m\rangle_{\mathcal{H}}\\
&\quad= \mathrm{i}\langle G_z(\varepsilon)\varphi, A\psi\rangle_{\mathcal{H}} - \mathrm{i}\langle A\varphi, G_z(\varepsilon)^*\psi\rangle_{\mathcal{H}}.
\end{aligned}$$

The lemma is proved. ∎

The strategy for the proof of Theorem 4.1 is standard and relies on the following abstract result about ordinary differential equations (see Lemma 3.3 of [15]).

LEMMA 4.8. *Let X be a Banach space, $\varepsilon_0 \in]0,1]$ and $f \in C^1(]0,\varepsilon_0],X)$. Suppose there exist $\gamma_1 \in [0,1]$, $\gamma_2 \in [0,1[$, $\gamma_3 \in \mathbb{R}$, and $c_1, c_2 > 0$ such that*

$$\forall \varepsilon \in]0,\varepsilon_0[, \quad \|f'(\varepsilon)\| \leqslant c_1 \varepsilon^{-\gamma_2}(1 + \|f(\varepsilon)\|^{\gamma_1}) \quad \textit{and} \quad \|f(\varepsilon)\| \leqslant c_2 \varepsilon^{-\gamma_3}.$$

Then f has a limit at 0 and there exists $c \geqslant 0$ which only depends on ε_0, γ_1, γ_2, γ_3, c_1 and c_2 such that

$$\forall \varepsilon \in]0,\varepsilon_0[, \quad \|f(\varepsilon)\| \leqslant c.$$

Now we can prove Theorem 4.1.

Proof of Theorem 4.1. *Step* 1. For $\varepsilon \in]0,1]$ we set $Q(\varepsilon) = \langle A \rangle^{-\delta} \langle \varepsilon A \rangle^{\delta - 1}$. According to the functional calculus we have

$$\|Q(\varepsilon)\|_{\mathcal{L}(\mathcal{H})} \leqslant 1 \quad \text{and} \quad \|AQ(\varepsilon)\|_{\mathcal{L}(\mathcal{H})} + \|Q(\varepsilon)A\|_{\mathcal{L}(\mathcal{H})} \lesssim \varepsilon^{\delta-1}. \tag{4.10}$$

Denoting by a prime the derivative with respect to ε we also have

$$\|Q'(\varepsilon)\|_{\mathcal{L}(\mathcal{H})} \lesssim \varepsilon^{\delta-1}. \tag{4.11}$$

Step 2. For $z \in \mathbb{C}_{I,+}$ we set

$$F_z(\varepsilon) = Q(\varepsilon) G_z(\varepsilon) Q(\varepsilon).$$

According to (4.10) and Proposition 4.5 applied with $Q = Q(\varepsilon)$ we have for $\varepsilon \in]0,\varepsilon_0]$ (ε_0 being given by Proposition 4.5)

$$\|F_z(\varepsilon)\| \leqslant \|G_z(\varepsilon)Q(\varepsilon)\| \lesssim 1 + \frac{\|F_z(\varepsilon)\|^{1/2}}{\sqrt{\alpha}\sqrt{\varepsilon}}, \tag{4.12}$$

and hence

$$\|F_z(\varepsilon)\| \lesssim \frac{1}{\alpha \varepsilon}. \tag{4.13}$$

Step 3. We now estimate the derivative of F:

$$F_z'(\varepsilon) = Q'(\varepsilon)G_z(\varepsilon)Q(\varepsilon) + Q(\varepsilon)G_z(\varepsilon)Q'(\varepsilon) + \mathrm{i}q(\varepsilon)G(\varepsilon)\Phi(B + \beta\Theta)\Phi G(\varepsilon)Q(\varepsilon).$$

Proposition 4.5 and (4.11) yield

$$\|Q'(\varepsilon)G_z(\varepsilon)Q(\varepsilon) + Q(\varepsilon)G_z(\varepsilon)Q'(\varepsilon)\| \lesssim \varepsilon^{\delta-1}\left(1 + \frac{\|F_z(\varepsilon)\|^{1/2}}{\sqrt{\alpha}\sqrt{\varepsilon}}\right) \tag{4.14}$$

and

$$\|Q(\varepsilon)G(\varepsilon)\Phi\Theta\Phi G(\varepsilon)Q(\varepsilon)\| \lesssim 1 + \|F_z(\varepsilon)\|_{\mathcal{L}(\mathcal{H})}. \tag{4.15}$$

For the remaining term we write in $\mathcal{L}(\mathcal{K},\mathcal{K}^*)$

$$\Phi B \Phi = B - \Phi B \Phi^\perp - \Phi^\perp B \Phi - \Phi^\perp B \Phi^\perp.$$

According to Proposition 4.5 we have

$$\|Q(\varepsilon)G(\varepsilon)(\Phi B \Phi^\perp + \Phi^\perp B \Phi + \Phi^\perp B \Phi^\perp)G(\varepsilon)Q(\varepsilon)\| \lesssim 1 + \frac{\|F_z(\varepsilon)\|}{\sqrt{\varepsilon}}.$$

Step 4. According to Lemma 4.7 we have on $\mathcal{L}(\mathcal{H})$:

$$Q(\varepsilon)G_z(\varepsilon)BG_z(\varepsilon)Q(\varepsilon) = \mathrm{i}q(\varepsilon)AG_z(\varepsilon)Q(\varepsilon) - \mathrm{i}q(\varepsilon)G_z(\varepsilon)AQ(\varepsilon) - \varepsilon Q(\varepsilon)G_z(\varepsilon)[M,A]_{\mathcal{K}}G_z(\varepsilon)Q(\varepsilon).$$

With (4.10), Proposition 4.5 and Lemma 4.3 we get

$$\|Q(\varepsilon)G_z(\varepsilon)BG_z(\varepsilon)Q(\varepsilon)\| \lesssim 1 + \alpha^{-1/2}\varepsilon^{\delta-3/2}\|F_z(\varepsilon)\|^{1/2} + \|F_z(\varepsilon)\|.$$

Together with (4.14) and (4.15) this gives

$$\|\alpha F_z'(\varepsilon)\| \lesssim \varepsilon^{\delta-1} + \varepsilon^{-1/2}\|\alpha F_z(\varepsilon)\| + \varepsilon^{\delta-3/2}\|\alpha F_z(\varepsilon)\|^{1/2}, \tag{4.16}$$

and hence, according to Lemma 4.8, we finally obtain

$$\|F_z(\varepsilon)\| \lesssim \frac{1}{\alpha}, \tag{4.17}$$

which gives the uniform resolvent estimates (4.1) when ε goes to 0.

Step 5. Now we prove the limiting absorption principle on I. Without loss of generality we can assume that $\delta \in]\frac{1}{2}, 1]$. We prove that there exists $C \geqslant 0$ such that for all $z, z' \in \mathbb{C}_{I,+}$ we have

$$\|\langle A\rangle^{-\delta}((H-z)^{-1}-(H-z')^{-1})\langle A\rangle^{-\delta}\|_{\mathcal{L}(\mathcal{H})} \lesssim \alpha^{-4\delta/(2\delta+1)}|z-z'|^{2\delta-1/(2\delta+1)}. \tag{4.18}$$

For any $c_0 > 0$, (4.18) is a direct consequence of the uniform estimate (4.1) as long as $|z - z'| \geqslant c_0\alpha$, so it is enough to prove (4.18) when $|z - z'| \leqslant c_0\alpha$ for some well chosen $c_0 > 0$. According to (4.16) and (4.17) we have

$$\|F_z'(\varepsilon)\| \lesssim \alpha^{-1}\varepsilon^{\delta-3/2},$$

and hence

$$\|F_z(\varepsilon) - F_z(0)\| \lesssim \alpha^{-1}\varepsilon^{\delta-1/2}.$$

Of course we have the same estimate for z'. Moreover, according to (4.12) we have for all $\varepsilon \in]0, \varepsilon_0]$

$$\left\|\frac{\partial}{\partial z}F_z(\varepsilon)\right\| = \|Q(\varepsilon)G_z(\varepsilon)^2Q(\varepsilon)\| \leqslant \|G_z(\varepsilon)Q(\varepsilon)\|^2 \lesssim \frac{1}{\alpha^2\varepsilon},$$

and hence

$$\|F_z(\varepsilon) - F_{z'}(\varepsilon)\| \lesssim \frac{|z - z'|}{\alpha^2\varepsilon}.$$

Given z and z' we take

$$\varepsilon = \left(\frac{|z - z'|}{\alpha}\right)^{2/(2\delta+1)}.$$

If c_0 was chosen small enough then $\varepsilon \in]0, \varepsilon_0]$, and we obtain

$$\|F_z(0) - F_{z'}(0)\| \lesssim \alpha^{-4\delta/(2\delta+1)}|z - z'|^{(2\delta-1)/(2\delta+1)},$$

which is exactly (4.18). Now for all $\lambda \in I$ the function

$$\mu \mapsto \langle A\rangle^{-\delta}(H - (\lambda + \mathrm{i}\mu))^{-1}\langle A\rangle^{-\delta}$$

has a limit when μ goes to 0^+. Taking the limit $\operatorname{Im} z, \operatorname{Im} z' \to 0^+$ in (4.18) proves that this limit is a Hölder-continuous function of index $\frac{2\delta-1}{2\delta+1}$. ∎

To conclude the proof of Theorem 4.1 it remains to give a proof of Lemma 4.3.

Proof of Lemma 4.3. The proof is inspired by those of Lemmas 1.2.1 and 1.2.8 in [3].

Step 1. For $\theta \in \mathbb{R}$ we set

$$\widetilde{H}_\theta = e^{i\theta A} \widetilde{H}_0 e^{-i\theta A} \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*).$$

We first prove that the map $\theta \mapsto \widetilde{H}_\theta$ is strongly C^1 and that for all $\theta, \tau \in \mathbb{R}$ and $\varphi \in \mathcal{K}$ we have in $\mathcal{K}^*$:

$$(\widetilde{H}_\tau - \widetilde{H}_\theta)\varphi = -\int_\theta^\tau e^{isA} B_0 e^{-isA} \varphi \, ds. \tag{4.19}$$

Let $\theta \in \mathbb{R}$ and $\varphi \in \mathcal{E}$. For $\varepsilon \in \mathbb{R}^*$ we have

$$\frac{\widetilde{H}_{\theta+\varepsilon} - \widetilde{H}_\theta}{\varepsilon} \varphi = e^{i(\theta+\varepsilon)A} \widetilde{H}_0 \frac{e^{-i\varepsilon A} - 1}{\varepsilon} e^{-i\theta A} \varphi + e^{i\theta A} \frac{e^{i\varepsilon A} - 1}{\varepsilon} \widetilde{H}_0 e^{-i\theta A} \varphi. \tag{4.20}$$

Since $e^{-i\theta A}\varphi \in \mathcal{E}$ we have

$$\frac{e^{-i\varepsilon A} - 1}{\varepsilon} e^{-i\theta A} \varphi \xrightarrow[\varepsilon \to 0]{\mathcal{K}} -iAe^{-i\theta A}\varphi,$$

and hence the first term in the right-hand side of (4.20) goes to $-ie^{i\theta A}\widetilde{H}_0 A e^{-i\theta A}\varphi$ in $\mathcal{K}^*$ when ε goes to 0. Now let $g = \widetilde{H}_0 e^{-i\theta A}\varphi \in \mathcal{K}^*$. Since $\mathcal{D}(A)$ is dense in $\mathcal{K}^*$, we can consider a sequence $(g_n)_{n\in\mathbb{N}} \in \mathcal{D}(A)^{\mathbb{N}}$ such that $g_n \to g$ in $\mathcal{K}^*$. For all $n \in \mathbb{N}$ we have in $\mathcal{H}$:

$$\frac{e^{i\varepsilon A} - 1}{\varepsilon} g_n - iAg_n = \frac{i}{\varepsilon} \int_0^\varepsilon (e^{i\tau A} - 1) A g_n \, d\tau.$$

In $\mathcal{E}^*$ we can let n go to infinity (we use the Lebesgue dominated convergence theorem for the right-hand side). We obtain that the equality holds in $\mathcal{E}^*$ when g_n is replaced by g, and hence the second term in the right-hand side of (4.20) goes to $ie^{i\theta A} A\widetilde{H}_0 e^{-i\theta A}\varphi$ in $\mathcal{E}^*$. This proves that the map $\theta \mapsto \widetilde{H}_\theta\varphi$ is differentiable with derivative $-e^{i\theta A}[\widetilde{H}_0, iA]e^{-i\theta A}\varphi \in \mathcal{E}^*$, and hence (4.19) holds in $\mathcal{L}(\mathcal{E}, \mathcal{E}^*)$. Since $B_0 = [\widetilde{H}_0, iA]$ extends to an operator in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$, this is the case for both terms in (4.19) and we have the equality in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$.

Step 2. On $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ we have $[\widetilde{H}_0, e^{i\theta A}] = (\widetilde{H}_0 - \widetilde{H}_\theta)e^{i\theta A}$ and hence for $t \in \mathbb{R}$ and $\theta \in \mathbb{R}^*$ we have in the strong sense in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$

$$\begin{aligned} e^{itH_0} \frac{e^{i\theta A} - 1}{i\theta} - \frac{e^{i\theta A} - 1}{i\theta} e^{itH_0} &= \frac{1}{i\theta} \int_0^t e^{isH_0} [i\widetilde{H}_0, e^{i\theta A}] e^{i(t-s)H_0} \, ds \\ &= \int_0^t e^{isH_0} \frac{\widetilde{H}_0 - \widetilde{H}_\theta}{\theta} e^{i\theta A} e^{i(t-s)H_0} \, ds. \end{aligned}$$

The operator $e^{i\theta A}$ goes strongly to 1 in $\mathcal{L}(\mathcal{K})$ and $\frac{e^{i\theta A}-1}{i\theta}$ converges strongly to A in $\mathcal{L}(\mathcal{E},\mathcal{K})$ and $\mathcal{L}(\mathcal{K}^*,\mathcal{E}^*)$. By (4.19), $\frac{\widetilde{H}_0-\widetilde{H}_\theta}{\theta}$ goes to B_0 strongly in $\mathcal{L}(\mathcal{K},\mathcal{K}^*)$ when θ goes to 0. Then, by the Lebesgue dominated convergence theorem we obtain in the strong sense in $\mathcal{L}(\mathcal{E},\mathcal{E}^*)$

$$[e^{itH_0}, A] = \int_0^t e^{isH_0} B_0 e^{i(t-s)H_0}\, ds.$$

But the right-hand side defines an operator in $\mathcal{L}(\mathcal{K},\mathcal{K}^*)$, so the operator on the left has an extension in $\mathcal{L}(\mathcal{K},\mathcal{K}^*)$ and

$$\|[e^{itH_0}, A]\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^*)} \lesssim |t| \|B_0\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^*)}. \tag{4.21}$$

Step 3. Let $\zeta \in \mathbb{C}_+$. We have

$$(H_0-\zeta)^{-1} = -i \int_0^{+\infty} e^{-it(H_0-\zeta)} dt,$$

so in the strong sense in $\mathcal{L}(\mathcal{E},\mathcal{E}^*)$

$$\begin{aligned}[(H_0-\zeta)^{-1}, iA] &= \int_{t=0}^{+\infty} [e^{-it(H_0-\zeta)}, A]\, dt = \int_{t=0}^{+\infty} e^{-it\zeta} \int_{s=0}^{t} e^{-isH_0} B_0 e^{-i(t-s)H_0}\, ds\, dt \\ &= \int_{s=0}^{+\infty} e^{-is(H_0-\zeta)} B_0 \int_{t=s}^{+\infty} e^{-i(t-s)(H_0-\zeta)}\, dt\, ds.\end{aligned}$$

This gives

$$[(H_0-\zeta)^{-1}, iA] = -(H_0-\zeta)^{-1} B_0 (H_0-\zeta)^{-1}.$$

We obtain the same result with a similar proof if $\mathrm{Im}(\zeta) < 0$. In particular the commutator $[(H_0-\zeta)^{-1}, A]$ extends to a bounded operator in $\mathcal{L}(\mathcal{K}^*,\mathcal{K})$ with

$$\|[(H_0-\zeta)^{-1}, A]\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})} \lesssim \frac{\|B_0\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^*)}}{|\,\mathrm{Im}(\zeta)|^2}. \tag{4.22}$$

With Lemma 2.3 we deduce in particular that $(H_0-\zeta)^{-1}$ maps continously $\mathcal{E}$ into itself. By duality, it defines a bounded operator on $\mathcal{E}^*$.

Step 4. Let $\psi : x \mapsto \phi(x)(x-i)^2$ and $\Psi = \psi(H_0)$. We have

$$\Phi = (H_0-i)^{-1} \Psi (H_0-i)^{-1}.$$

On $\mathcal{L}(\mathcal{E},\mathcal{E}^*)$ we have

$$[\Psi, A] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} [e^{itH_0}, A] \widehat{\psi}(t)\, dt.$$

The right-hand side extends to an operator in $\mathcal{L}(\mathcal{K},\mathcal{K}^*)$. Then this is also the case for the left-hand side, and moreover

$$\|[\Psi, A]\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^*)} \lesssim \|B_0\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^*)} \int_{\mathbb{R}} |t\widehat{\psi}(t)| \, \mathrm{d}t.$$

As above, we deduce that Ψ leaves $\mathcal{E}$ invariant and extends to a bounded operator on $\mathcal{E}^*$. Then in $\mathcal{L}(\mathcal{E},\mathcal{E}^*)$

$$\begin{aligned}[\Phi, A] = [(H_0 - \mathrm{i})^{-1}, A]\Psi(H_0 - \mathrm{i})^{-1} + (H_0 - \mathrm{i})^{-1}[\Psi, A](H_0 - \mathrm{i})^{-1} \\ + (H_0 - \mathrm{i})^{-1}\Psi[(H_0 - \mathrm{i})^{-1}, A].\end{aligned}$$

This proves that $[\Phi, A] \in \mathcal{L}(\mathcal{K}^*,\mathcal{K})$ (so Φ defines bounded operators on $\mathcal{E}$ and $\mathcal{E}^*$) and

$$\|[\Phi, A]\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})} \lesssim \|B_0\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^*)}.$$

Step 5. Finally we can write in $\mathcal{L}(\mathcal{E},\mathcal{E}^*)$

$$[M, A] = [\Phi, A](B + \beta\Theta)\Phi + \Phi[B + \beta\Theta, A]\Phi + \Phi(B + \beta\Theta)[\Phi, A].$$

All the terms of the right-hand side extend to operators in $\mathcal{L}(\mathcal{K}^*,\mathcal{K})$. With the above estimates and the assumptions of Definition 2.6, we have the proof. ∎

REMARK 4.9. We use the notation of Section 3. Let $I \subset \mathbb{R}_+^* \setminus \mathcal{T}$ be a compact interval. Let $\delta > \frac{1}{2}$. By Proposition 3.2 and Theorem 4.1 we obtain a uniform bound for the operator

$$\langle A_x \rangle^{-\delta} (H_a - z)^{-1} \langle A_x \rangle^{-\delta}$$

when $z \in \mathbb{C}_{I,+}$. As usual, we can replace the weight $\langle A_x \rangle^{-\delta}$ by $\langle x \rangle^{-\delta}$. Since the operators A_x and x only act in the x directions, we can follow the same proof as in the Euclidean space (see for instance Lemma 8.2 in [29]).

5. MULTIPLE COMMUTATORS ESTIMATES

In this section we generalize the multiple resolvent estimates known for a self-adjoint operator (see [14], [15]) or for the perturbation by a dissipative operator (see [5], [37]).

Let $N \geqslant 2$ be fixed for all this section. We will use the notation of Definition 2.8. Thus the symbol " $\lesssim$ " will stand for " $\leqslant C$ " where C is a constant which depends on C_Θ, I, J, δ, β and Υ_N.

For $n \in \{1, \dots, N\}$ and $\varepsilon \in]0, 1]$ we set

$$C_n(\varepsilon) = \sum_{j=1}^{n} \frac{(-\mathrm{i}\varepsilon)^j}{j!} B_j \in \mathcal{L}(\mathcal{K},\mathcal{K}^*). \tag{5.1}$$

In order to prove multiple resolvent estimates, we first need some estimates for the inverse of $(\widetilde{H} + C_n(\varepsilon) - z)$. It is not clear that this operator has an inverse, since for $n \geqslant 3$ there is an anti-dissipative term in $C_n(\varepsilon)$, but it will be the case for ε small enough. The following result generalizes Lemma 3.1 in [15] (see also Lemma 3.1 in [37]) to our setting.

PROPOSITION 5.1. *Suppose A is a conjugate operator for H up to order N on J with bounds $(\alpha, \beta, \Upsilon_N)$.*

(i) *There exists $\varepsilon_N > 0$ such that for $n \in \{1, \dots, N\}$, $z \in \mathbb{C}_{I,+}$ and $\varepsilon \in]0, \varepsilon_N]$ the operator $(\widetilde{H} + C_n(\varepsilon) - z)$ has a bounded inverse in $\mathcal{L}(\mathcal{K}^*, \mathcal{K})$, which we denote by $G_z^n(\varepsilon)$.*

(ii) *For $n \in \mathbb{N}$, $z \in \mathbb{C}_{I,+}$ and $\varepsilon \in]0, \varepsilon_N]$ we have*

$$\|G_z^n(\varepsilon)\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})} \lesssim \frac{1}{\alpha\varepsilon} \quad \textit{and} \quad \|G_z^n(\varepsilon)\langle A\rangle^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{K})} \lesssim \frac{1}{\alpha\sqrt{\varepsilon}}.$$

(iii) *The function $\varepsilon \in]0, \varepsilon_N[\mapsto G_z^n(\varepsilon)$ is differentiable in $\mathcal{L}(\mathcal{K}^*, \mathcal{K})$. Moreover in $\mathcal{L}(\mathcal{D}(A), \mathcal{D}(A)^*)$ we have the equality*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} G_z^n(\varepsilon) = [G_z^n(\varepsilon), A] - \mathrm{i}\frac{(-\mathrm{i}\varepsilon)^n}{n!} G_z^n(\varepsilon) B_{n+1} G_z^n(\varepsilon).$$

For the proof of Proposition 5.1 we need the following lemma, inspired by the standard technique for factored perturbations (see [16]).

LEMMA 5.2. *Let $T \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*)$. Assume that T is bounded with bounded inverse. Let $P_1 \in \mathcal{L}(\mathcal{H}, \mathcal{K}^*)$ and $P_2 \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ be such that $\|P_2 T^{-1} P_1\|_{\mathcal{L}(\mathcal{H})} < 1$. Then $T + P_1 P_2 \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ has a bounded inverse given by $T^{-1} - T^{-1} P_1 \Gamma^{-1} P_2 T^{-1} \in \mathcal{L}(\mathcal{K}^*, \mathcal{K})$, where $\Gamma = 1 + P_2 T^{-1} P_1 \in \mathcal{L}(\mathcal{H})$.*

Proof of Lemma 5.2. The assumptions ensure that Γ is bounded on $\mathcal{H}$ with bounded inverse, so the operator $R = T^{-1} - T^{-1} P_1 \Gamma^{-1} P_2 T^{-1}$ is well-defined in $\mathcal{L}(\mathcal{K}^*, \mathcal{K})$. We only have to check that R is indeed an inverse for $T + P_1 P_2$. On $\mathcal{K}^*$ we have

$$\begin{aligned}(T + P_1 P_2) R &= 1 + P_1 P_2 T^{-1} - P_1 \Gamma^{-1} P_2 T^{-1} - P_1 P_2 T^{-1} P_1 \Gamma^{-1} P_2 T^{-1} \\ &= 1 + P_1 (1 - \Gamma^{-1} - P_2 T^{-1} P_1 \Gamma^{-1}) P_2 T^{-1} \\ &= 1 + P_1 (1 - (1 + P_2 T^{-1} P_1) \Gamma^{-1}) P_2 T^{-1} = 1.\end{aligned}$$

Similarly we have on $\mathcal{K}$:

$$\begin{aligned}R(T + P_1 P_2) &= 1 + T^{-1} P_1 P_2 - T^{-1} P_1 \Gamma^{-1} P_2 - T^{-1} P_1 \Gamma^{-1} P_2 T^{-1} P_1 P_2 \\ &= 1 + T^{-1} P_1 (1 - \Gamma^{-1} - \Gamma^{-1} P_2 T^{-1} P_1) P_2 = 1. \quad \blacksquare\end{aligned}$$

Proof of Proposition 5.1. We use the notation introduced in Section 4.

Step 1. Let $\varepsilon_0 > 0$ be given by Proposition 4.5. The operator $\Phi\Theta\Phi$ is bounded and self-adjoint on $\mathcal{H}$. It is also non-negative, so its square root $\sqrt{\Phi\Theta\Phi}$ is well-defined as a bounded operator on $\mathcal{H}$. As in Proposition 4.5, we write $\mathcal{K}_0$ either for $\mathcal{H}$ or $\mathcal{K}$. Then for $Q \in \mathcal{L}(\mathcal{K}_0^*)$, $z \in \mathbb{C}_{I,+}$, $\varepsilon \in]0, \varepsilon_0]$ and $\varphi \in \mathcal{K}_0^*$ we have according to Proposition 4.5:

$$\langle \Theta\Phi G_z(\varepsilon)Q\varphi, \Phi G_z(\varepsilon)Q\varphi\rangle \lesssim \left(\|Q\|^2_{\mathcal{L}(\mathcal{K}_0^*)} + \|Q^* G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)}\right)\|\varphi\|^2_{\mathcal{K}_0^*}.$$

This proves that

$$\|\sqrt{\Phi\Theta\Phi}G_z(\varepsilon)Q\|_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)} \lesssim \|Q\|_{\mathcal{L}(\mathcal{K}_0^*)} + \|Q^* G_z(\varepsilon)Q\|^{1/2}_{\mathcal{L}(\mathcal{K}_0^*,\mathcal{K}_0)}. \tag{5.2}$$

Applied with $Q = \sqrt{\Phi\Theta\Phi} \in \mathcal{L}(\mathcal{H})$, this gives

$$\sup_{z\in\mathbb{C}_{I,+},\, \varepsilon\in]0,\varepsilon_0]} \|\sqrt{\Phi\Theta\Phi}G_z(\varepsilon)\sqrt{\Phi\Theta\Phi}\|_{\mathcal{L}(\mathcal{H})} < +\infty. \tag{5.3}$$

Step 2. For $z \in \mathbb{C}_{I,+}$ and $\varepsilon \in]0, \varepsilon_\Theta]$ (where $\varepsilon_\Theta \in]0, \varepsilon_0]$ is chosen small enough) we can apply Lemma 5.2 with $T = (\widetilde{H} - \mathrm{i}\varepsilon M - z) \in \mathcal{L}(\mathcal{K},\mathcal{K}^*)$, $P_1 = \mathrm{i}\varepsilon\beta\sqrt{\Phi\Theta\Phi} \in \mathcal{L}(\mathcal{H})$ and $P_2 = \sqrt{\Phi\Theta\Phi} \in \mathcal{L}(\mathcal{H})$. We obtain that the operator $(\widetilde{H} - \mathrm{i}\varepsilon\Phi B\Phi - z)$ has a bounded inverse $G_z^\Theta(\varepsilon) \in \mathcal{L}(\mathcal{K}^*,\mathcal{K})$, given by

$$G_z^\Theta(\varepsilon) = G_z(\varepsilon) - \mathrm{i}\varepsilon\beta G_z(\varepsilon)\sqrt{\Phi\Theta\Phi}\Gamma_z^\Theta(\varepsilon)^{-1}\sqrt{\Phi\Theta\Phi}G_z(\varepsilon), \tag{5.4}$$

where

$$\Gamma_z^\Theta(\varepsilon) = 1 + \mathrm{i}\varepsilon\beta\sqrt{\Phi\Theta\Phi}G_z(\varepsilon)\sqrt{\Phi\Theta\Phi} \in \mathcal{L}(\mathcal{H}).$$

In particular $\Gamma_z^\Theta(\varepsilon)^{-1}$ is bounded in $\mathcal{L}(\mathcal{H})$ uniformly with respect to $z \in \mathbb{C}_{I,+}$ and $\varepsilon \in]0, \varepsilon_\Theta]$. Corollary 4.6 and estimate (5.2) applied with $Q = \mathrm{Id}_{\mathcal{K}^*}$ give

$$\|\sqrt{\Phi\Theta\Phi}G_z(\varepsilon)\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{H})} \lesssim \frac{1}{\sqrt{\alpha}\sqrt{\varepsilon}}.$$

With the similar estimate for $G_z(\varepsilon)\sqrt{\Phi\Theta\Phi}$ and (5.4) we obtain

$$\|G_z^\Theta(\varepsilon)\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})} \lesssim \frac{1}{\alpha\varepsilon}. \tag{5.5}$$

With Proposition 4.5 we can check similarly that

$$\|G_z^\Theta(\varepsilon)\Phi^\perp\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})} \lesssim \frac{1}{\sqrt{\alpha}\sqrt{\varepsilon}} \quad \text{and} \tag{5.6}$$

$$\|G_z^\Theta(\varepsilon)\langle A\rangle^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{K})} \lesssim \frac{1}{\alpha\sqrt{\varepsilon}}. \tag{5.7}$$

Step 3. Now we want to apply similarly Lemma 5.2 with $T = (\widetilde{H} - \mathrm{i}\varepsilon\Phi B\Phi - z)$, $P_1 = \mathrm{i}\varepsilon\Phi^\perp B\langle H_0\rangle^{-1/2}$ and $P_2 = \langle H_0\rangle^{1/2}\Phi$. According to (5.6) we have

$$\varepsilon\|\langle H_0\rangle^{1/2}\Phi G_z^\Theta(\varepsilon)\Phi^\perp B\langle H_0\rangle^{-1/2}\|_{\mathcal{L}(\mathcal{H})} \lesssim \varepsilon\|\Phi G_z^\Theta(\varepsilon)\Phi^\perp\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})}\|B\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^*)} \lesssim \sqrt{\varepsilon}.$$

So if $\varepsilon_\perp \in]0, \varepsilon_\Theta]$ is chosen small enough we can apply Lemma 5.2 for $\varepsilon \in]0, \varepsilon_\perp]$: for all $z \in \mathbb{C}_{I,+}$ and $\varepsilon \in]0, \varepsilon_\perp]$ the operator $(\widetilde{H} - z - \mathrm{i}\varepsilon B\Phi)$ has a bounded inverse $G_z^\perp(\varepsilon) \in \mathcal{L}(\mathcal{K}^*,\mathcal{K})$ given by

$$G_z^\perp(\varepsilon) = G_z^\Theta(\varepsilon) - \mathrm{i}\varepsilon G_z^\Theta(\varepsilon)\Phi^\perp B\langle H_0\rangle^{-1/2}\Gamma_z^\perp(\varepsilon)^{-1}\langle H_0\rangle^{1/2}\Phi G_z^\Theta(\varepsilon),$$

where

$$\Gamma_z^\perp(\varepsilon) = 1 + \mathrm{i}\varepsilon\langle H_0\rangle^{1/2}\Phi G_z^\Theta(\varepsilon)\Phi^\perp B\langle H_0\rangle^{-1/2}.$$

Then, as above we use (5.5), (5.6) and (5.7) to prove

$$\|G_z^{\perp}(\varepsilon)\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})} \lesssim \frac{1}{\alpha\varepsilon}, \tag{5.8}$$

$$\|G_z^{\perp}(\varepsilon)\Phi^{\perp}\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})} \lesssim \frac{1}{\sqrt{\alpha}\sqrt{\varepsilon}}, \quad \text{and} \tag{5.9}$$

$$\|G_z^{\perp}(\varepsilon)\langle A\rangle^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{K})} \lesssim \frac{1}{\alpha\sqrt{\varepsilon}}. \tag{5.10}$$

Step 4. In order to prove the existence of $G_z^1(\varepsilon)$, it remains to apply Lemma 5.2 with $T = (\widetilde{H} - \mathrm{i}\varepsilon B\Phi - z)$, $P_1 = \mathrm{i}\varepsilon B\langle H_0\rangle^{-1/2}$ and $P_2 = \langle H_0\rangle^{1/2}\Phi^{\perp}$. We have

$$\varepsilon\|\langle H_0\rangle^{1/2}\Phi^{\perp}G_z^{\Theta}(\varepsilon)B\langle H_0\rangle^{-1/2}\|_{\mathcal{L}(\mathcal{H})} \lesssim \varepsilon\|\Phi^{\perp}G_z^{\Theta}(\varepsilon)\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})}\|B\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^*)} \lesssim \sqrt{\varepsilon}.$$

So if $\varepsilon_1 \in]0,\varepsilon_{\perp}]$ is chosen small enough we can apply Lemma 5.2, which proves that for $z \in \mathbb{C}_{I,+}$ and $\varepsilon \in]0,\varepsilon_1]$ the operator $(\widetilde{H} - \mathrm{i}\varepsilon B - z)$ has a bounded inverse $G_z^1(\varepsilon) \in \mathcal{L}(\mathcal{K}^*,\mathcal{K})$ given by

$$G_z^1(\varepsilon) = G_z^{\perp}(\varepsilon) - \mathrm{i}\varepsilon G_z^{\perp}(\varepsilon)B\langle H_0\rangle^{-1/2}\Gamma_z^1(\varepsilon)^{-1}\langle H_0\rangle^{1/2}\Phi^{\perp}G_z^{\perp}(\varepsilon),$$

where

$$\Gamma_z^1(\varepsilon) = 1 + \mathrm{i}\varepsilon\langle H_0\rangle^{1/2}\Phi^{\perp}G_z^{\Theta}(\varepsilon)B\langle H_0\rangle^{-1/2}.$$

Moreover we have

$$\|G_z^1(\varepsilon)\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})} \lesssim \frac{1}{\alpha\varepsilon} \quad \text{and} \tag{5.11}$$

$$\|G_z^1(\varepsilon)\langle A\rangle^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{K})} \lesssim \frac{1}{\alpha\sqrt{\varepsilon}}. \tag{5.12}$$

Step 5. For $n \in \{2,\dots,N\}$ we have

$$\begin{aligned}\|\langle H_0\rangle^{1/2}G_z^1(\varepsilon)(C_n(\varepsilon) - C_1(\varepsilon))\langle H_0\rangle^{-1/2}\|_{\mathcal{L}(\mathcal{H})} &\leqslant \sum_{j=2}^{n}\varepsilon^j\|G_z^1(\varepsilon)\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})}\|B_j\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^*)}\\ &\lesssim \varepsilon^2 \times \frac{1}{\alpha\varepsilon} \times \alpha \lesssim \varepsilon.\end{aligned}$$

Thus for $\varepsilon \in]0,\varepsilon_N]$, ε_N chosen small enough, we can apply Lemma 5.2 with $T = \widetilde{H} + C_1(\varepsilon) - z$, $P_1 = (C_n(\varepsilon) - C_1(\varepsilon))\langle H_0\rangle^{-1/2}$ and $P_2 = \langle H_0\rangle^{1/2}$. This proves that the operator $\widetilde{H} + C_n(\varepsilon) - z$ has a bounded inverse in $\mathcal{L}(\mathcal{K}^*,\mathcal{K})$, given by

$$\begin{aligned}G_z^n(\varepsilon) = G_z^1(\varepsilon) - G_z^1(\varepsilon)(C_n(\varepsilon) - C_1(\varepsilon))\langle H_0\rangle^{-1/2}(1 + G_z^1(\varepsilon)(C_n(\varepsilon)\\ - C_1(\varepsilon)))^{-1}\langle H_0\rangle^{1/2}G_z^1(\varepsilon).\end{aligned}$$

This proves the first statement, and the estimates are proved as above.

Step 6. Let $\varepsilon \in]0,\varepsilon_N[$. For $\widetilde{\varepsilon} \in]\frac{\varepsilon}{2},\varepsilon_N[$ we have

$$G_z^n(\widetilde{\varepsilon}) - G_z^n(\varepsilon) = -G_z^n(\widetilde{\varepsilon})(C_n(\widetilde{\varepsilon}) - C_n(\varepsilon))G_z^n(\varepsilon).$$

Since C_n is a continuous function in $\mathcal{L}(\mathcal{K},\mathcal{K}^*)$ and G_z^n is uniformly bounded in $\mathcal{L}(\mathcal{K}^*,\mathcal{K})$ (by a constant which depends on α) on $]\frac{\varepsilon}{2},\varepsilon_N[$, the map G_z^n is continuous in $\mathcal{L}(\mathcal{K}^*,\mathcal{K})$. Then we divide this equality by $\widetilde{\varepsilon}-\varepsilon$ and let $\widetilde{\varepsilon}$ go to ε. We obtain that G_z^n is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}G_z^n(\varepsilon) = -G_z^n(\varepsilon)C_n'(\varepsilon)G_z^n(\varepsilon).$$

The derivative $C_n'(\varepsilon)$ is well-defined in $\mathcal{L}(\mathcal{K},\mathcal{K}^*)$. In the sense of forms on $\mathcal{E}$ we can check that

$$C_n'(\varepsilon) = [\widetilde{H}+C_n(\varepsilon)-z,A] - \frac{(-\mathrm{i}\varepsilon)^n}{n!}[B_n,A].$$

But the right-hand side extends to an operator in $\mathcal{L}(\mathcal{K},\mathcal{K}^*)$, and the last statement of the proposition follows. ∎

The following two results generalize Theorems 3.2 and 3.5 in [14].

THEOREM 5.3. *Suppose A is a conjugate operator for H up to order N on J with bounds $(\alpha,\beta,\Upsilon_N)$. Let $\delta_1,\delta_2 \geqslant 0$ be such that $\delta_1+\delta_2 < N-1$. Let I be a compact subinterval of $\mathring{J}$. Then there exists $c \geqslant 0$ which only depends on C_Θ, J, I, δ_1, δ_2, β and Υ_N such that for all $z \in \mathbb{C}_{I,+}$ we have*

$$\|\langle A\rangle^{\delta_1}\mathbf{1}_{\mathbb{R}_-}(A)(H-z)^{-1}\mathbf{1}_{\mathbb{R}_+}(A)\langle A\rangle^{\delta_2}\| \leqslant \frac{c}{\alpha}.$$

Moreover for $\mathrm{Re}(z) \in \mathring{J}$ fixed this operator has a limit when $\mathrm{Im}(z) \searrow 0$. This limit defines in $\mathcal{L}(\mathcal{H})$ a Hölder-continuous function of index $\frac{N-1-\delta_1-\delta_2}{N+1}$ with respect to $\mathrm{Re}(z)$.

Proof. Let ε_N be given by Proposition 5.1. For $z \in \mathbb{C}_{I,+}$ and $\varepsilon \in]0,\varepsilon_N]$ we set

$$F_z^N(\varepsilon) = \langle A\rangle^{\delta_1}\mathrm{e}^{\varepsilon A}\mathbf{1}_{\mathbb{R}_-}(A)G_z^N(\varepsilon)\mathbf{1}_{\mathbb{R}_+}(A)\mathrm{e}^{-\varepsilon A}\langle A\rangle^{\delta_2}.$$

According to Proposition 5.1, the functional calculus and the fact that

$$\|[B_N,A]\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^*)} \lesssim \alpha$$

we have

$$\begin{aligned}\|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}F_z^N(\varepsilon)\| &= \frac{\varepsilon^N}{N!}\|\langle A\rangle^{\delta_1}\mathrm{e}^{\varepsilon A}\mathbf{1}_{\mathbb{R}_-}(A)G_z^N(\varepsilon)[B_N,A]G_z^N(\varepsilon)\mathbf{1}_{\mathbb{R}_+}(A)\mathrm{e}^{-\varepsilon A}\langle A\rangle^{\delta_2}\| \\ &\lesssim \varepsilon^{-\delta_1}\times\alpha^{-1}\varepsilon^{N-2}\times\varepsilon^{-\delta_2} = \frac{\varepsilon^{N-\delta_1-\delta_2-2}}{\alpha}.\end{aligned}$$

Since $N-\delta_1-\delta_2-2 > -1$, this proves that $F_z^N(\varepsilon)$ is uniformly bounded (we do not have to use Lemma 4.8 here). Now let $z,z' \in \mathbb{C}_{I,+}$ and $\varepsilon \in]0,\varepsilon_0]$. The previous estimates give

$$\|F_z^N(\varepsilon)-F_z^N(0)\| \leqslant \frac{c}{\alpha}\varepsilon^{N-1-\delta_1-\delta_2} \quad\text{and}\quad \|F_z^N(\varepsilon)-F_{z'}^N(\varepsilon)\| \leqslant \frac{c}{\alpha^2}\varepsilon^{-(\delta_1+\delta_2+2)}|z-z'|.$$

We get the second statement as we did for Theorem 4.1, taking

$$\varepsilon = \alpha^{-1/(N+1)}|z-z'|^{1/(N+1)}. \quad ∎$$

THEOREM 5.4. *Suppose A is a conjugate operator for H up to order N on J with bounds $(\alpha, \beta, \Upsilon_N)$. Let $\delta \in]\frac{1}{2}, N[$. Then there exists $c \geqslant 0$ which only depends on C_Θ, J, I, δ_1, δ_2, β and Υ_N such that for all $z \in \mathbb{C}_{I,+}$ we have*

$$\|\langle A\rangle^{-\delta}(H-z)^{-1}\mathbf{1}_{\mathbb{R}_+}(A)\langle A\rangle^{\delta-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{c}{\alpha},$$

and for $\mathrm{Re}(z) \in \mathring{J}$ *fixed this operator has a limit when* $\mathrm{Im}(z) \searrow 0$. *This limit defines in $\mathcal{L}(\mathcal{H})$ a Hölder-continuous function with respect to* $\mathrm{Re}(z)$. *Moreover we have similar results for the operator*

$$\langle A\rangle^{\delta-1}\mathbf{1}_{\mathbb{R}_-}(A)(H-z)^{-1}\langle A\rangle^{-\delta}.$$

Proof. We follow the proof given in [14]. It relies itself on the results of [25]. We also refer to [36] for a proof in the dissipative case (perturbation by a dissipative operator). The case of a dissipative perturbation in the sense of forms does not rise new difficulties, so we omit the details. ∎

Now that we have Theorems 4.1, 5.3 and 5.4 we can follow the idea developped in Section 5 of [5]. The purpose is not only to prove uniform estimates for the powers of the resolvent, but also to allow inserted factors. This is motivated by the wave equation. Indeed, the derivatives of the corresponding resolvent are not its powers in this case (see Example 5.7 below).

Let $n \in \{1, \dots, N\}$. We consider $\Phi_0 \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, $\Phi_1, \dots, \Phi_{n-1} \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ and $\Phi_n \in \mathcal{L}(\mathcal{H}, \mathcal{K}^*)$. We assume (inductively) on $m \in \{1, \dots, N\}$ that the operator

$$\mathrm{ad}_{\mathrm{i}A}^m(\Phi_0) := [\mathrm{ad}_{\mathrm{i}A}^{m-1}(\Phi_0), \mathrm{i}A]$$

(with $\mathrm{ad}_{\mathrm{i}A}^0(\Phi_0) = \Phi_0$), at least defined as an operator in $\mathcal{L}(\mathcal{E}, \mathcal{E}^*)$, can be extended to an operator in $\mathcal{L}(\mathcal{K}, \mathcal{H})$. We assume similarly that the commutators $\mathrm{ad}_{\mathrm{i}A}^m(\Phi_j)$ for $m \in \{1, \dots, N\}$ and $j \in \{1, \dots, n-1\}$ extend to operators in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ and finally that the commutators $\mathrm{ad}_{\mathrm{i}A}^m(\Phi_n)$ for $m \in \{1, \dots, N\}$ extend to operators in $\mathcal{L}(\mathcal{H}, \mathcal{K}^*)$. Then for $j \in \{1, \dots, n-1\}$ we set

$$\|\Phi_j\|_{\mathcal{C}_N(A,\mathcal{K},\mathcal{K}^*)} = \sum_{m=0}^{N} \|\mathrm{ad}_{\mathrm{i}A}^m(\Phi_j)\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^*)}.$$

We similarly define $\|\Phi_0\|_{\mathcal{C}_N(A,\mathcal{K},\mathcal{H})}$ and $\|\Phi_n\|_{\mathcal{C}_N(A,\mathcal{H},\mathcal{K}^*)}$, and then

$$\|(\Phi_0, \dots, \Phi_n)\|_{\mathcal{C}_N^n} = \|\Phi_0\|_{\mathcal{C}_N(A,\mathcal{K},\mathcal{H})}\|\Phi_n\|_{\mathcal{C}_N(A,\mathcal{H},\mathcal{K}^*)} \prod_{j=1}^{n-1} \|\Phi_j\|_{\mathcal{C}_N(A,\mathcal{K},\mathcal{K}^*)}.$$

For $z \in \mathbb{C}_+$ we set

$$\mathcal{R}_n(z) = \Phi_0(H-z)^{-1}\Phi_1(H-z)^{-1}\cdots\Phi_{n-1}(H-z)^{-1}\Phi_n. \tag{5.13}$$

The statement is the following.

THEOREM 5.5. *Suppose that the self-adjoint operator A is conjugate to the maximal dissipative operator H on J up to order N with bounds $(\alpha, \beta, \Upsilon_N)$. Let $I \subset \mathring{J}$ be a* *compact interval. Let* $\delta \in]n - \frac{1}{2}, N[$ *and* $\delta_1, \delta_2 \geqslant 0$ *such that* $\delta_1 + \delta_2 < N - n$*. Then there exists* $c \geqslant 0$ *such that:*

$$\|\langle A\rangle^{-\delta}\mathcal{R}_n(z)\langle A_\lambda\rangle^{-\delta}\| \leqslant \frac{c}{\alpha^n}\|(\Phi_0,\dots,\Phi_n)\|_{\mathcal{C}_N^n},$$
$$\|\langle A\rangle^{\delta-n}\mathbf{1}_{\mathbb{R}_-}(A)\mathcal{R}_n(z)\langle A\rangle^{-\delta}\| \leqslant \frac{c}{\alpha^n}\|(\Phi_0,\dots,\Phi_n)\|_{\mathcal{C}_N^n},$$
$$\|\langle A\rangle^{-\delta}\mathcal{R}_n(z)\mathbf{1}_{\mathbb{R}_+}(A)\langle A\rangle^{\delta-n}\| \leqslant \frac{c}{\alpha^n}\|(\Phi_0,\dots,\Phi_n)\|_{\mathcal{C}_N^n} \quad \textit{and}$$
$$\|\langle A\rangle^{\delta_1}\mathbf{1}_{\mathbb{R}_-}(A)\mathcal{R}_n(z)\mathbf{1}_{\mathbb{R}_+}(A)\langle A\rangle^{\delta_2}\| \leqslant \frac{c}{\alpha^n}\|(\Phi_0,\dots,\Phi_n)\|_{\mathcal{C}_N^n}.$$

Proof. We can follow the proof of the analogous Theorem 5.14 in [5]. We only briefly recall the strategy. With the identity

$$(H-z)^{-1} = (H-\mathrm{i})^{-1} + (z-\mathrm{i})(H-\mathrm{i})^{-2} + (z-\mathrm{i})^2(H-\mathrm{i})^{-1}(H-z)^{-1}(H-\mathrm{i})^{-1},$$

we see that we can assume without loss of generality that the operators Φ_j and their commutators with A are in $\mathcal{L}(\mathcal{H})$. Then the idea is to start from the estimates for a single resolvent (see Theorems 4.1, 5.3 and 5.4), to prove analog estimates with $(H-z)^{-1}$ replaced by an operator of the form $\Phi_j(H-z)^{-1}\Phi_k$ (for this we use the commutation properties between Φ_j and A), and finally we use Lemma 5.4 in [5] to obtain the multiple resolvent estimates with inserted factors. We omit the details and refer to the proof of Theorem 5.14 in [5]. ∎

REMARK 5.6. With the same idea we could even prove uniform estimates for an operator of the form

$$\mathcal{R}(z) = \Phi_0(H_1 - z)^{-1}\Phi_1(H_2 - z)^{-1}\cdots\Phi_{n-1}(H_n - z)^{-1}\Phi_n,$$

where $H_1,\dots,H_n$ are different maximal dissipative operators of the form discussed in Section 2 with uniform constant C_Θ in (2.2) and with the same form domain $\mathcal{K}$, under the assumption that A is conjugated to H_k on J with bounds $(\alpha_k, \beta, \Upsilon_N)$ for all $k \in \{1,\dots,n\}$. Then the quotient α^n is replaced by $\alpha_1\cdots\alpha_n$ in the estimates of the theorem.

EXAMPLE 5.7. We consider the wave equation (1.6) on the half-space (3.5). Assume that $w_0 = 0$ on $\partial\Omega$. Let w be the solution of (1.6). For $\mu > 0$ we set $w_\mu(t) = \mathbf{1}_{\mathbb{R}_+}(t)\mathrm{e}^{-t\mu}w(t)$. Then the inverse Fourier transform of w_μ,

$$\check{w}_\mu(\tau) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\tau}w_\mu(t)\,\mathrm{d}t = \int_0^{+\infty}\mathrm{e}^{\mathrm{i}t(\tau+\mathrm{i}\mu)}w(t)\,\mathrm{d}t,$$

is a solution of the problem

$$\begin{cases}(-\Delta - z^2)\check{w}_\mu(\tau) = -\mathrm{i}zw_0 + w_1 & \text{on } \Omega,\\ \partial_\nu\check{w}_\mu(\tau) = \mathrm{i}za\check{w}_\mu(\tau) & \text{on } \partial\Omega,\end{cases}$$

where $z = \tau + \mathrm{i}\mu$. In other words, we have

$$\check{w}_\mu(\tau) = R(z)(-\mathrm{i}zw_0 + w_1) \quad \text{where } R(z) = (H_{az} - z^2)^{-1}.$$

In order to study the properties of $\check{w}_\mu(\tau)$ and hence those of $w(t)$ we have to prove uniform resolvent estimates for the derivative of $R(z)$ when $\mathrm{Im}(z) \searrow 0$ (see for instance Theorem 1.2 in [5] for the wave equation on $\mathbb{R}^d$). We can check that for $z \in \mathbb{C}_+$ we have

$$R'(z) = R(z)(\mathrm{i}\Theta + 2z)R(z),$$

where $\Theta \in \mathcal{L}(H^1(\Omega), H^1(\Omega))$ is the operator corresponding to the imaginary part q_Θ of q_a (see (3.2)). Following Proposition 5.9 in [5] we can check that for $n \in \mathbb{N}^*$ the derivative $R^{(n)}(z)$ is a linear combination of terms of the form

$$z^k R(z)\Theta^{j_1} R(z)\Theta^{j_2} R(z) \cdots \Theta^{j_m} R(z),$$

where $m \in \{0, \dots, n\}$ (there are $m+1$ factors $R(z)$), $k \in \mathbb{N}$, $j_1, \dots, j_m \in \{0, 1\}$, $\Theta^1 = \Theta$, $\Theta^0 = \mathrm{Id}$ and $n = 2m - k - (j_1 + \cdots + j_m)$. The difference is that Θ is not a bounded operator on L^2. However, we have checked the commutation properties between Θ and A in the proof of Proposition 3.2, so with Theorem 5.5 we can prove the following result.

PROPOSITION 5.8. *Let $n \in \mathbb{N}$ and assume that* (3.4) *holds for $N \geqslant n$. Let $\delta > n + \frac{1}{2}$ and let I be a compact subset of $\mathbb{R}_+^*$. Then there exists $C \geqslant 0$ such that for all $z \in \mathbb{C}_{I,+}$ we have*

$$\|\langle x \rangle^{-\delta} R^{(n)}(z) \langle x \rangle^{-\delta}\|_{\mathcal{L}(L^2)} \leqslant C.$$

6. ABSOLUTELY CONTINUOUS SPECTRUM

In this section we discuss the properties of the absolutely continuous subspace for a dissipative operator. We recall from [7] the following definition.

DEFINITION 6.1. Let H be a maximal dissipative operator on a Hilbert space $\mathcal{H}$. The absolutely continuous subspace $\mathcal{H}_{\mathrm{ac}}(H)$ of H is the closure in $\mathcal{H}$ of

$$\mathcal{H}_{\mathrm{ac}}^*(H) := \Big\{ \varphi \in \mathcal{H} \,:\, \exists C_\varphi \geqslant 0, \forall \psi \in \mathcal{H}, \int_0^{+\infty} |\langle \mathrm{e}^{-\mathrm{i}tH}\varphi, \psi\rangle_{\mathcal{H}}|^2 \, \mathrm{d}t \leqslant C_\varphi \|\psi\|_{\mathcal{H}}^2 \Big\}.$$

For a self-adjoint operator this definition coincides with the usual definition involving the spectral measure (see for instance Proposition 1.7, Theorem 1.3 and Corollary 1.4 in [30]).

In the self-adjoint case, the uniform resolvent estimates and the $L^2(\mathbb{R}_+, \mathcal{H})$ norm of the solution of the time-dependant problem are linked by the theory of relatively smooth operators in the sense of Kato (see [16] and Section XIII.7 of [32]). It is less known that this link remains valid for dissipative operators.

In order to extend the self-adjoint theory of relative smoothness for a dissipative operator H, we use a self-adjoint dilation of H. For the general theory of self-adjoint dilations we refer to [26]. Here we only recall that a maximal dissipative operator H on a Hilbert space $\mathcal{H}$ always has a self-adjoint dilation. This means that there exists a self-adjoint operator $\widehat{H}$ on some Hilbert space $\widehat{\mathcal{H}}$ (which contains $\mathcal{H}$ as a subspace) such that on $\mathcal{L}(\mathcal{H})$ we have

$$\begin{aligned}
\forall z \in \mathbb{C}_+, \qquad & P_{\mathcal{H}}(\widehat{H}-z)^{-1} I_{\mathcal{H}} = (H-z)^{-1},\\
\forall z \in \mathbb{C}_+, \qquad & P_{\mathcal{H}}(\widehat{H}-\bar{z})^{-1} I_{\mathcal{H}} = (H^*-\bar{z})^{-1},\\
\forall t \geqslant 0, \qquad & P_{\mathcal{H}} e^{-it\widehat{H}} I_{\mathcal{H}} = e^{-itH},\\
\forall t \geqslant 0, \qquad & P_{\mathcal{H}} e^{it\widehat{H}} I_{\mathcal{H}} = e^{itH^*},
\end{aligned}$$

where $P_{\mathcal{H}} \in \mathcal{L}(\widehat{\mathcal{H}}, \mathcal{H})$ denotes the orthogonal projection of $\widehat{\mathcal{H}}$ on $\mathcal{H}$ and $I_{\mathcal{H}} \in \mathcal{L}(\mathcal{H}, \widehat{\mathcal{H}})$ is the embedding of $\mathcal{H}$ in $\widehat{\mathcal{H}}$. An explicit example of (minimal) self-adjoint dilation for the dissipative Schrödinger operator on $\mathbb{R}^d$ is given in [28].

PROPOSITION 6.2. *Let Q be a closed operator on $\mathcal{H}$. Assume that there exists $C \geqslant 0$ such that for all $z \in \mathbb{C}_+$ and $\varphi \in \mathcal{D}(Q^*)$ we have*

$$\langle ((H-z)^{-1} - (H^*-\bar{z})^{-1}) Q^*\varphi, Q^*\varphi \rangle_{\mathcal{H}} \leqslant C \|\varphi\|_{\mathcal{H}}^2.$$

Then for $\psi \in \mathcal{H}$ we have $e^{-itH}\psi \in \mathcal{D}(Q)$ for almost all $t \geqslant 0$ and

$$\int_0^{+\infty} \|Q e^{-itH}\psi\|_{\mathcal{H}}^2 \, dt \leqslant C \|\psi\|_{\mathcal{H}}^2.$$

We also have $e^{itH^*}\psi \in \mathcal{D}(Q)$ for almost all $t \geqslant 0$ and

$$\int_0^{+\infty} \|Q e^{itH^*}\psi\|_{\mathcal{H}}^2 \, dt \leqslant C \|\psi\|_{\mathcal{H}}^2.$$

Proof. Let $\widehat{H}$ be a self-adjoint dilation of H on a Hilbert space $\widehat{\mathcal{H}}$ which contains $\mathcal{H}$ as a subspace. We can write $\widehat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}^\perp$. We extend Q as an operator $\widehat{Q}$ on $\widehat{\mathcal{H}}$ by 0 on $\mathcal{H}^\perp$. Then $\widehat{Q}$ is a closed operator on $\widehat{\mathcal{H}}$ with domain $\mathcal{D}(\widehat{Q}) = \mathcal{D}(Q) \oplus \mathcal{H}^\perp$. Then for all $z \in \mathbb{C}_+$ and $\widehat{\varphi} = (\varphi, \varphi^\perp), \widehat{\psi} = (\psi, \psi^\perp) \in \mathcal{D}(\widehat{Q})$ we have

$$\begin{aligned}
\langle ((\widehat{H}-z)^{-1} - (\widehat{H}-\bar{z})^{-1}) \widehat{Q}^*\widehat{\varphi}, \widehat{Q}^*\widehat{\psi} \rangle_{\widehat{\mathcal{H}}} &= \langle ((H-z)^{-1} - (H^*-\bar{z})^{-1}) Q^*\varphi, Q^*\psi \rangle_{\mathcal{H}}\\
&\leqslant C \|\varphi\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} \leqslant C \|\widehat{\varphi}\|_{\widehat{\mathcal{H}}} \|\widehat{\psi}\|_{\widehat{\mathcal{H}}}.
\end{aligned}$$

Let $\widehat{\zeta} \in \widehat{\mathcal{H}}$. According to Theorem XIII.25 in [32] we have $e^{-it\widehat{H}}\widehat{\zeta} \in \mathcal{D}(\widehat{Q})$ for almost all $t \in \mathbb{R}$ and

$$\int_{\mathbb{R}} \|\widehat{Q} e^{-it\widehat{H}} \widehat{\zeta}\|_{\widehat{\mathcal{H}}}^2 \, dt \leqslant C \|\widehat{\zeta}\|_{\widehat{\mathcal{H}}}^2.$$

Now let $\varphi \in \mathcal{H}$ and $\widehat{\varphi} = (\varphi, 0) \in \widehat{\mathcal{H}}$. We have $\mathrm{e}^{-\mathrm{i}tH}\varphi = P_{\mathcal{H}}\mathrm{e}^{-\mathrm{i}t\widehat{H}}\widehat{\varphi} \in P_{\mathcal{H}}\mathcal{D}(\widehat{Q}) = \mathcal{D}(Q)$ for almost all $t \geqslant 0$ and moreover

$$\int_0^{+\infty} \|Q\mathrm{e}^{-\mathrm{i}tH}\varphi\|_{\mathcal{H}}^2 \,\mathrm{d}t = \int_0^{+\infty} \|\widehat{Q}\mathrm{e}^{-\mathrm{i}t\widehat{H}}\widehat{\varphi}\|_{\widehat{\mathcal{H}}}^2 \,\mathrm{d}t \leqslant C\|\widehat{\varphi}\|_{\widehat{\mathcal{H}}}^2 = C\|\varphi\|_{\mathcal{H}}^2.$$

We conclude similarly for the integral of $\|Q\mathrm{e}^{\mathrm{i}tH^*}\varphi\|_{\mathcal{H}}^2$. ∎

COROLLARY 6.3. *Under the assumptions of Proposition* 6.2 *we have* $\mathrm{Ran}(Q^*) \subset \mathcal{H}_{\mathrm{ac}}^*(H)$.

Proof. Let $\varphi \in \mathrm{Ran}(Q^*)$ and $\zeta \in \mathcal{H}$ be such that $\varphi = Q^*\zeta$. Then for $\psi \in \mathcal{H}$ we have

$$\int_0^{+\infty} |\langle \mathrm{e}^{-\mathrm{i}tH}\varphi, \psi\rangle_{\mathcal{H}}|^2 \,\mathrm{d}t \leqslant \int_0^{+\infty} \|\zeta\|_{\mathcal{H}}^2 \|Q\mathrm{e}^{\mathrm{i}tH^*}\psi\|_{\mathcal{H}}^2 \,\mathrm{d}t \leqslant C\|\zeta\|_{\mathcal{H}}^2\|\psi\|_{\mathcal{H}}^2. \quad ∎$$

Theorem 4.1 gives an estimate as in Proposition 6.2 with $Q = \langle A\rangle^{-\delta}$ but only for $z \in \mathbb{C}_{I,+}$ for some interval I. In order to obtain an estimate for all $z \in \mathbb{C}_+$ we have to localize spectrally. For this we are going to use a function of the self-adjoint part H_0 of H. We first prove the following lemma.

LEMMA 6.4. *Let H_0, $\widetilde{H}_0$ and $\mathcal{K}$ be as in Section* 2. *Let A be a self-adjoint operator on $\mathcal{H}$. Assume that the first assumption of Definition* 2.4 *and the commutator estimate* (2.5) *hold. Then for $\delta \in [-1, 1]$ and $\chi \in C_0^\infty(\mathbb{R})$ the operator $\langle A\rangle^\delta \chi(H_0)\langle A\rangle^{-\delta}$ extends to a bounded operator on $\mathcal{H}$.*

Proof. We consider an almost analytic extension $\widetilde{\chi}$ of χ (see [8], [10]):

$$\widetilde{\chi}(x + \mathrm{i}y) = \psi(y) \sum_{k=0}^{2} \chi^{(k)}(x) \frac{(\mathrm{i}y)^k}{k!}$$

where $\psi \in C_0^\infty(\mathbb{R}, [0, 1])$ is supported on $[-2, 2]$ and equal to 1 on $[-1, 1]$. We have

$$\frac{\partial\widetilde{\chi}}{\partial\overline{\zeta}}(x + \mathrm{i}y) = \frac{\mathrm{i}\psi'(y)}{2} \sum_{k=0}^{2} \chi^{(k)}(x) \frac{(\mathrm{i}y)^k}{k!} + \frac{\psi(y)}{2}\chi^{(3)}(x)\frac{(\mathrm{i}y)^2}{2},$$

and in particular for $\zeta \in \mathbb{C}$

$$\left|\frac{\partial\widetilde{\chi}}{\partial\overline{\zeta}}(\zeta)\right| \lesssim |\operatorname{Im}\zeta|^2 \mathbf{1}_{\{\mathrm{Re}(\zeta)\in\mathrm{supp}(\chi), |\operatorname{Im}(\zeta)|\leqslant 2\}}(\zeta). \tag{6.1}$$

Thus we can write the Helffer–Sjöstrand formula for $\chi(H_0)$:

$$\chi(H_0) = -\frac{1}{\pi} \int_{\zeta = x+\mathrm{i}y \in \mathbb{C}} \frac{\partial\widetilde{\chi}}{\partial\overline{\zeta}}(\zeta)(\widetilde{H}_0 - \zeta)^{-1} \,\mathrm{d}x\,\mathrm{d}y.$$

With (4.22) and (6.1) we see that the operator $[\chi(H_0), \mathrm{i}A]$ extends to an operator in $\mathcal{L}(\mathcal{K}^*, \mathcal{K})$ and hence in $\mathcal{L}(\mathcal{H})$. This proves that $A\chi(H_0)\langle A\rangle^{-1}$ is bounded on

$\mathcal{H}$. Since for $\varphi \in \mathcal{H}$ we have

$$\|\langle A\rangle\chi(H_0)\langle A\rangle^{-1}\varphi\|_{\mathcal{H}}^2 = \|\chi(H_0)\langle A\rangle^{-1}\varphi\|_{\mathcal{H}}^2 + \|A\chi(H_0)\langle A\rangle^{-1}\varphi\|_{\mathcal{H}}^2,$$

the operator $\langle A\rangle\chi(H_0)\langle A\rangle^{-1}$ is also bounded on $\mathcal{H}$. This gives the result when $\delta = 1$. Since it clearly holds for $\delta = 0$, we obtain the case $\delta \in [0,1]$ by interpolation, and finally the general case follows by duality. ∎

REMARK 6.5. Let $N \in \mathbb{N}^*$. Similarly to Definition 2.7, assume inductively that the commutators $\mathrm{ad}_{iA}^n(\widetilde{H}_0)$ extend to bounded operators on $\mathcal{L}(\mathcal{K},\mathcal{K}^*)$ for $n = 1,\dots,N$. Then with a similar proof we can show that the conclusion of Lemma 6.4 holds for any $\delta \in [-N,N]$.

Now we can prove the main result of this section.

THEOREM 6.6. *Assume that A is a conjugate operator to H on the open interval J, and let $\delta > \frac{1}{2}$. Then* $\mathrm{Ran}(\mathbf{1}_J(H_0)\langle A\rangle^{-\delta}) \subset \mathcal{H}_{\mathrm{ac}}(H)$.

Proof. Without loss of generality, we can assume that $\delta \in]\frac{1}{2},1]$. Let I and I' be compact intervals such that $I \subset \mathring{I}' \subset I' \subset J$. Let $\chi \in C_0^\infty(\mathbb{R})$ be supported in $\mathring{I}$ and equal to 1 on a neighborhood of I. According to Lemma 6.4 and Theorem 4.1 (and Remark 4.2) the operator

$$\langle A\rangle^{-\delta}\chi(H_0)(H-z)^{-1}\chi(H_0)\langle A\rangle^{-\delta}$$

and its adjoint are bounded in $\mathcal{L}(\mathcal{H})$ uniformly in $z \in \mathbb{C}_{I',+}$. Then for $z \in \mathbb{C}_{\mathbb{R}\setminus I',+}$ and $\varphi,\psi \in \mathcal{H}$ we have by the resolvent identity (see (2.4)):

$$\begin{aligned}
&|\langle\langle A\rangle^{-\delta}\chi(H_0)(H-z)^{-1}\chi(H_0)\langle A\rangle^{-\delta}\varphi,\psi\rangle_{\mathcal{H}}| \\
&\quad\leqslant |\langle\langle A\rangle^{-\delta}\chi(H_0)(H_0-z)^{-1}\chi(H_0)\langle A\rangle^{-\delta}\varphi,\psi\rangle_{\mathcal{H}}| \\
&\qquad + q_\Theta((H-z)^{-1}\chi(H_0)\langle A\rangle^{-\delta}\varphi,(H_0-\bar{z})^{-1}\chi(H_0)\langle A\rangle^{-\delta}\psi) \\
&\quad\lesssim \|\varphi\|\|\psi\| + q_\Theta((H-z)^{-1}\chi(H_0)\langle A\rangle^{-\delta}\varphi)^{1/2}q_\Theta((H_0-\bar{z})^{-1}\chi(H_0)\langle A\rangle^{-\delta}\psi)^{1/2}.
\end{aligned}$$

According to Proposition 4.4 and (2.2) we have

$$\begin{aligned}
&\|\langle A\rangle^{-\delta}\chi(H_0)(H-z)^{-1}\chi(H_0)\langle A\rangle^{-\delta}\| \\
&\qquad\lesssim 1 + \|\langle A\rangle^{-\delta}\chi(H_0)(H-z)^{-1}\chi(H_0)\langle A\rangle^{-\delta}\|^{1/2},
\end{aligned}$$

and hence

$$\|\langle A\rangle^{-\delta}\chi(H_0)(H-z)^{-1}\chi(H_0)\langle A\rangle^{-\delta}\| \lesssim 1.$$

Since we have the same estimate for $(H^*-\bar{z})^{-1}$ instead of $(H-z)^{-1}$, we conclude with Corollary 6.3 that $\mathrm{Ran}(\chi(H_0)\langle A\rangle^{-\delta}) \subset \mathcal{H}_{\mathrm{ac}}^*(H)$. Since $\mathcal{H}_{\mathrm{ac}}(H_a)$ is closed in $\mathcal{H}$ by definition, the result follows. ∎

We go back to the Schrödinger operator on the dissipative wave guide discussed in Section 3, see (1.4)–(1.5). In [38] we have proved that for an everywhere effective absorption index the norm of the solution of the Schrödinger equation (1.7) decays exponentially, which implies in particular that $\mathcal{H}_{\mathrm{ac}}(H_a) =$

$L^2(\Omega)$. In general we cannot expect such a fast decay but the result concerning the $L^2_t(\mathbb{R}_+, L^2(\Omega))$ norm remains valid.

PROPOSITION 6.7. *With the notation of Section* 3 *we have* $\mathcal{H}_{\mathrm{ac}}(H_a) = L^2(\Omega)$.

Proof. We recall that $\mathcal{T}$ is the (discrete) set of thresholds. Let $\delta > \frac{1}{2}$. Let $J \subset \mathbb{R}$ be open with $\bar{J} \subset \mathbb{R} \setminus \mathcal{T}$. Then by Theorem 6.6 we have $\mathrm{Ran}(\mathbf{1}_J(H_0)\langle A_x\rangle^{-\delta}) \subset \mathcal{H}_{\mathrm{ac}}(H_a)$. Since H_0 has no eigenvalue, the union of these sets for all suitable J is dense in $L^2(\Omega)$. Since $\mathcal{H}_{\mathrm{ac}}(H_a)$ is closed in $L^2(\Omega)$, the result follows. ∎

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