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DETERMINANTS ASSOCIATED TO TRACES ON OPERATOR BIMODULES

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ABSTRACT. Given a II_1-factor $\mathcal{M}$ with tracial state τ and given an $\mathcal{M}$-bimodule $\mathcal{E}(\mathcal{M},\tau)$ of operators affiliated to $\mathcal{M}$ we show that traces on $\mathcal{E}(\mathcal{M},\tau)$ (namely, linear functionals that are invariant under unitary conjugation) are in bijective correspondence with rearrangement-invariant linear functionals on the corresponding symmetric function space E. We also show that, given a positive trace φ on $\mathcal{E}(\mathcal{M},\tau)$, the map $\det_\varphi : \mathcal{E}_{\log}(\mathcal{M},\tau) \to [0,\infty)$ defined by $\det_\varphi(T) = \exp(\varphi(\log|T|))$ when $\log|T| \in \mathcal{E}(\mathcal{M},\tau)$ and 0 otherwise, is multiplicative on the $*$-algebra $\mathcal{E}_{\log}(\mathcal{M},\tau)$ that consists of all affiliated operators T such that $\log_+(|T|) \in \mathcal{E}(\mathcal{M},\tau)$. Finally, we show that all multiplicative maps on the invertible elements of $\mathcal{E}_{\log}(\mathcal{M},\tau)$ arise in this fashion.

KEYWORDS: *Determinant, von Neumann algebra,* II_1*-factor, noncommutative function space.*

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1. INTRODUCTION

Let $\mathcal{M}$ be a von Neumann algebra factor of type II_1, with tracial state τ. Assume $\mathcal{M}$ has separable predual. The Fuglede–Kadison determinant [8], is the multiplicative map $\Delta_\tau : \mathcal{M} \to [0,\infty)$ defined by

$$(1.1)\qquad \Delta_\tau(T) = \lim_{\varepsilon\to 0^+} \exp(\tau(\log(|T|+\varepsilon))).$$

In this paper, we prove multiplicativity of analogous determinants corresponding to arbitrary positive traces on arbitrary $\mathcal{M}$-bimodules of affiliated operators.

Choose any normal representation of $\mathcal{M}$ on a Hilbert space and let $\mathcal{S}(\mathcal{M},\tau)$ be the $*$-algebra of (possibly unbounded) operators on the Hilbert space affiliated to $\mathcal{M}$. This algebra, often called the Murray–von Neumann algebra of $\mathcal{M}$, is independent of the representation. See, for example, Section 6 of [11] for an exposition of this theory. Let $\mathrm{Proj}(\mathcal{M})$ denote the set of projections (i.e., self-adjoint idempotents) in $\mathcal{M}$. For $A \in \mathcal{S}(\mathcal{M},\tau)$ and $t \in (0,1)$, $\mu(t,A)$ denotes the generalized singular number of A, defined by

$$\mu(t,A) = \inf\{\|A(1-p)\| : p \in \mathrm{Proj}(\mathcal{M}),\ \tau(p) \leqslant t\},$$

where $\|\cdot\|$ is the operator norm. This goes back to Murray and von Neumann; see, for example, Section 2.3 of [14] for some basic theory. We will write simply $\mu(A)$ for the function $t \mapsto \mu(t,A)$, which is nonincreasing and right continuous.

Let E be a complex vector space of measurable functions on $[0,1]$ with the property that if f and g are measurable functions with $f^* \leqslant g^*$ and $g \in E$, then $f \in E$, where f^* denotes the decreasing rearrangement of $|f|$. Following [14], we will call such a space E a Calkin function space. Note that $f \in E$ implies that the dilation $D_2 f$ lies in E, where $D_2 f(t) = f(\frac{t}{2})$. In particular, every nonzero Calkin function space contains $L_\infty[0,1]$. The corresponding $\mathcal{M}$-bimodule $\mathcal{E}(\mathcal{M},\tau)$ is the set of all $A \in \mathcal{S}(\mathcal{M},\tau)$ such that $\mu(A) \in E$. This correspondence, sometimes called the Calkin correspondence in the setting of $(\mathcal{M},\tau)$, is a bijection from the set of all Calkin function spaces onto the set of all operator $\mathcal{M}$-bimodules, by which we mean subspaces of $\mathcal{S}(\mathcal{M},\tau)$ that are closed under left and right multiplication by elements of $\mathcal{M}$, and it goes back to Guido and Isola [9]. See Theorem 2.4.4 of [14] for the formulation used here. An equivalent version of this is also described in [4]. Note that if $\mathcal{A} \subseteq \mathcal{M}$ is any unital abelian von Neumann subalgebra that is diffuse (i.e., has no minimal projections), then the $*$-algebra $\mathcal{S}(\mathcal{A},\tau\restriction_{\mathcal{A}})$ of affiliated operators is naturally embedded in $\mathcal{S}(\mathcal{M},\tau)$ and, upon identifying $\mathcal{A}$ with $L_\infty(0,1)$, the elements of $\mathcal{S}(\mathcal{A},\tau\restriction_{\mathcal{A}})$ are naturally identified with measurable functions on $(0,1)$. Under these identifications, we have $E = \mathcal{S}(\mathcal{A},\tau\restriction_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M},\tau)$.

By a trace on $\mathcal{E}(\mathcal{M},\tau)$, we mean a linear functional φ of $\mathcal{E}(\mathcal{M},\tau)$ such that $\varphi(UAU^*) = \varphi(A)$ for every $A \in \mathcal{E}(\mathcal{M},\tau)$ and every unitary $U \in \mathcal{M}$. A functional φ_0 of E is said to be rearrangement-invariant if $\varphi_0(f) = \varphi_0(g)$ whenever $f,g \in E$, $f,g \geqslant 0$ and $f^* = g^*$.

The difficult half of the following result is essentially proved in [13]. The proof of the other half is similar to the proof of Lemma 9.4 of [6].

THEOREM 1.1. *Let $\mathcal{M}$ be a II_1-factor with separable predual. Let E be a Calkin function space and let $\mathcal{E}(\mathcal{M},\tau)$ be the corresponding $\mathcal{M}$-bimodule. There is a bijection from the set of all traces of $\mathcal{E}(\mathcal{M},\tau)$ onto the set of all rearrangement-invariant functionals of E, whereby a trace φ of $\mathcal{E}(\mathcal{M},\tau)$ is mapped to a functional φ_0 of E satisfying*

$$\varphi_0(\mu(A)) = \varphi(A) \quad \text{whenever } A \in \mathcal{E}(\mathcal{M},\tau) \text{ and } A \geqslant 0. \tag{1.2}$$

Proof. Suppose $\varphi_0 : E \to \mathbb{C}$ is a rearrangement-invariant linear functional. By the proof of (part of) Theorem 5.2 of [13], there is a trace $\varphi : \mathcal{E}(\mathcal{M},\tau) \to \mathbb{C}$ satisfying (1.2). The statement of that theorem includes additional assumptions about E, namely, that it carries a rearrangement-invariant complete norm. However, the proof found in [13] is valid, verbatim, in the more general situation considered here.

Suppose $\varphi : \mathcal{E}(\mathcal{M}, \tau) \to \mathbb{C}$ is a trace. We will now show that for any $A \in \mathcal{E}(\mathcal{M}, \tau)$ that is positive, $\varphi(A)$ depends only on $\mu(A)$. Indeed, let $A_1, A_2 \in \mathcal{E}(\mathcal{M}, \tau)$ be such that $A_1, A_2 \geqslant 0$ and $\mu(A_1) = \mu(A_2)$. Set

$$B_k = \sum_{n \geqslant 0} n 1_{[n,n+1)}(A_k), \quad C_k = A_k - B_k, \quad k = 1, 2.$$

Clearly, positive operators B_1 and B_2 have discrete spectrum and $\mu(B_1) = \mu(B_2)$. Since $\mathcal{M}$ is a factor, one can choose a unitary element $U \in \mathcal{M}$ such that $B_1 = UB_2U^{-1}$. Clearly, $\varphi(B_1) = \varphi(UB_2U^{-1}) = \varphi(B_2)$. By Theorem 2.3 in [7], we have $\varphi\restriction_{\mathcal{M}} = c_\varphi \tau\restriction_{\mathcal{M}}$ for a constant c_φ. For bounded positive operators C_1 and C_2, we have $\mu(C_1) = \mu(C_2)$ and also, therefore,

$$\varphi(C_1) = c_\varphi \tau(C_1) = c_\varphi \tau(C_2) = \varphi(C_2).$$

Thus, we get

$$\varphi(A_1) = \varphi(B_1) + \varphi(C_1) = \varphi(B_2) + \varphi(C_2) = \varphi(A_2).$$

Let $\mathcal{A}$ be any unital, diffuse, abelian von Neumann subalgebra of $\mathcal{M}$. As described above, E is naturally identified with $\mathcal{S}(\mathcal{A}, \tau\restriction_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$, and restricting φ to this subalgebra yields a linear functional φ_0 on E, which is rearrangement-invariant and satisfies (1.2), because of the fact that $\varphi(A)$ depends only on $\mu(A)$ for all $A \geqslant 0$. Using (1.2), we see that the functional φ_0 does not depend on $\mathcal{A}$, namely, does not depend on which copy of E we chose in $\mathcal{E}(\mathcal{M}, \tau)$.

Finally, as φ is uniquely determined by φ_0 and the condition (1.2), we see that the map $\varphi \mapsto \varphi_0$ is the desired bijection. ∎

For convenience, we will use also φ, instead of φ_0, to denote the functional on E corresponding to a trace φ on $\mathcal{E}(\mathcal{M}, \tau)$.

For example, taking E to be the function space L_1 of complex-valued functions on $[0, 1]$ that are integrable with respect to Lebesgue measure, the corresponding bimodule is $\mathcal{L}_1(\mathcal{M}, \tau)$. Moreover, the functional $f \mapsto \int_0^1 f(t)\,\mathrm{d}t$ on L_1 corresponds to the usual trace τ on $\mathcal{L}_1(\mathcal{M}, \tau)$. Other examples of traces on bimodules are provided by the Dixmier traces on Marcinkiewicz bimodules, which are of interest in noncommutative geometry. See, for example, [2], [3] and [12]; particularly, consider the treatment of functionals supported at zero, but adapted to the case of a II_1-factor $\mathcal{M}$, namely, corresponding to function spaces on $[0, 1]$. A specific case (essentially, taken from [3]) is found in Example 3.3.

The Fuglede–Kadison determinant mentioned at the start of this introduction is actually naturally defined on the space, often denoted $\mathcal{L}_{\log}(\mathcal{M}, \tau)$, of all $T \in \mathcal{S}(\mathcal{M}, \tau)$ such that $\log_+(|T|) \in \mathcal{L}_1(\mathcal{M}, \tau)$, where $\log_+(t) = \max(\log(t), 0)$.

See [10] for a development of Δ_τ in this generality, including a proof of multiplicativity.

In the rest of this paper, we will for the most part consider only *positive* traces φ, namely, those satisfying

$$A \geqslant 0 \Rightarrow \varphi(A) \geqslant 0$$

(the exception being Lemma 2.8). Positive traces correspond, under the rubric of Theorem 1.1, to positive rearrangement-invariant linear functionals. In the following, we use the function $\log_-(t) = -\min(\log(t), 0)$; thus, $\log = \log_+ - \log_-$.

DEFINITION 1.2. Let $\mathcal{M}$ be a II_1-factor and consider a positive trace φ on an $\mathcal{M}$-bimodule $\mathcal{E}(\mathcal{M},\tau)$. Let $\mathcal{E}_{\log}(\mathcal{M},\tau)$ be the set of all $T \in \mathcal{S}(\mathcal{M},\tau)$ such that $\log_+(|T|) \in \mathcal{E}(\mathcal{M},\tau)$ and for such T let

$$\det_\varphi(T) = \begin{cases} \exp(\varphi(\log(|T|))) & \ker T = \{0\} \text{ and } \log_-(|T|) \in E, \\ 0 & \ker T = \{0\} \text{ and } \log_-(|T|) \notin E, \\ 0 & \ker T \neq \{0\}. \end{cases}$$

Thus, in the case $E = L_1$ and $\varphi = \tau$, we have the Fuglede–Kadison determinant: $\det_\tau = \Delta_\tau$. The natural domain of this determinant by the above rubric should be written $\mathcal{L}_{1,\log}(\mathcal{M},\tau)$, but we will write $\mathcal{L}_{\log}(\mathcal{M},\tau)$ for this, in keeping with earlier convention (cf. [5], [6]).

The main result of this paper is the following.

THEOREM 1.3. *For an arbitrary Calkin function space E on $[0,1]$ and arbitrary positive trace φ on the corresponding bimodule $\mathcal{E}(\mathcal{M},\tau)$, the set $\mathcal{E}_{\log}(\mathcal{M},\tau)$ is a $*$-subalgebra of $\mathcal{S}(\mathcal{M},\tau)$ and, if $A, B \in \mathcal{E}_{\log}(\mathcal{M},\tau)$, then*

$$\det_\varphi(AB) = \det_\varphi(A)\det_\varphi(B). \tag{1.3}$$

The proof, presented in the next section, relies on Fuglede and Kadison's result [8] that Δ_τ is multiplicative on $\mathcal{M}$ and on the characterization from [4] of sums of $(\mathcal{E}(\mathcal{M},\tau)),\mathcal{M})$-commutators. Thus, a special case of this proof yields an alternative proof of Haagerup and Schultz's result [10] about the extension of the Fuglede–Kadison determinant to $\mathcal{L}_{\log}(\mathcal{M},\tau)$.

REMARK 1.4. It is immediate that $\det_\varphi(1) = 1$ and, for $T \in \mathcal{E}_{\log}(\mathcal{M},\tau)$, $\det_\varphi(T) = 0$ if and only if T fails to be invertible in $\mathcal{E}_{\log}(\mathcal{M},\tau)$.

REMARK 1.5. In the case that $\varphi = 0$, we clearly have, for $T \in \mathcal{E}_{\log}(\mathcal{M},\tau)$,

$$\det_\varphi(T) = \begin{cases} 1 & \text{if } T \text{ is invertible in } \mathcal{E}_{\log}(\mathcal{M},\tau), \\ 0 & \text{otherwise.} \end{cases}$$

However, if $\varphi \neq 0$, then $\det_\varphi$ is onto $[0,\infty)$.

REMARK 1.6. It is not difficult to see, in the case $\varphi = \tau$, that Definition 1.2 agrees with the definition by equation (1.1), in fact even for all $T \in \mathcal{L}_{\log}(\mathcal{M},\tau)$.

However, the analoguous statement is not true for general traces φ. In fact, it obviously fails when $\varphi = 0$, (see Remark 1.5, above). See Example 3.3 for specific examples of this failure when $\varphi \neq 0$.

We are grateful to Amudhan Krishnaswamy-Usha for asking us a question that led to the next result.

PROPOSITION 1.7. *For an arbitrary Calkin function space E on $[0,1]$ and an arbitrary map*

$$m : \mathcal{E}_{\log}(\mathcal{M},\tau) \to [0,\infty)$$

that is multiplicative, order-preserving and nonzero, there exists a positive trace φ on $\mathcal{E}(\mathcal{M},\tau)$ such that $m(X) = \det_\varphi(X)$ for every invertible element X in $\mathcal{E}_{\log}(\mathcal{M},\tau)$.

We will show (in Proposition 3.2) that we cannot hope for m to agree with $\det_\varphi$ on all of $\mathcal{E}_{\log}(\mathcal{M},\tau)$.

The proofs of Theorem 1.3 and Proposition 3.2 are contained in the next two sections.

2. PROOF OF THEOREM 1.3

Let us begin by describing some further notation and standard conventions.

(i) $S(0,1)$ will denote the set of all complex-valued Borel measurable functions on $[0,1]$ and L_∞ will denote the set of all essentially bounded elements of $S(0,1)$. As usual, we consider functions that are equal almost everywhere to be the same.

(ii) We will apply the Borel functional calculus to self-adjoint elements $T \in \mathcal{S}(\mathcal{M},\tau)$, and will also use the standard notation $T_+ = \max(T,0)$ and $T_- = -\min(T,0)$.

(iii) For self-adjoint $A \in \mathcal{S}(\mathcal{M},\tau)$, we consider its eigenvalue function (or spectral scale), defined for $t \in (0,1)$ by

$$\lambda(t,A) = \inf\{s \in \mathbb{R} : \tau(1_{(s,\infty)}(A)) \leqslant t\},$$

where, in accordance with notation for Borel functional calculus, $1_{(s,\infty)}(A)$ denotes the spectral projection of A associated to the interval (s,∞). This also goes back to Murray and von Neumann. We will write simply $\lambda(A)$ for the function $t \mapsto \lambda(t,A)$, which is nonincreasing and right continuous. Note that, if $A \geqslant 0$, then $\lambda(A) = \mu(A)$. Moreover, when $a \leqslant b$, with $a \leqslant \lim_{t\to 0} \lambda(t,A)$ and $b \geqslant \lim_{t\to 1} \lambda(t,A)$, we have

$$\tau(A\,1_{[a,b]}(A)) = \int_c^d \lambda(t,A)\,\mathrm{d}t, \tag{2.1}$$

$$\tau(1_{[a,b]}(A)) = d - c, \tag{2.2}$$

where

$$c = \inf\{s : \lambda(s, A) \leqslant b\}, \quad d = \sup\{s : \lambda(s, A) \geqslant a\}.$$

For any $T \in \mathcal{S}(\mathcal{M}, \tau)$, since $\mu(T) = \mu(|T|) = \lambda(|T|)$, from (2.2), we get

$$\tau(1_{[0,\mu(t,T)]}(|T|)) \geqslant 1 - t. \tag{2.3}$$

(iv) The following inequalities are standard (see, for example, Corollary 2.3.16 of [14]): for all $A, B \in \mathcal{S}(\mathcal{M}, \tau)$, if $s, t > 0$ and $s + t < 1$, then

$$\mu(s + t, A + B) \leqslant \mu(s, A) + \mu(t, B), \tag{2.4}$$

$$\mu(s + t, AB) \leqslant \mu(s, A)\mu(t, B). \tag{2.5}$$

(v) If a function f on $(0, 1)$ is right-continuous and monotone, then we will let $\widetilde{f}$ denote left-continuous version, namely,

$$\widetilde{f}(x) = \lim_{t \to x^-} f(t). \tag{2.6}$$

LEMMA 2.1. *Let $T, S \in \mathcal{S}(\mathcal{M}, \tau)$ be self-adjoint. Then for every $t \in (0, \frac{1}{4})$, we have*

$$\Big| \int_{2t}^{1-2t} \Big(\log(\mu(u, e^T e^S)) - \lambda(u, T) - \lambda(u, S) \Big) \, du \Big| \leqslant 8t(\mu(t, T) + \mu(t, S)).$$

Proof. Fix $t \in (0, \frac{1}{4})$ and, using the continuous functional calculus, set

$$T_0 = \min\{T_+, \mu(t, T)\} - \min\{T_-, \mu(t, T)\},$$
$$S_0 = \min\{S_+, \mu(t, S)\} - \min\{S_-, \mu(t, S)\}.$$

We have

$$T - T_0 = (T_+ - \mu(t, T))_+ - (T_- - \mu(t, T))_+,$$
$$|T - T_0| = (T_+ - \mu(t, T))_+ + (T_- - \mu(t, T))_+ = (|T| - \mu(t, T))_+.$$

Thus, we have $(T - T_0)1_{[0,\mu(t,T)]}(|T|) = 0$ and, using (2.3), we get $\mu(t, T - T_0) = 0$; similarly, we have $\mu(t, S - S_0) = 0$. Using (2.5), for every $u \in (2t, 1)$ we have

$$\mu(u, e^T e^S) = \mu(u, e^{T-T_0} \cdot e^{T_0} e^{S_0} \cdot e^{S-S_0}) \leqslant \mu(t, e^{T-T_0})\mu(u - 2t, e^{T_0} e^{S_0})\mu(t, e^{S-S_0}),$$
$$\mu(u, e^{T_0} e^{S_0}) = \mu(u, e^{T_0-T} \cdot e^T e^S \cdot e^{S_0-S}) \leqslant \mu(t, e^{T_0-T})\mu(u - 2t, e^T e^S)\mu(t, e^{S_0-S}),$$

Since $\mu(t, e^{T-T_0}) \leqslant 1$ and $\mu(t, e^{T_0-T}) \leqslant 1$ and similarly for $S - S_0$, we get

$$\mu(u, e^T e^S) \leqslant \mu(u - 2t, e^{T_0} e^{S_0}), \quad \mu(u, e^{T_0} e^{S_0}) \leqslant \mu(u - 2t, e^T e^S).$$

Thus, for $u \in (2t, 1 - 2t)$, we have

$$\mu(u + 2t, e^{T_0} e^{S_0}) \leqslant \mu(u, e^T e^S) \leqslant \mu(u - 2t, e^{T_0} e^{S_0}).$$

It follows that

$$\int_{4t}^{1} \log(\mu(u, e^{T_0} e^{S_0})) du \leqslant \int_{2t}^{1-2t} \log(\mu(u, e^T e^S)) du \leqslant \int_{0}^{1-4t} \log(\mu(u, e^{T_0} e^{S_0})) du. \tag{2.7}$$

Since $-\mu(t,T) \leqslant T_0 \leqslant \mu(t,T)$ and similarly for S_0, we also have

$$\mathrm{e}^{-\mu(t,T)-\mu(t,S)} \leqslant \mu(\mathrm{e}^{T_0}\mathrm{e}^{S_0}) \leqslant \mathrm{e}^{\mu(t,T)+\mu(t,S)}.$$

Thus,

$$\|\log(\mu(\mathrm{e}^{T_0}\mathrm{e}^{S_0}))\|_\infty \leqslant \mu(t,T)+\mu(t,S).$$

In particular,

$$\Big|\int_0^{4t} \log(\mu(u,\mathrm{e}^{T_0}\mathrm{e}^{S_0}))\,\mathrm{d}u\Big| \leqslant 4t\|\log(\mu(\mathrm{e}^{T_0}\mathrm{e}^{S_0}))\|_\infty \leqslant 4t(\mu(t,T)+\mu(t,S)),$$

$$\Big|\int_{1-4t}^{1} \log(\mu(u,\mathrm{e}^{T_0}\mathrm{e}^{S_0}))\,\mathrm{d}u\Big| \leqslant 4t\|\log(\mu(\mathrm{e}^{T_0}\mathrm{e}^{S_0}))\|_\infty \leqslant 4t(\mu(t,T)+\mu(t,S)).$$

Using (2.7), we get

$$\Big|\int_{2t}^{1-2t} \log(\mu(u,\mathrm{e}^{T}\mathrm{e}^{S}))\,\mathrm{d}u - \int_0^{1} \log(\mu(u,\mathrm{e}^{T_0}\mathrm{e}^{S_0}))\,\mathrm{d}u\Big| \leqslant 4t(\mu(t,T)+\mu(t,S)).$$

Since the Fuglede–Kadison determinant Δ_τ is multiplicative on $\mathcal{M}$, we have

$$\int_0^1 \log(\mu(u,\mathrm{e}^{T_0}\mathrm{e}^{S_0}))\,\mathrm{d}u = \log(\Delta_\tau(\mathrm{e}^{T_0}\mathrm{e}^{S_0}))$$
$$= \log(\Delta_\tau(\mathrm{e}^{T_0})) + \log(\Delta_\tau(\mathrm{e}^{S_0})) = \tau(T_0)+\tau(S_0).$$

But using

$$\Big|\tau(T_0) - \int_{2t}^{1-2t} \lambda(u,T)\,\mathrm{d}u\Big| \leqslant 4t\mu(t,T),$$

and the same also for S, the assertion follows. ∎

In the following, we use the notation (2.6) for the left-continuous versions of monotone functions. (Though, as elements of E, $\mu(T)$ and the left-continuous version $\widetilde{\mu}(T)$ are identified, these functions $\mu(T)$ and similarly $\lambda(T)$ are of interest aside from their membership in E, and for correctness at all points of $(0,1)$ we must use their left-continuous versions in the following inequalities and elsewhere below.)

LEMMA 2.2. *If $S,T \in \mathcal{S}(\mathcal{M},\tau)$ are self-adjoint, then for all $u \in (0,1)$, we have*

$$-\widetilde{\mu}(\tfrac{1-u}{2},T) - \widetilde{\mu}(\tfrac{1-u}{2},S) \leqslant \log(\mu(u,\mathrm{e}^T\mathrm{e}^S)) \leqslant \mu(\tfrac{u}{2},T)+\mu(\tfrac{u}{2},S). \tag{2.8}$$

Proof. Using (2.5), we get

$$\mu(u,\mathrm{e}^T\mathrm{e}^S) \leqslant \mu(\tfrac{u}{2},\mathrm{e}^T)\mu(\tfrac{u}{2},\mathrm{e}^S) \leqslant \mu(\tfrac{u}{2},\mathrm{e}^{T_+})\mu(\tfrac{u}{2},\mathrm{e}^{S_+}) = \mathrm{e}^{\mu(\frac{u}{2},T_+)+\mu(\frac{u}{2},S_+)} \tag{2.9}$$
$$\leqslant \mathrm{e}^{\mu(\frac{u}{2},T)+\mu(\frac{u}{2},S)},$$

which yields the right-most inequality in (2.8). Replacing S with $-T$ and T with $-S$ in (2.9), we get

$$\mu(u, e^{-S}e^{-T}) \leqslant e^{\mu(\frac{u}{2}, T_-)+\mu(\frac{u}{2}, S_-)}, \quad \widetilde{\mu}(u, e^{-S}e^{-T}) \leqslant e^{\widetilde{\mu}(\frac{u}{2}, T_-)+\widetilde{\mu}(\frac{u}{2}, S_-)}. \tag{2.10}$$

As is well known and easy to show,

$$\mu(u, e^{T}e^{S}) = \frac{1}{\widetilde{\mu}(1-u, e^{-S}e^{-T})}.$$

Thus, replacing u with $1-u$ in (2.10), we get

$$\mu(u, e^{T}e^{S}) \geqslant e^{-\widetilde{\mu}(\frac{1-u}{2}, T_-)-\widetilde{\mu}(\frac{1-u}{2}, S_-)} \geqslant e^{-\widetilde{\mu}(\frac{1-u}{2}, T)-\widetilde{\mu}(\frac{1-u}{2}, S)},$$

which yields the left-most inequality in (2.8). ∎

The next lemma is a combination of Theorems 3.3.3 and 3.3.4 from [14].

LEMMA 2.3. *If $S, T \in \mathcal{M}$ are positive, then*

$$\int_0^t \mu(u, T+S)\,du \leqslant \int_0^t (\mu(u,T)+\mu(u,S))\,du \leqslant \int_0^{2t} \mu(u, T+S)\,du.$$

Proof. This follows easily from the fact that, for a positive operator, T, we have

$$\int_0^t \mu(u,T)\,du = \sup\{\tau(pT) : p \in \mathrm{Proj}(\mathcal{M}),\ \tau(p) \leqslant t\}. \quad ∎$$

For every function $f \in S(0,1)$ that is bounded on compact subsets of $(0,1)$, define

$$(\Psi f)(t) = \begin{cases} \frac{1}{t}\int_t^{1-t} f(s)\,ds & 0 < t < \frac{1}{2}, \\ 0 & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Clearly, Ψf is continuous on $(0,1]$ and Ψ is linear. Note that Ψ is defined on every function arising as $\mu(A)$ or $\lambda(A)$ for $A \in \mathcal{S}(\mathcal{M},\tau)$.

LEMMA 2.4. *Let $S, T \in \mathcal{E}(\mathcal{M},\tau)$ be positive. Then*

$$\Psi(\mu(T+S)-\mu(T)-\mu(S)) \in E.$$

Proof. First suppose $S, T \in \mathcal{M}$ are positive. From Lemma 2.3 and the fact that $\tau(T) = \int_0^1 \mu(u,T)\,du$, we have

$$\int_{2t}^1 \mu(u, T+S)\,du \leqslant \int_t^1 (\mu(u,T)+\mu(u,S))\,du \leqslant \int_t^1 \mu(u, T+S)\,du. \tag{2.11}$$

For arbitrary positive $S, T \in \mathcal{S}(\mathcal{M}, \tau)$, set $T_n = \min\{T, n\}$ and $S_n = \min\{S, n\}$. Since $\mu(T_n) \uparrow \mu(T)$, $\mu(S_n) \uparrow \mu(S)$ and $\mu(T_n + S_n) \uparrow \mu(T + S)$, it follows from the monotone convergence principle that (2.11) also holds. From (2.11), we have

$$\Big|\int_t^1 (\mu(u, T+S) - \mu(u, T) - \mu(u, S))\, \mathrm{d}u\Big| \leqslant \int_t^{2t} \mu(u, T+S)\, \mathrm{d}u \leqslant t\mu(t, T+S).$$

Thus, for $t \in (0, \frac{1}{2})$, we have

$$\begin{aligned} g\Big| &\int_t^{1-t} (\mu(u, T+S) - \mu(u, T) - \mu(u, S))\, \mathrm{d}u\, g\Big| \\ &\leqslant g\Big|\int_t^1 (\mu(u, T+S) - \mu(u, T) - \mu(u, S))\, \mathrm{d}u\, g\Big| \\ &\qquad + g\Big|\int_{1-t}^1 (\mu(u, T+S) - \mu(u, T) - \mu(u, S))\, \mathrm{d}u\, g\Big| \\ &\leqslant t\mu(t, T+S) + t\mu(1-t, T+S) + t\mu(1-t, T) + t\mu(1-t, S) \leqslant 4t\mu(t, T+S). \end{aligned}$$

This concludes the proof. ∎

LEMMA 2.5. *Let $T \in \mathcal{S}(\mathcal{M}, \tau)$ be self-adjoint. Then*

$$\Psi(\lambda(T) - \mu(T_+) + \mu(T_-)) \in L_\infty.$$

Proof. If $T_+ = 0$ or $T_- = 0$, then $\lambda(T) = \mu(T_+) - \mu(T_-)$. Suppose $T_+ \neq 0$ and $T_- \neq 0$. Let t_0 be the trace of the support projection of T_+. We have

$$\lambda(u, T) = \begin{cases} \mu(u, T_+) & u \in (0, t_0), \\ -\widetilde{\mu}(1-u, T_-) & u \in [t_0, 1). \end{cases}$$

It follows that, for all sufficiently small t, we have

$$\begin{aligned} t(\Psi\lambda(T))(t) &= \int_t^{t_0} \lambda(u, T)\mathrm{d}u + \int_{t_0}^{1-t} \lambda(u, T)\mathrm{d}u = \int_t^{t_0} \mu(u, T_+)\mathrm{d}u - \int_{t_0}^{1-t} \mu(1-u, T_-)\mathrm{d}u \\ &= \int_t^{t_0} \mu(u, T_+)\, \mathrm{d}u - \int_t^{1-t_0} \mu(u, T_-)\, \mathrm{d}u \\ &= \int_t^1 (\mu(u, T_+) - \mu(u, T_-))\, \mathrm{d}u = t(\Psi(\mu(T_+) - \mu(T_-)))(t), \end{aligned}$$

where the last equality holds because the integrand is zero when u is sufficiently close to 1. Thus, $\Psi(\lambda(T) - \mu(T_+) + \mu(T_-))(t)$ vanishes for all t sufficiently small. Since this function is continuous on $(0, 1]$, it is bounded. ∎

LEMMA 2.6. *Let $S, T \in \mathcal{E}(\mathcal{M}, \tau)$ be self-adjoint. Then*

$$\Psi(\lambda(T) + \lambda(S) - \lambda(T+S)) \in E.$$

Proof. We have

$$(T+S)_+ - (T+S)_- = T_+ - T_- + S_+ - S_-.$$

Therefore,

$$(T+S)_+ + T_- + S_- = (T+S)_- + T_+ + S_+.$$

Denote the above quantity by A. From Lemma 2.4, we obtain

$$\Psi(\mu(A) - \mu((T+S)_+) - \mu(T_-) - \mu(S_-)) \in E,$$
$$\Psi(\mu(A) - \mu((T+S)_-) - \mu(T_+) - \mu(S_+)) \in E.$$

Subtracting those formulae, we obtain

$$\Psi(\mu((T+S)_+) - \mu((T+S)_-) - \mu(T_+) + \mu(T_-) - \mu(S_+) + \mu(S_-)) \in E.$$

The assertion follows now from Lemma 2.5 as applied to the operators T, S and $T+S$, and the fact that E contains L_∞. ∎

In the next result, the notation $[\mathcal{E}(\mathcal{M}, \tau), \mathcal{M}]$ denotes the space spanned by the set of all commutators of the form $[S, T] = ST - TS$, for $S \in \mathcal{M}$ and $T \in \mathcal{E}(\mathcal{M}, \tau)$. It amounts to a reformulation of a special case of Theorem 4.6 in [4].

THEOREM 2.7. *Let $T \in \mathcal{E}(\mathcal{M}, \tau)$ be self-adjoint. Then $T \in [\mathcal{E}(\mathcal{M}, \tau), \mathcal{M}]$ if and only if $\Psi\lambda(T) \in E$.*

Proof. By Theorem 4.6 of [4], $T \in [\mathcal{E}(\mathcal{M}, \tau), \mathcal{M}]$ if and only if the function

$$r \mapsto \frac{1}{r}\tau(1_{[0,\mu(r,T))]}(|T|)T)$$

belongs to E. Thus, it will suffice to show that the function

$$r \mapsto \frac{1}{r}\tau(1_{[0,\mu(r,T))]}(|T|)T) - \Psi\lambda(T)(r) \tag{2.12}$$

belongs to E. First suppose $T_- = 0$. Then, using $\lambda(T) = \mu(T)$ and (2.1), we have

$$\tau(1_{[0,\mu(r,T))]}(|T|)T) = \int_{r'}^{1} \mu(t, T)\,\mathrm{d}t,$$

where $r' = \inf\{s : \mu(s, T) \leqslant \mu(r, T)\}$. Thus $r' \leqslant r$ and, for $0 < r < \frac{1}{2}$,

$$\left|\tau(1_{[0,\mu(r,T))]}(|T|)T) - \int_{r}^{1-r} \lambda(t, T)\,\mathrm{d}t\right| \leqslant (r - r')\mu(r, T) + r\mu(1-r, T) \leqslant 2r\mu(r, T),$$

which implies that the function (2.12) belongs to E.

If $T_+ = 0$, then we may of course replace T by $-T$ and we are done.

Suppose $T_+ \neq 0$ and $T_- \neq 0$. Letting, $t_0 = \inf\{t : \lambda(t, T_+) \geqslant 0\}$, we have $0 < t_0 < 1$ and

$$\lambda(t,T) = \begin{cases} \mu(t,T_+) & 0 < t < t_0, \\ \widetilde{\mu}(1-t,T_-) & t_0 \leqslant t < 1. \end{cases}$$

For $r < t_0$, we have

$$\begin{aligned} \tau(1_{[0,\mu(r,T)]}(|T|)T) &= \tau(1_{[-\mu(r,T),\mu(r,T)]}(T)T) \\ &= \tau(1_{[0,\mu(r,T)]}(T_+)T_+) - \tau(1_{[0,\mu(r,T)]}(T_-)T_-) \\ &= \int_{r'}^{t_0} \lambda(t,T)\,\mathrm{d}t + \int_{t_0}^{1-r''} \lambda(t,T)\,\mathrm{d}t, \end{aligned}$$

where

$$r' = \inf\{s : \mu(s,T_+) \leqslant \mu(r,T)\} \tag{2.13}$$

$$r'' = \inf\{s : \mu(s,T_-) \leqslant \mu(r,T)\}. \tag{2.14}$$

Since $\mu(r,T_\pm) \leqslant \mu(r,T)$, we have $r', r'' \leqslant r$. Thus, we have

$$\begin{aligned} \Big|\tau(1_{[0,\mu(r,T)]}(|T|)T) - \int_r^{1-r} \lambda(t,T)\,\mathrm{d}t\Big| &= \Big|\int_{r'}^{r} \lambda(t,T)\,\mathrm{d}t + \int_{1-r}^{1-r''} \lambda(t,T)\,\mathrm{d}t\Big| \\ &\leqslant \int_{r'}^{r} \mu(t,T_+)\,\mathrm{d}t + \int_{r''}^{r} \mu(t,T_-)\,\mathrm{d}t \\ &\leqslant (r-r')\mu(r',T_+) + (r-r'')\mu(r'',T_-) \leqslant 2r\mu(r,T), \end{aligned}$$

where for the last inequality we used (2.13)–(2.14). This shows that the function (2.12) belongs to E and, thus, completes the proof. ∎

LEMMA 2.8. *Let $\varphi : \mathcal{E}(\mathcal{M},\tau) \to \mathbb{C}$ be a trace. If $T \in \mathcal{E}(\mathcal{M},\tau)$ is self-adjoint and is such that $\Psi\lambda(T) \in E$, then $\varphi(T) = 0$.*

Proof. It follows from Theorem 2.7 that $T \in [\mathcal{E}(\mathcal{M},\tau), \mathcal{M}]$. Since φ is a trace, it follows that $\varphi(T) = 0$. ∎

Proof of Theorem 1.3. For $A \in \mathcal{S}(\mathcal{M},\tau)$, we have that $A \in \mathcal{E}_{\log}(\mathcal{M},\tau)$ if and only if $\log_+ \mu(A) \in E$, and this is, in turn, equivalent to $\log(1+\mu(A)) \in E$. Using the basic equalities (2.4)–(2.5), we easily see that for $A, B \in \mathcal{E}_{\log}(\mathcal{M},\tau)$, we have

$$\log(1+\mu(A+B)) \leqslant \log(1+D_2\mu(A)+D_2\mu(B)) \leqslant \log((1+D_2\mu(A))(1+D_2\mu(B))),$$
$$\log(1+\mu(AB)) \leqslant \log(1+D_2\mu(A)D_2\mu(B)) \leqslant \log((1+D_2\mu(A))(1+D_2\mu(B))),$$

where $(D_2 f)(t) = f(\frac{t}{2})$. But since $\log(1+D_2\mu(A)) + \log(1+D_2\mu(B)) \in E$, these imply that $A+B$ and AB belong to $\mathcal{E}_{\log}(\mathcal{M},\tau)$. From this, one easily sees that $\mathcal{E}_{\log}(\mathcal{M},\tau)$ is a $*$-subalgebra of $\mathcal{S}(\mathcal{M},\tau)$.

It remains to show that $\det_\varphi$ is multiplicative. Letting $A,B\in\mathcal{E}_{\log}(\mathcal{M},\tau)$, we will show (1.3). We may, without loss of generality, assume $A,B\geqslant 0$. Indeed, we have $\mu(AB)=\mu(|A||B^*|)$. Thus, if the assertion holds for positive operators, then we will have

$$\det_\varphi(AB)=\det_\varphi(|A||B^*|)=\det_\varphi(|A|)\det_\varphi(|B^*|)=\det_\varphi(A)\det_\varphi(B).$$

Suppose first that $\log(A),\log(B)\in\mathcal{E}(\mathcal{M},\tau)$. Denote, for brevity, $T=\log(A)$ and $S=\log(B)$. It follows from Lemma 2.2 that $\log(|AB|)\in E$.

Using Lemma 2.1 and replacing t with $\frac{1}{2}t$, for all $t\in(0,\frac{1}{2})$, we get

$$\Big|\int_t^{1-t}(\log(\mu(u,\mathrm{e}^T\mathrm{e}^S))-\lambda(u,T)-\lambda(u,S))\,\mathrm{d}u\Big|\leqslant 4t(\mu(\tfrac{t}{2},T)+\mu(\tfrac{t}{2},S)).$$

In particular, we have

$$\Psi(\log(\mu(\mathrm{e}^T\mathrm{e}^S))-\lambda(T)-\lambda(S))\in E.$$

It follows from Lemma 2.6 that

$$\Psi(\lambda(\log(|\mathrm{e}^T\mathrm{e}^S|)-T-S))\in E.$$

Using Lemma 2.8, we conclude that

$$\varphi(\log(|\mathrm{e}^T\mathrm{e}^S|)-T-S)=0.$$

This implies (1.3) for our A,B.

If B has a nonzero kernel, then so does AB and (1.3) holds.

Suppose now that $\ker B$ is zero but $\log_-(B)\notin E$. Then, of course, we have $\lim_{t\to 1}\mu(t,B)=0$. If $\ker AB\neq\{0\}$, then (1.3) holds, so suppose $\ker AB=\{0\}$. We have, from (2.5), for all $t\in(0,\frac{1}{2})$,

$$\mu(1-t,AB)\leqslant\mu(t,A)\mu(1-2t,B)$$

and, thus,

$$\log(\mu(1-t,AB))\leqslant\log(\mu(t,A))+\log(\mu(1-2t,B)).$$

So, for sufficiently small $t>0$,

$$\begin{aligned}\log_-\mu(1-t,AB)+\log_+\mu(t,A)&\geqslant-\log\mu(1-t,AB)+\log\mu(t,A)\\&\geqslant-\log\mu(1-2t,B)=\log_-\mu(1-2t,B).\end{aligned}$$

Since the function $t\mapsto\log_-\mu(1-2t,B)$ is not in E, while the function $t\mapsto\log_+\mu(t,A)$ does belong to E, we conclude that the function $t\mapsto\log_-\mu(1-t,AB)$ does not belong to E. Therefore, the function $\log_-(\mu(AB))$ does not belong to E and both left- and right-hand sides of (1.3) are zero. This concludes the proof of (1.3) in the degenerate case. ∎

3. PROOF OF PROPOSITION 1.7 AND SOME EXAMPLES

LEMMA 3.1. *Let $m : \mathcal{E}_{\log}(\mathcal{M}, \tau) \to \mathbb{R}$ be multiplicative and order-preserving. Then for every $T \in \mathcal{E}_{\log}(\mathcal{M}, \tau)$, $m(T)$ depends only on $\mu(T)$.*

Proof. We may without loss of generality assume m is not identically zero. Thus, $m(1) = 1$. By Theorem 1 of [1], every unitary element is a product of multiplicative commutators of unitaries (in fact, of symmetries) and it follows that m sends the entire unitary group of $\mathcal{M}$ to 1. Thus, by employing the polar decomposition, we have

$$\forall\, T \in \mathcal{E}_{\log}(\mathcal{M}, \tau), \quad m(T) = m(|T|).$$

It, therefore, suffices to prove the assertion for positive operators.

Let $0 \leqslant T, S \in \mathcal{E}_{\log}(\mathcal{M}, \tau)$ be such that $\mu(T) = \mu(S)$. Set

$$T_\varepsilon = \sum_{n\in\mathbb{Z}} (1+\varepsilon)^n 1_{((1+\varepsilon)^n,(1+\varepsilon)^{n+1})}(T), \quad S_\varepsilon = \sum_{n\in\mathbb{Z}} (1+\varepsilon)^n 1_{((1+\varepsilon)^n,(1+\varepsilon)^{n+1})}(S).$$

For a given n, positive operators T_ε and S_ε have discrete spectrum and $\mu(T_\varepsilon) = \mu(S_\varepsilon)$. Since $\mathcal{M}$ is a factor, one can choose a unitary operator $U_\varepsilon \in \mathcal{M}$ such that $S_\varepsilon = U_\varepsilon T_\varepsilon U_\varepsilon^{-1}$. Thus,

$$m(S_\varepsilon) = m(U_\varepsilon T_\varepsilon U_\varepsilon^{-1}) = m(U_\varepsilon) m(T_\varepsilon) m(U_\varepsilon)^{-1} = m(T_\varepsilon).$$

Clearly,

$$S_\varepsilon \leqslant S \leqslant (1+\varepsilon) S_\varepsilon, \quad T_\varepsilon \leqslant T \leqslant (1+\varepsilon) T_\varepsilon.$$

Since m is order preserving, it follows that

$$m(S) \leqslant m(1+\varepsilon) m(S_\varepsilon) = m(1+\varepsilon) m(T_\varepsilon) \leqslant m(1+\varepsilon) m(T).$$

Since m is order preserving, it follows that $m(1+\varepsilon) \searrow 1$ as $\varepsilon \searrow 0$. Passing $\varepsilon \to 0$, we obtain $m(S) \leqslant m(T)$. Similarly, $m(T) \leqslant m(S)$. Thus, $m(S) = m(T)$ and the proof is complete. ∎

Proof of Proposition 1.7. Since the map m is multiplicative and not identically zero, we must have $m(1) = 1$. By Lemma 3.1, $m(T)$ depends only on $\mu(T)$ for all $T \in \mathcal{E}_{\log}(\mathcal{M}, \tau)$.

Let $\mathcal{A}$ be any unital, diffuse, abelian von Neumann subalgebra of $\mathcal{M}$. As in the proof of Theorem 1.3, E is naturally identified with $\mathcal{S}(\mathcal{A}, \tau\restriction_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$. Given real-valued $f \in E$, let $T \in \mathcal{S}(\mathcal{A}, \tau\restriction_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$ be the corresponding self-adjoint operator. Note that e^T is an invertible element of $\mathcal{E}_{\log}(\mathcal{M}, \tau)$ and, thus, $m(\mathrm{e}^T) > 0$. We define

$$\varphi_0(f) = \log m(\mathrm{e}^T). \tag{3.1}$$

We will show that φ_0 is $\mathbb{R}$-linear. First, given $f_1, f_2 \in E$ and the corresponding self-adjoint $T_1, T_2 \in \mathcal{S}(\mathcal{A}, \tau\restriction_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$, since T_1 and T_2 commute, we have

$$\varphi_0(f_1+f_2) = \log m(\mathrm{e}^{T_1+T_2}) = \log m(\mathrm{e}^{T_1}\mathrm{e}^{T_2}) = \log(m(\mathrm{e}^{T_1}) m(\mathrm{e}^{T_2})) = \varphi_0(f_1) + \varphi_0(f_2),$$

i.e., φ_0 preserves addition. From this, we easily see that $\varphi_0(rf) = r\varphi_0(f)$ for every rational number r and real-valued $f \in E$. This last fact is, of course, equivalent to

$$m(\mathrm{e}^{rT}) = m(\mathrm{e}^{T})^r \tag{3.2}$$

for every self-adjoint $T \in \mathcal{S}(\mathcal{A}, \tau\restriction_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$ and every rational number r. When $T \geqslant 0$, using the order-preserving property of m, we obtain from this that (3.2) holds for every $r \in \mathbb{R}$, and similarly when $T \leqslant 0$. For arbitrary self-adjoint $T \in \mathcal{S}(\mathcal{A}, \tau\restriction_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$, writing $T = T_+ - T_-$ for T_+ and T_- positive elements of $\mathcal{S}(\mathcal{A}, \tau\restriction_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$, in the usual way, we get, for all $r \in \mathbb{R}$,

$$\begin{aligned} m(\mathrm{e}^{rT}) &= m(\mathrm{e}^{rT_+ + (-r)T_-}) = m(\mathrm{e}^{rT_+})m(\mathrm{e}^{(-r)T_-}) = m(\mathrm{e}^{T_+})^r m(\mathrm{e}^{T_-})^{-r} \\ &= (m(\mathrm{e}^{T_+}\mathrm{e}^{-T_-}))^r = m(\mathrm{e}^{T})^r. \end{aligned}$$

Thus (3.2) holds for all self-adjoint T and all $r \in \mathbb{R}$, and it follows that $\varphi_0(rf) = r\varphi_0(f)$ for all real-valued $f \in E$ and all $r \in \mathbb{R}$. Thus, we have defined an $\mathbb{R}$-linear functional φ_0 on the space of real-valued elements of E. Complexification extends φ_0 to a $\mathbb{C}$-linear functional on E.

We now observe that φ_0 is rearrangement-invariant. If $f \in E$ and $f \geqslant 0$ and if $T \in \mathcal{S}(\mathcal{A}, \tau\restriction_{\mathcal{A}}) \cap \mathcal{E}(\mathcal{M}, \tau)$ is the corresponding element, then $\mu(\mathrm{e}^T) = \mathrm{e}^{f^*}$, where f^* is the nondecreasing rearrangement of f. Since $m(\mathrm{e}^T)$ depends only on $\mu(\mathrm{e}^T)$, we see that $\varphi_0(f) = \varphi_0(f^*)$ and, thus, φ_0 is rearrangement-invariant.

By Theorem 1.1, there is a unique trace φ on $\mathcal{E}(\mathcal{M}, \tau)$ such that $\varphi(T) = \varphi_0(\mu(T))$ whenever $T \in \mathcal{E}(\mathcal{M}, \tau)$ is positive. Suppose X is an invertible element of $\mathcal{E}_{\log}(\mathcal{M}, \tau)$ and let us observe that $m(X) = \det_\varphi(X)$. Since $m(X) = m(|X|)$ and likewise for $\det_\varphi$, we may without loss of generality assume $X \geqslant 0$. Thus, there is a self-adjoint $T = \log(X) \in \mathcal{E}(\mathcal{M}, \tau)$ such that $X = \mathrm{e}^T$. Thus, by (3.1), we have

$$m(X) = \mathrm{e}^{\varphi_0(\lambda(T))} = \mathrm{e}^{\varphi(T)} = \det_\varphi(X),$$

as required. ∎

The following shows that Proposition 1.7 cannot be improved to obtain $m = \det_\varphi$ on all of $\mathcal{E}_{\log}(\mathcal{M}, \tau)$.

PROPOSITION 3.2. *Let E be a symmetric function space. Consider a strictly larger symmetric function space F. If ψ is an arbitrary positive trace on $\mathcal{F}(\mathcal{M}, \tau)$, then*

$$\det_\psi\restriction_{\mathcal{E}_{\log}(\mathcal{M},\tau)} \neq \det_\varphi$$

for each positive trace φ on $\mathcal{E}(\mathcal{M}, \tau)$.

Proof. To see this, fix $0 \leqslant T \in \mathcal{F}(\mathcal{M}, \tau)$ such that $T \notin \mathcal{E}(\mathcal{M}, \tau)$. Take $X = \mathrm{e}^{-T}$. Then X is bounded, so belongs to $\mathcal{E}_{\log}(\mathcal{M}, \tau)$. Moreover, $X^{-1} = \mathrm{e}^T$ belongs to $\mathcal{F}_{\log}(\mathcal{M}, \tau)$, but X is not invertible in $\mathcal{E}_{\log}(\mathcal{M}, \tau)$. Thus, we have

$$\det_\psi(X) = \mathrm{e}^{-\psi(T)} \neq 0 = \det_\varphi(X). \quad ∎$$

See Remark 1.6 for the relevance of the following example.

EXAMPLE 3.3. We give a class of examples of a nonzero trace φ on a bimodule $\mathcal{E}(\mathcal{M},\tau)$ and $T \in \mathcal{E}(\mathcal{M},\tau)$ such that $\varphi \neq 0$ but

$$\det_\varphi(T) \neq \lim_{\varepsilon\to 0^+} \det_\varphi(|T|+\varepsilon). \tag{3.3}$$

Let ψ be an increasing, continuous, concave function on the interval $[0,1]$ satisfying

$$\lim_{t\to 0} \frac{\psi(2t)}{\psi(t)} = 1.$$

For example, take $\psi(t) = \frac{1}{2-\log(t)}$. Let $E = M_\psi$ be the Marcinkiewicz space

$$E = \Big\{ f \in S(0,1) : \sup_{0<t<1} \frac{1}{\psi(t)} \int_0^t f^*(s)\,\mathrm{d}s < \infty \Big\},$$

where f^* is the decreasing rearrangement of $|f|$. Let $\mathcal{E}(\mathcal{M},\tau)$ be the corresponding $\mathcal{M}$-bimodule. By Example 2.5(ii) of [3], there is a positive, rearrangement-invariant, linear functional φ on E that vanishes on $E \cap L_\infty$, but satisfies $\varphi(\psi') = 1$. For $f \in E$ with $f \geqslant 0$, $\varphi(f)$ is realized as a particular sort of generalized limit as $t \to 0$ of $\frac{1}{\psi(t)} \int_0^t f^*(s)\,\mathrm{d}s$. Let φ denote also the trace on $\mathcal{E}(\mathcal{M},\tau)$, according to Theorem 1.1. Thus, we have $\det_\varphi(T) = 1$ whenever $T \in \mathcal{M}$ is bounded and has bounded inverse. Consequently, if $T \in \mathcal{M}$ fails to be invertible in $\mathcal{E}_{\log}(\mathcal{M},\tau)$, for example, because it has a nonzero kernel, then, by Definition 1.2, $\det_\varphi(T) = 0$, but the right-hand-side of (3.3) is equal to 1.

The class of examples considered above involved non-invertible elements of $\mathcal{E}_{\log}(\mathcal{M},\tau)$. However, (3.3) can also fail when T is invertible in $\mathcal{E}_{\log}(\mathcal{M},\tau)$. For example, take $T \geqslant 0$ such that $\mu(T)(t) = \exp(-\psi'(1-t))$. In particular, T is bounded. Then $\det_\varphi(T) = \mathrm{e}^{-1}$ but again the right-hand-side of (3.3) is equal to 1.

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