J. OPERATOR THEORY
78:2(2017), 247–279
doi: 10.7900/jot.2016jul26.2121

EXHAUSTIVE FAMILIES OF REPRESENTATIONS AND SPECTRA OF PSEUDODIFFERENTIAL OPERATORS

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Communicated by Stefaan Vaes

ABSTRACT. A family of representations $\mathcal{F}$ of a C^*-algebra A is *exhaustive* if every irreducible representation of A is weakly contained in some $\phi \in \mathcal{F}$. Such an $\mathcal{F}$ has the property that "$a \in A$ is invertible if and only if $\phi(a)$ is invertible for any $\phi \in \mathcal{F}$". The regular representations of amenable, second countable, locally compact groupoids form an exhaustive family of representations. If A is a separable C^*-algebra, a family $\mathcal{F}$ of representations of A is exhaustive if and only if it is strictly spectral. We consider also unbounded operators. A typical application is to parametric pseudodifferential operators.

KEYWORDS: *Operator spectrum, essential spectrum, C^*-algebra, representations of C^*-algebra, self-adjoint operator, pseudodifferential operator, Cayley transform.*

MSC (2010): Primary 46K10; Secondary 35P05, 47L90, 58H05, 81Q10.

1. INTRODUCTION

A typical result in spectral theory of N-body Hamiltonians [15], [24], [25], [35] associates to the Hamiltonian H a family of other operators H_ϕ, $\phi \in \mathcal{F}$, such that the essential spectrum $\mathrm{Spec}_{\mathrm{ess}}(H)$ of H is obtained in terms of the usual spectra $\mathrm{Spec}(H_\phi)$ of H_ϕ as the closure of the union of the later:

$$\mathrm{Spec}_{\mathrm{ess}}(H) = \overline{\bigcup_{\phi \in \mathcal{F}}} \mathrm{Spec}(H_\phi). \tag{1.1}$$

It was noticed that sometimes the closure is not necessary, and one of the motivations of our paper is to clarify this issue. Our approach is based on the well known fact that the operators H_ϕ are obtained as homomorphic images (in a suitable sense) of the operator H, that is $H_\phi = \phi(H)$, where the morphisms ϕ are part of a suitable family of representations $\mathcal{F}$ of a certain C^*-algebra associated to H. This justifies the study of families of representations. See for example [25] for an illustration of this approach. To simplify our terminology, by a *morphism* or a *representation* of a C^*-algebras, we shall always mean a $*$-*morphism*, respectively, a $*$-*representation* of that C^*-algebra into another C^*-algebra.

Another, related, motivation comes from the characterization of Fredholm integral operators [15], [38], [44], [49], [51], [52], [53] and of Toeplitz operators [10]. We are especially interested in the approach to this question using groupoids [17], [18], [30], [31], [54]. More precisely, for suitable manifolds M and for differential operators D on M compatible with the geometry, there was devised a procedure to associate to M the following data: (i) spaces Z_α, $\alpha \in I$; (ii) groups G_α, $\alpha \in I$; and (iii) G_α-invariant differential operators D_α acting on $Z_\alpha \times G_\alpha$. This data can be used to characterize the Fredholm property of D as follows. Let m be the order of D, then

$$D : H^s(M) \to H^{s-m}(M) \text{ is Fredholm} \Leftrightarrow D \text{ is elliptic and } D_\alpha \text{ is invertible for all } \alpha \in I. \tag{1.2}$$

Moreover, the spaces Z_α and the groups G_α are independent of D. If M is compact (without boundary), then the index I is empty (so there are no D_αs). In general, for non-compact manifolds, the conditions on the operators D_α are, nevertheless, necessary. The non-compact geometries to which this characterization of Fredholm operators applies include: asymptotically euclidean manifolds, asymptotically hyperbolic manifolds, manifolds with poly-cylindrical ends, and many others (see [39], [40] for surveys). Again, the operators D_α are homomorphic images of the operator D, which leads us again to the study of families of representations.

The results in [24], [25], [30] mentioned above are the main motivation for this work, which is a purely theoretical one on the representation theory of C^*-algebras, even though the applications are to spectral theory and pseudodifferential operators.

Our main results concern "exhaustive families of representations", a concept that we introduce and study in this paper. To explain our results, let us discuss first the important, related concept of a "strictly spectral family of representations". Recall from [48] that *a strictly spectral family* of representations $\mathcal{F}$ of a unital C^*-algebra A is a set of representations with the property that $a \in A$ is invertible if and only if $\phi(a)$ is invertible for all $\phi \in \mathcal{F}$. This concept is directly applicable to the problems mentioned in the beginning of this introduction. It is equivalent to the concept of a *strictly norming family* of representations [23], [48], a concept that we recall in the main body of the paper. In practice, it is not straightforward to check that a family of representations is strictly spectral or strictly norming. Motivated by this, we introduce an *exhaustive family* of representations of A as a set $\mathcal{F}$ with the property that every irreducible representation of A is weakly contained in a representation $\phi \in \mathcal{F}$. Exhaustive families of representations turn out to have many useful properties.

Here are the contents of the sections of the paper and our main results. In the following section, the second section, we discuss some results on faithful family of representations in preparation and as motivation for the study of exhaustive families of representations, which is the main thrust of the third section. Thus, in the third section, we discuss and prove various basic properties of exhaustive families. We also discuss their relation with strictly spectral families of representations. We prove that the C^*-algebras of groupoids $\mathcal{G}$ that satisfy the Effros–Hahn conjecture and have amenable isotropy groups have the property that the family of regular representations $\mathcal{R} = \{\pi_x\}$ is exhaustive (here x is ranging through the units of $\mathcal{G}$). We notice that an example due to Voiculescu (private communication) shows that this result is not true in general. In the fourth section we provide a necessary and sufficient condition for a family of representations of A to be exhaustive in terms of the topology on the primitive ideal spectrum $\mathrm{Prim}(A)$ of A. In particular, we show that for a separable C^*-algebra, a set of representations of A is strictly spectral if, and only if, it is exhaustive. We also provide an example of a strictly spectral family that is not exhaustive in the non-separable case. The fifth section contains some material that allows us to treat also unbounded operators affiliated to a C^*-algebra. The last section, the sixth, contains a typical application of our results to parametric families of differential operators. This type of operators arises in the analysis on manifolds with corners (more precisely, in the case of manifolds with poly-cylindrical ends). In that case, we recover the fact that an operator compatible with the geometry is invertible if and only if its Mellin transform is invertible. Due to the fact that the main applications are to areas other than the study of C^*-algebras, we have writen the paper with an eye towards the non-specialist in C^*-algebras. In particular, in addition to the relevant references, we have also included a few short proofs of some known (or essentially known) results.

2. C^*-ALGEBRAS AND THEIR PRIMITIVE IDEAL SPECTRUM

We begin with a review of some needed general C^*-algebra results. Our main reference is [20], but see also [14], [45], [49], [50], [55], [56]. Recall that a *C^*-algebra* is a complex algebra A together with a conjugate linear involution $*$ and a complete norm $\|\cdot\|$ such that $(ab)^* = b^*a^*$, $\|ab\| \leqslant \|a\|\|b\|$, and $\|a^*a\| = \|a\|^2$, for all $a, b \in A$. (The fact that $*$ is an involution means that $a^{**} = a$.) In particular, a C^*-algebra is also a Banach algebra. Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the space of linear, bounded operators on $\mathcal{H}$. One of the main reasons why C^*-algebras are important in applications is that every norm-closed subalgebra $A \subset \mathcal{L}(\mathcal{H})$ that is also closed under taking Hilbert space adjoints is a C^*-algebra. Abstract C^*-algebras have many non-trivial properties that can then be used to study the concretely given algebra A. Conversely, every abstract C^*-algebra is isometrically isomorphic to a norm closed subalgebra of $\mathcal{L}(\mathcal{H})$ (the Gelfand–Naimark theorem, see Theorem 2.6.1 of [20]). A *representation* of a C^*-algebra A on the Hilbert space $\mathcal{H}_\pi$ is a morphism $\pi : A \to \mathcal{L}(\mathcal{H}_\pi)$ to the algebra of bounded operators on $\mathcal{H}_\pi$. (Recall that, in this paper, by a morphism of C^*-algebras, we shall always mean a $*$-*morphism*.) We shall use the fact that every morphism ϕ of C^*-algebras (and hence any representation of a C^*-algebra) has norm $\|\phi\| \leqslant 1$. Consequently, every bijective morphism of C^*-algebras is an isometric isomorphism, and, in particular

$$\|\phi(a)\| = \|a + \ker(\phi)\|_{A/\ker(\phi)}. \tag{2.1}$$

A two-sided ideal $I \subset A$ is called *primitive* if it is the kernel of an irreducible representation. We shall denote by $\mathrm{Prim}(A)$ the set of primitive ideals of A. For any two-sided ideal $J \subset A$, we have that its primitive ideal spectrum $\mathrm{Prim}(J)$ identifies with the set of all the primitive ideals of A *not* containing the two-sided ideal $J \subset A$. It turns out then that the sets of the form $\mathrm{Prim}(J)$, where J ranges through the set of two-sided ideals $J \subset A$, define a topology on $\mathrm{Prim}(A)$, called the *Jacobson topology* on $\mathrm{Prim}(A)$. If $A = C(K)$, the algebra of continuous functions on a compact space K, then K and $\mathrm{Prim}(A)$ are canonically homeomorphic. See Example 5.6 for a slightly more involved example.

Throughout this paper, we shall denote by A a generic C^*-algebra. Also, by $\phi : A \to \mathcal{L}(\mathcal{H}_\phi)$ we shall denote generic representations of A. For any representation ϕ of A, we define its support, $\mathrm{supp}(\phi) \subset \mathrm{Prim}(A)$ as the complement of $\mathrm{Prim}(\ker(\phi))$, that is, $\mathrm{supp}(\phi) := \mathrm{Prim}(A) \setminus \mathrm{Prim}(\ker(\phi))$ is the set of primitive ideals of A *containing* $\ker(\phi)$.

REMARK 2.1. The irreducible representations of A do not form a set (there are too many of them). The *unitary equivalence classes* of irreducible representations of A do form a set however, which we shall denote by $\widehat{A}$. By $\pi : A \to \mathcal{L}(\mathcal{H}_\pi)$ we shall denote an arbitrary *irreducible* representation of A. There exists then by definition a surjective map

$$can : \widehat{A} \to \mathrm{Prim}(A) \tag{2.2}$$

that associates to (the class of) each irreducible representation $\pi \in \widehat{A}$ its kernel $\ker(\pi)$. For each $a \in A$ and each irreducible representation π of A, the algebraic properties of $\pi(a)$ depend only on the kernel of π. That yields a well defined function

$$can : \widehat{A} \ni \pi \to \|\pi(a)\| \in [0, \|a\|], \tag{2.3}$$

which descends to a well defined function

$$n_a : \mathrm{Prim}(A) \ni \pi \to \|\pi(a)\| \in [0, \|a\|], \quad n_a(\ker(\pi)) = \|\pi(a)\|, \tag{2.4}$$

because if ϕ_1 and ϕ_2 are representations of A with the same kernel, then $\|\phi_1(a)\| = \|\phi_2(a)\|$ for all $a \in A$.

A C^*-algebra is *type I* if and only if the surjection $can : \widehat{A} \to \mathrm{Prim}(A)$ of equation (2.2) is, in fact, a bijection [20] (a deep result). Then the discussion of Remark 2.1 becomes unnecessary and several arguments below will be (slightly) simplified since we will not have to make distinction between equivalence classes of irreducible representations and their kernels. Fortunately, many (if not all) of the C^*-algebras that arise in the study of pseudodifferential operators and of other practical questions are type I C^*-algebras. In spite of this, it seems unnatural at this time to restrict our study to type I C^*-algebras. Therefore, we will not assume that A is a type I C^*-algebra, unless this assumption is really needed. When A is a type I C^*-algebra, we will identify $\widehat{A}$ and $\mathrm{Prim}(A)$.

We shall need the following simple (and well known) lemma [20].

LEMMA 2.2. *The map* $n_a : \mathrm{Prim}(A) \ni I \to \|a + I\|_{A/I} \in [0, \|a\|]$ *is lower semi-continuous, that is, the set* $\{I \in \mathrm{Prim}(A) : \|a + I\|_{A/I} > t\}$ *is open for any* $t \in \mathbb{R}$.

We include the simple proof for the benefit of the non-specialist.

Proof. Let us fix $t \in \mathbb{R}$. Since n_a takes on non-negative values, we may assume $t \geqslant 0$. Let then $\chi : [0, \infty) \to [0, 1]$ be a continuous function that is zero on $[0, t^2]$ but is > 0 on (t^2, ∞) and let $b = \chi(a^*a)$, which is defined using the functional calculus with continuous functions. If $\phi : A \to \mathcal{L}(\mathcal{H}_\phi)$ is a representation of A, then we have that $\|\phi(a)\|^2 = \|\phi(a^*a)\| \leqslant t^2$ if and only if

$$\chi(\phi(a^*a)) = \phi(\chi(a^*a)) = \phi(b) = 0.$$

Let then J be the (closed) two sided ideal generated by b, that is, $J := \overline{AbA}$. Then

$$\begin{aligned}\{I \in \mathrm{Prim}(A) : \|a + I\|_{A/I} \leqslant t\} &= \{I \in \mathrm{Prim}(A) : b \in I\} \\ &= \{I \in \mathrm{Prim}(A) : J \subset I\} = \mathrm{Prim}(A) \setminus \mathrm{Prim}(J),\end{aligned}$$

is hence a closed set. Consequently, $\{I \in \mathrm{Prim}A : \|a + I\|_{A/I} > t\}$ is open, as claimed. ∎

For any unital C^*-algebra A and any $a \in A$, we denote by $\mathrm{Spec}_A(a)$ the *spectrum* of a in A, defined by

$$\mathrm{Spec}_A(a) := \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible in } A\}.$$

It is known that $\mathrm{Spec}_A(a)$ is, in fact, independent of the C^*-algebra A [20]. (See next.) It is also known classically that $\mathrm{Spec}_A(a)$ is compact and non-empty, unlike in the case of unbounded operators [20]. For A non-unital, we let $\mathrm{Spec}(a) := \mathrm{Spec}_{\widetilde{A}}(a)$.

We shall need the following general property of C^*-algebras [20].

LEMMA 2.3. *Let $A_1 \subset B$ be two C^*-algebras and $a \in A_1$ be such that it has an inverse in B, denoted a^{-1}. Then $a^{-1} \in A_1$. In particular, the spectrum of a is independent of the C^*-algebra in which we compute it:*

$$\mathrm{Spec}_{A_1}(a) = \mathrm{Spec}_B(a) =: \mathrm{Spec}(a). \tag{2.5}$$

We need the following general fact [20].

REMARK 2.4. Let A be a C^*-algebra and $J \subset A$ be a two-sided ideal, then we have that $\mathrm{Prim}(A)$ is the disjoint union of $\mathrm{Prim}(J)$ and $\mathrm{Prim}(A/J)$ [20]. This correspondence sends a primitive ideal I of A to $I \cap J$, if $I \cap J \neq J$, and otherwise (i.e. if $J \subset I$) it sends I to I/J, which is an ideal of A/J.

3. FAITHFUL FAMILIES

Let $\mathcal{F}$ be a set of representations of A. We say that the family $\mathcal{F}$ is *faithful* if the direct sum representation $\rho := \bigoplus_{\phi \in \mathcal{F}} \phi$ is injective. The results of this subsection are for the most part very well-known, see for instance [48], but we include them for the purpose of later reference and in order to compare them with the properties of exhaustive families and strictly norming families. We have the following well known result that will serve us as a model for characterization of "strictly norming families" of representations in the next subsection.

PROPOSITION 3.1. *Let $\mathcal{F}$ be a family of representations of the C^*-algebra A. The following are equivalent:*

(i) *The family $\mathcal{F}$ is faithful.*

(ii) *The union* $\bigcup_{\phi \in \mathcal{F}} \operatorname{supp}(\phi)$ *is dense in* $\operatorname{Prim}(A)$.

(iii) $\|a\| = \sup_{\phi \in \mathcal{F}} \|\phi(a)\|$ *for all* $a \in A$.

Proof. (i) $\Rightarrow$ (ii) We proceed by contradiction. Let us assume that (i) is true, but that (ii) is not true. That is, we assume that $\bigcup_{\phi \in \mathcal{F}} \operatorname{supp}(\phi)$ is not dense in $\operatorname{Prim}(A)$. Then there exists a non empty open set $\operatorname{Prim}(J) \subset \operatorname{Prim}(A)$ that does not intersect $\bigcup_{\phi \in \mathcal{F}} \operatorname{supp}(\phi)$, where $J \subset A$ is a non-trivial two-sided ideal. Then $J \neq 0$ is contained in the kernel of $\bigoplus_{\phi \in \mathcal{F}} \phi$ and hence $\mathcal{F}$ is not faithful. This is a contradiction, and hence (ii) must be true if (i) is true.

(ii) $\Rightarrow$ (iii) For a given $a \in A$, the map sending the kernel $\ker \pi$ of an irreducible representation π to $\|\pi(a)\|$ is a lower semi-continuous function from $\operatorname{Prim}(A)$ to $[0, \infty)$, by Lemma 2.2. Moreover, for any $a \in A$ there exists an irreducible representation π_a such that $\|\pi_a(a)\| = \|a\|$ [20]. Hence, for every $\varepsilon > 0$, $\{\pi \in \operatorname{Prim}(A) : \|\pi(a)\| > \|a\| - \varepsilon\}$ is a non empty open set (it contains $\ker \pi_a$) and then it contains some $\pi \in \bigcup_{\phi \in \mathcal{F}} \operatorname{supp}(\phi)$, since the later set was assumed to be dense in $\operatorname{Prim}(A)$. Let $\phi \in \mathcal{F}$ be such that $\ker(\pi) \supset \ker(\phi)$. Then

$$\|a\| \geqslant \|\phi(a)\| \geqslant \|\pi(a)\| > \|a\| - \varepsilon,$$

where the first inequality is due to the general fact that representations of C^*-algebras have norm $\leqslant 1$ and the second one is due to the fact that

$$\|\phi(a)\| = \|a + \ker(\phi)\|_{A/\ker(\phi)} \geqslant \|a + \ker(\pi)\|_{A/\ker(\pi)} = \|\pi(a)\|,$$

by equation (2.1). Consequently, $\|a\| = \sup_{\phi\in\mathcal{F}} \|\phi(a)\|$, as desired.

(iii) $\Rightarrow$ (i) Let $\rho := \bigoplus_{\phi\in\mathcal{F}} \phi : A \to \bigoplus_{\phi\in\mathcal{F}} \mathcal{L}(H_\phi)$. We need to show that ρ is injective. The norm on $\bigoplus_{\phi\in\mathcal{F}} \mathcal{L}(H_\phi)$ is the sup norm, $\|(T_\phi)_{\phi\in\mathcal{F}}\| = \sup_{\phi\in\mathcal{F}} \|T_\phi\|$. Therefore $\|\rho(a)\| = \sup_{\phi\in\mathcal{F}} \|\phi(a)\| = \|a\|$, since we are assuming (iii). Consequently, ρ is isometric, and hence it is injective. ∎

In the next proposition we shall need to assume that A is unital (that is, that it has a unit $1 \in A$). This assumption is not very restrictive since, given any non-unital C^*-algebra A_0, the algebra with adjoint unit $A = \widetilde{A}_0 := A_0 \oplus \mathbb{C}$ has a unique C^*-algebra norm.

We shall need the following remark on extensions of representations.

REMARK 3.2. Let B be a C^*-algebra and $I \subset B$ be a closed two-sided ideal. Recall from Proposition 2.10.4 in [20] that any representation $\pi : I \to \mathcal{L}(\mathcal{H})$ extends to a unique representation $\pi : B \to \mathcal{L}(\mathcal{K}) \subset \mathcal{L}(\mathcal{H})$, $\mathcal{K} = \overline{\pi(I)\mathcal{H}}$ (the closure is actually not needed by the Cohen–Hewitt factorization theorem). This extension is an instance of the Rieffel induction [47] corresponding to I, regarded as an A–I bimodule.

In particular, we shall use this remark in order to deal with non-unital algebras as follows.

NOTATION 3.3. Let A be a C^*-algebra and let us denote by $A' := A$ if A has a unit and by $A' := \widetilde{A} := A \oplus \mathbb{C}$ if A does not have a unit. Let $\chi_0 : \widetilde{A} \to \mathbb{C}$ be the canonical projection. Then, if $\mathcal{F}$ is a set of representations of A, we let $\mathcal{F}' := \mathcal{F}$ if A has a unit and $\mathcal{F}' := \mathcal{F} \cup \{\chi_0\}$ if A does not have a unit. By implicitly extending the representations of A to A', we have that $\mathcal{F}'$ is a set of representations of $\widetilde{A}$.

Using this notation, we have the following result.

PROPOSITION 3.4. *Let $\mathcal{F}$ be a faithful family of nondegenerate representations of a C^*-algebra A. An element $a \in A'$ is invertible if and only if $\phi(a)$ is invertible in $\mathcal{L}(\mathcal{H}_\phi)$ for all $\phi \in \mathcal{F}'$ and the set $\{\|\phi(a)^{-1}\| : \phi \in \mathcal{F}'\}$ is bounded.*

Proof. By replacing A with A', we may assume that A is unital. Since each $\phi \in \mathcal{F}$ is nondegenerate, if a is invertible, $\phi(a)$ also is invertible and $\|\phi(a)^{-1}\| = \|\phi(a^{-1})\| \leqslant \|a^{-1}\|$ is hence bounded.

Conversely, let ρ be the direct sum of all the representations $\phi \in \mathcal{F}$, that is,

$$\rho := \bigoplus_{\phi\in\mathcal{F}} \phi : A \longrightarrow \bigoplus_{\phi\in\mathcal{F}} \mathcal{L}(H_\phi). \tag{3.1}$$

If $\|\phi(a)\|$ is invertible for all $\phi \in \mathcal{F}$ and there exists M independent of ϕ such that $\|\phi(a)^{-1}\| \leqslant M$, then $b := (\phi(a)^{-1})_{\phi\in\mathcal{F}}$ is a well defined element in $B :=$

$\bigoplus_{\phi \in \mathcal{F}} \mathcal{L}(\mathcal{H}_\phi)$ and b is an inverse for $\rho(a)$ in B. Let $A_1 := \rho(A)$. Then $\rho(a) \in A_1$ is invertible in B. Then observe that since ρ is continuous, injective, and surjective morphism of C^*-algebras, it defines an isomorphism of algebras $A \to A_1$. We then conclude that a is invertible in A as well. ∎

The following is a converse of the above proposition. Recall that $a \in A$ is called *normal* if $aa^* = a^*a$.

PROPOSITION 3.5. *Let $\mathcal{F}$ be a family of representations of a unital C^*-algebra A with the following property:*

> *"If $a \in A$ is such that $\phi(a)$ is invertible in $\mathcal{L}(\mathcal{H}_\phi)$ for all $\phi \in \mathcal{F}$ and the set $\{\|\phi(a)^{-1}\| : \phi \in \mathcal{F}\}$ is bounded, then a is invertible in A."*

Then the family $\mathcal{F}$ is faithful.

Proof. Clearly, the family $\mathcal{F}$ is not empty, since otherwise all elements of A would be invertible, which is not possible. Let us assume, by contradiction, that the family $\mathcal{F}$ is *not* faithful. Then, by Proposition 3.1(ii), there exists a non-empty open set $V \subset \mathrm{Prim}(A)$ that does not intersect $\bigcup_{\phi \in \mathcal{F}} \mathrm{supp}(\phi)$. Let $J \subset A$, $J \neq 0$, be the (closed) two-sided ideal corresponding to V, that is, $V = \mathrm{Prim}(J)$. Since $\mathcal{F}$ is non-empty, we have $J \neq \mathrm{Prim}(A)$. Then every $\phi \in \mathcal{F}$ is such that $\phi = 0$ on J. Let $a \in J$, $a \neq 0$. By replacing a with $a^*a \in J$, we can assume $a \geqslant 0$. Let $\lambda \in \mathrm{Spec}(a)$, $\lambda \neq 0$. Such a λ exists since a is normal and non-zero. Let $c := \lambda - a$. Then, for any $\phi \in \mathcal{F}$, $\phi(c) = \lambda \in \mathbb{C}$ is invertible and $\|\phi(c)^{-1}\| = \lambda^{-1}$ is bounded. However, c is not invertible (in any C^*-algebra containing it) since it belongs to the non-trivial ideal J. ∎

Recall that $\mathcal{C}_0(X)$ is the set of continuous functions on X that have vanishing limit at infinity. Then $\mathcal{C}_0(X)$ is a commutative C^*-algebra, and all commutative C^*-algebras are of this form.

EXAMPLE 3.6. Let μ_α, $\alpha \in I$, be a family of positive, regular Borel measures on a locally compact space X. Let ϕ_α be the corresponding multiplication representation of the C^*-algebra $\mathcal{C}_0(X) \to \mathcal{L}(L^2(X, \mu_\alpha))$. We have $\mathrm{supp}(\phi_\alpha) = \mathrm{supp}(\mu_\alpha)$ and the family $\mathcal{F} := \{\phi_\alpha : \alpha \in I\}$ is faithful if and only if $\bigcup_{\alpha \in I} \mathrm{supp}(\mu_\alpha)$ is dense in X. In particular, if each μ_α is the Dirac measure concentrated at some $x_\alpha \in X$, then $\phi_\alpha(f) = f(x_\alpha) =: \mathrm{ev}_{x_\alpha}(f) \in \mathbb{C}$ and $\mathrm{supp}(\mu_\alpha) = \{x_\alpha\}$. We shall henceforth identify $x_\alpha \in X$ with the corresponding evaluation irreducible representation ev_{x_α}. Then we have that

$$\mathcal{F} = \{\mathrm{ev}_{x_\alpha} : \alpha \in I\} \text{ is faithful } \Leftrightarrow \{x_\alpha : \alpha \in I\} \text{ is dense in } X.$$

This example extends right away to C^*-algebras of the form $\mathcal{C}_0(X; \mathcal{K})$ of functions with values compact operators on some given Hilbert space.

We conclude our discussion of faithful families with the following result. We denote by $\overline{\bigcup} S_\alpha := \overline{\bigcup_\alpha} S_\alpha$ the closure of the union of the family of sets S_α.

PROPOSITION 3.7. *Let $\mathcal{F}$ be a family of representations of a unital C^*-algebra A. Then $\mathcal{F}$ is faithful if and only if for any normal $a \in A$,*

$$\mathrm{Spec}(a) = \overline{\bigcup_{\phi \in \mathcal{F}}} \mathrm{Spec}(\phi(a)). \tag{3.2}$$

Proof. Let us assume first that the family $\mathcal{F}$ is faithful and that a is normal. Since we have that $\mathrm{Spec}(\phi_0(a)) \subset \mathrm{Spec}(a)$ for any representation ϕ_0 of A, it is enough to show that $\mathrm{Spec}(a) \subset \overline{\bigcup_{\phi \in \mathcal{F}}} \mathrm{Spec}(\phi(a))$. Let us assume the contrary and let $\lambda \in \mathrm{Spec}(a) \setminus \overline{\bigcup_{\phi \in \mathcal{F}}} \mathrm{Spec}(\phi(a))$. By replacing a with $a - \lambda$, we can assume that $\lambda = 0$. We thus have that $\phi(a)$ is invertible for all $\phi \in \mathcal{F}$, but a is not invertible (in A). Moreover, $\|\phi(a)^{-1}\| \leqslant \delta^{-1}$, where δ is the distance from $\lambda = 0$ to the spectrum of $\phi(a)$, by the properties of the functional calculus for normal operators. This is however a contradiction by Proposition 3.4, which implies that a must be invertible in A as well.

To prove the converse, let us assume that $\mathrm{Spec}(a) \subset \overline{\bigcup_{\phi \in \mathcal{F}}} \mathrm{Spec}(\phi(a))$, for all normal elements $a \in A$. Let J be a non-trivial (closed *selfadjoint*) two-sided ideal on which all the representations $\phi \in \mathcal{F}$ vanish. We have to show that $J = 0$, which would imply that $\mathcal{F}$ is faithful. Let $a \in J$ be a normal element. Then $\mathrm{Spec}(a) \subset \overline{\bigcup_{\phi \in \mathcal{F}}} \mathrm{Spec}(\phi(a)) = \{0\}$. Since a is normal we deduce $a = 0$ and hence J has no normal element other than 0. Then, for any $a \in J$, we can write $a = 1/2(a + a^*) + 1/2(a - a^*)$, the sum of two normal elements in J because J is selfadjoint. Therefore $1/2(a + a^*) = 1/2(a - a^*) = 0$, and hence $a = 0$ and $J = 0$. ∎

We refer to [4], [11], [31], [39], [45], [54] for background material on groupoids. The following is well known, but is useful in order to set up the terminology and to introduce some concepts to be used below.

EXAMPLE 3.8. Let $\mathcal{G}$ be a locally compact groupoid with units M and with Haar system (λ_x), $x \in M$. If $d : \mathcal{G} \to M$ denotes the domain map $\mathcal{G} \to M$, then we denote $\mathcal{G}_A := d^{-1}(A)$, $A \subset M$, and $\mathcal{G}_x := d^{-1}(x)$, $x \in M$. We recall that λ_x has support $\mathcal{G}_x$ (and is right invariant and continuous in a natural sense). The *regular representation* π_x of $C^*(\mathcal{G})$ then acts on $L^2(\mathcal{G}_x, \lambda_x)$ by left convolution. Let $\mathcal{R} := \{\pi_x : x \in M\}$ be the set of regular representations of $C^*(\mathcal{G})$, the C^*-algebra associated to $\mathcal{G}$. Let I be the intersection of all the kernels of the representations π_x. Then the set $\mathcal{R}$ is a faithful set of representations of $C^*_{\mathrm{r}}(\mathcal{G}) \simeq C^*(\mathcal{G})/I$, the reduced C^*-algebra of $\mathcal{G}$. In general, $\mathcal{R}$ will not be a faithful family of representations of $C^*(\mathcal{G})$, unless the canonical projection $C^*(\mathcal{G}) \to C^*_{\mathrm{r}}(\mathcal{G})$ is an isomorphism.

4. EXHAUSTIVE AND STRICLTY NORMING FAMILIES

Let us notice that Example 3.6 shows that the "sup" in the relation $\|a\| = \sup_{\phi\in\mathcal{F}} \|\phi(a)\|$ (Proposition 3.1) may not be attained. It also shows that the closure of the union in equation (3.2) is needed. Sometimes, in applications, one does obtain however the stronger version of these results (that is, that the sup is attained and that the closure is not needed), see [15], [25], for example. Moreover, the condition that the norms of $\phi(a)^{-1}$ be uniformly bounded (in ϕ) for any fixed $a \in A$ is inconvenient and often not needed in applications. For this reason, we introduce now a new class of sets of representations of A, the class of "exhaustive sets of representations", a class that has some additional properties. The concept of an exhaustive set of representations turns out to be closely related to the concept of a "strictly spectral set of representations", introduced by Roch [48], which we discus first.

4.1. STRICTLY SPECTRAL SETS OF REPRESENTATIONS. We now recall the concepts of strictly spectral and strictly norming families of representations [23], [48]. See also [44], [49].

DEFINITION 4.1 (Roch). Let $\mathcal{F}$ be a set of representations of a *unital* C^*-algebra A.

(i) We shall say that $\mathcal{F}$ is *strictly spectral* if

$$\text{“}a \in A \text{ is invertible } \Leftrightarrow \phi(a) \text{ is invertible for any } \phi \in \mathcal{F}\text{”}.$$

(ii) We shall say that $\mathcal{F}$ is *strictly norming* if, for any $a \in A$, there exists $\phi \in \mathcal{F}$ such that $\|a\| = \|\phi(a)\|$.

EXAMPLE 4.2. By classical results [20], the set of all irreducible representations of a C^*-algebra is strictly norming. A proof of this well-known fact is contained in [23]. See also Theorem 4.4.

The classes of strictly spectral and strictly norming sets of representations actually coincide (see Theorem 4.4 below). Before discussing that result, however, we need to extend the above definitions to the non-unital case.

REMARK 4.3. Using the notation introduced in Notation 3.3, we obtain then the following form of the definition of a strictly spectral family:

> "The family $\mathcal{F}$ is *strictly spectral* if, for any $a \in A$, the element $1 + a$ is invertible in $\widetilde{A} := A \oplus \mathbb{C}$ if and only if $1 + \phi(a)$ is invertible for any $\phi \in \mathcal{F}$".

Similarly, the definition of a strictly norming family becomes:

> "$\mathcal{F}$ is *strictly norming* if, for any $a \in A$ and $\lambda \in \mathbb{C}$, either there exists $\phi \in \mathcal{F}$ such that $\|\lambda + a\| = \|\lambda + \phi(a)\|$ or $\|\lambda + a\| = |\lambda|$".

The following result was proved in the unital case in [48]. See also [23].

THEOREM 4.4 (Roch). *Let $\mathcal{F}$ be a set of non-degenerate representations of a unital C^*-algebra A. Then $\mathcal{F}$ is strictly norming if and only if it is strictly spectral.*

Proof. The unital case was proved already. If A does not have a unit, then we simply replace A with A' and $\mathcal{F}$ with $\mathcal{F}'$ (see the notation introduced in Notation 3.3) to reduce to the unital case. ∎

Clearly, a strictly spectral family of representations will consist only of non-degenerate representations, but this does not hold true for a strictly norming family.

We now give some examples of how exhaustive and strictly norming sets of representations are useful for invertibility questions. The following characterization of Fredholm operators is a consequence of the definitions.

COROLLARY 4.5. *Let $1 \in A \subset \mathcal{L}(\mathcal{H})$ be a sub-C^*-algebra of bounded operators on the Hilbert space $\mathcal{H}$ containing the algebra of compact operators on $\mathcal{H}$, $\mathcal{K} = \mathcal{K}(\mathcal{H})$. Let $\mathcal{F}$ be a strictly spectral family of representations of $A/\mathcal{K}$. We then have the following characterization of Fredholm operators $a \in A$:*

$a \in A$ is Fredholm if and only if $\phi(a)$ is invertible in for all $\phi \in \mathcal{F}$.

The following proposition is the analog of Proposition 3.7 in the framework of strictly norming families.

THEOREM 4.6. *Let $\mathcal{F}$ be a family of representations of a unital C^*-algebra A. Then $\mathcal{F}$ is strictly spectral if and only if for any $a \in A$,*

$$\operatorname{Spec}(a) = \bigcup_{\phi \in \mathcal{F}} \operatorname{Spec}(\phi(a)). \tag{4.1}$$

Proof. Let us assume first that the family $\mathcal{F}$ is strictly spectral. We proceed in analogy with the proof of Proposition 3.7. Since $\operatorname{Spec}(\phi_0(a)) \subset \operatorname{Spec}(a)$ for any representation ϕ_0 of A, it is enough to show that $\operatorname{Spec}(a) \subset \bigcup_{\phi \in \mathcal{F}} \operatorname{Spec}(\phi(a))$. Let us assume the contrary and let $\lambda \in \operatorname{Spec}(a) \setminus \bigcup_{\phi \in \mathcal{F}} \operatorname{Spec}(\phi(a))$. By replacing a with $a - \lambda$, we can assume that $\lambda = 0$. We thus have that $\phi(a)$ is invertible for all $\phi \in \mathcal{F}$, but a is not invertible (in A), contradicting the assumption that $\mathcal{F}$ is strictly spectral.

To prove the converse, let us assume that $\operatorname{Spec}(a) \subset \bigcup_{\phi \in \mathcal{F}} \operatorname{Spec}(\phi(a))$ for all $a \in A$. Let us assume that $a \in A$ and that $\phi(a)$ is invertible for all $\phi \in \mathcal{F}$. Then $0 \notin \bigcup_{\phi \in \mathcal{F}} \operatorname{Spec}(\phi(a))$. Since $\operatorname{Spec}(a) \subset \bigcup_{\phi \in \mathcal{F}} \operatorname{Spec}(\phi(a))$, we have that $0 \notin \operatorname{Spec}(a)$, and hence a is invertible. Thus the family $\mathcal{F}$ is strictly spectral. ∎

4.2. EXHAUSTIVE FAMILIES OF REPRESENTATIONS. It is not always easy to check that a family of representations is strictly spectral (or strictly norming, for that matter). For this reason, we introduce a slightly more restrictive class of families of representations, the class of exhaustive families of representations. It is convenient to do this for ideals first.

DEFINITION 4.7. Let A be a C^*-algebra, possibly without unit, and let $\mathcal{I}$ a set of (closed, two-sided) ideals $I \subset A$. We say that $\mathcal{I}$ is *exhaustive* if, by definition, for any irreducible representation π of A, there exists $I \in \mathcal{I}$ such that $I \subset \ker(\pi)$.

We shall typically work with families of representations $\mathcal{F}$. We consider, nevertheless, the case of families of morphisms as well. We thus have the following closely related definition.

DEFINITION 4.8. Let $\mathcal{F}$ be a set of morphisms $\phi : A \to B_\phi$ of a (not necessarily unital) C^*-algebra A. The algebras B_ϕ are not fixed. We shall say that $\mathcal{F}$ is *exhaustive* if the family of ideals $\{\ker(\phi) : \phi \in \mathcal{F}\}$ is exhaustive. Similarly, a set of unitary equivalence classes of representations $\mathcal{F}$ of A is exhaustive if the corresponding set of kernels is exhaustive.

The following simple remark is sometimes useful.

REMARK 4.9. Let ϕ be a representation of A. Recall that $\mathrm{supp}(\phi)$ is the set of primitive ideals of A that contain $\ker(\phi)$. Moreover, $\ker(\phi)$ depends only on the unitary equivalence class of ϕ. We then see that $\mathcal{F}$ is exhaustive if and only if $\mathrm{Prim}(A) = \bigcup_{\phi\in\mathcal{F}} \mathrm{supp}(\phi)$.

Recall that we denote by $A' := A$ if A has a unit and $A' := \widetilde{A} := A \oplus \mathbb{C}$, the algebra of adjoint unit, if A does not have a unit.

PROPOSITION 4.10. *Let A be a possibly non-unital C^*-algebra and let $\mathcal{F}$ be a family of representations of A. We denote by $\mathcal{F}' = \mathcal{F}$ if A has a unit and by $\mathcal{F}' := \mathcal{F} \cup \{\chi_0\}$, where $\chi_0 : A' = A \oplus \mathbb{C} \to \mathbb{C}$ is the canonical projection (as in Notation* 3.3*). Then we have:*

(i) *$\mathcal{F}$ is an exhaustive set of representations of A if and only if $\mathcal{F}'$ is an exhaustive set of representations of A'.*

(ii) *$\mathcal{F}$ is a strictly spectral set of representations of A if and only if $\mathcal{F}'$ is a strictly spectral set of representations of A'.*

Proof. To prove (i), we only need to consider the case when A does not have a unit. The result then follows from Remark 4.9 and from the relation $\mathrm{Prim}(A') = \mathrm{Prim}(\widetilde{A}) = \mathrm{Prim}(A) \cup \{\ker(\chi_0)\}$, where, we recall, $\ker(\chi_0) = A$. The other statement is really the corresponding definitions. ∎

REMARK 4.11. Let $\mathcal{F}_i$, $i = 1, 2$, be two families of representations of A. Let denote by $\mathcal{I}_i := \{\ker(\phi) : \phi \in \mathcal{F}_i\}$. We assume that $\mathcal{I}_1 = \mathcal{I}_2$. Then the families $\mathcal{F}_i$ are at the same time exhaustive or not. The same is true for the properties of being strictly norming, or strictly spectral. So these properties are really properties of a family of ideals of A rather than of families of representations of A. Nevertheless, it is customary to work with families of representations rather than families of ideals. In the same way, we can consider the analogous properties of families of *morphisms* of C^*-algebras.

Let us record the following simple facts, for further use.

PROPOSITION 4.12. *Let $\mathcal{F}$ be a set of representations of a C^*-algebra. If $\mathcal{F}$ is exhaustive, then $\mathcal{F}$ is strictly spectral and hence also strictly norming. If $\mathcal{F}$ is strictly norming, then it is also faithful.*

Proof. Let A be the given C^*-algebra. Let us prove first that any exhaustive family $\mathcal{F}$ is strictly norming. Indeed, let $a \in A'$. Then there exists an irreducible representation π_a of A' such that $\|\pi(a)\| = \|a\|$ [20]. Unless $1 \notin A$ and $\pi = \chi_0$, where $\chi_0 : A' = A \oplus \mathbb{C} \to \mathbb{C}$ is the projection, there will exist $\phi \in \mathcal{F}$ such that $\pi \in \operatorname{supp}(\phi)$. Then, as in the proof of (ii) $\Rightarrow$ (iii) in Proposition 3.1, we have that $\|a\| = \|\pi(a)\| \leqslant \|\phi(a)\| \leqslant \|a\|$. Hence $\|\phi(a)\| = \|a\|$. On the other hand, if $1 \notin A$ and $\pi = \chi_0$, then let $a = \lambda + a_0$, with $\lambda \in \mathbb{C}$ and $a_0 \in A$. Then $\|\lambda + a_0\| = \|a\| = \|\pi(a)\| = |\lambda|$. Since any strictly norming family is invertibility preserving, by Theorem 4.4, the first part of the proposition follows.

Let us prove first that any strictly norming family $\mathcal{F}$ is faithful. Indeed, let us consider the representation $\rho := \bigoplus_{\phi\in\mathcal{F}} \phi : A \to \bigoplus_{\phi\in\mathcal{F}} \mathcal{L}(H_\phi)$. By the definition of a strictly norming family of representations, the representation ρ is isometric. Therefore it is injective and consequently $\mathcal{F}$ is faithful. ∎

We summarize the above proposition in

$$\mathcal{F} \text{ exhaustive } \Rightarrow \mathcal{F} \text{ strictly norming } \Rightarrow \mathcal{F} \text{ faithful.}$$

In the next two examples we will see that there exist faithful families that are not strictly norming and strictly norming families that are not exhaustive.

EXAMPLE 4.13. We consider again the framework of Example 3.6 and consider only families of *irreducible representations*. Thus $A = \mathcal{C}_0(X)$, for a locally compact space X. The irreducible representations of A then identify with the points of X, since $X \simeq \operatorname{Prim}(A) = \widehat{A}$. A family $\mathcal{F}$ of *irreducible* representations of A thus identifies with a subset $\mathcal{F} \subset X$. We then have that a family $\mathcal{F} \subset X$ of irreducible representations of $A = \mathcal{C}_0(X)$ is faithful if and only if $\mathcal{F}$ is *dense* in X. On the other hand, a family of irreducible representations of $A = \mathcal{C}_0(X)$ is exhaustive if and only if $\mathcal{F} = X$.

The relation between exhaustive and strictly norming families is not so simple. We begin with the following remark on the above example.

REMARK 4.14. If, in Example 4.13, X is moreover *metrisable*, then every strictly norming family $\mathcal{F} \subset X$ is also exhaustive, because for any $x \in X$, there exists a compactly supported, continuous function $\psi_x : X \to [0,1]$ such that $\psi_x(x) = 1$ and $\psi_x(y) < 1$ for $y \neq x$ (we can do that by arranging that $\psi_x(y) =$

$1 - d(x, y)$, for $d(x, y)$ small, and use the Tietze extension theorem. In general, however, it is not true that any strictly norming family is exhaustive. Indeed, let I be an uncountable set and $X = [0, 1]^I$. Let $x \in X$ be arbitrary, then the family $\mathcal{F} := X \setminus \{x\}$ is strictly norming but is not exhaustive. Indeed, let $f : X \to [0, 1]$ be a continuous function such that $f(x) = 1$. Since f depends on a countable number of variables, the set $\{f = 1\}$ will not be reduced to x alone. See also Theorem 5.4.

We conclude this subsection with the following result that is relevant for the next subsection. See also [49] and the comment at the end of this subsection. The results in that book can be used to give a proof of the following results for strictly spectral families (which are essentially contained in that book). For the benefit of the reader, we include nevertheless the short, direct proofs, since we are also interested in exhaustive families.

PROPOSITION 4.15. *Let $I \subset A$ be an ideal of a C^*-algebra. Let $\mathcal{F}_I$ be a set of nondegenerate representations of I and $\mathcal{F}_{A/I}$ be a set of representations of A/I. Let $\mathcal{F} := \mathcal{F}_I \cup \mathcal{F}_{A/I}$, regarded as a family of representations of A. If $\mathcal{F}_I$ and $\mathcal{F}_{A/I}$ are both exhaustive, then $\mathcal{F}$ is also exhaustive. The same result holds by replacing "exhaustive" with "strictly norming".*

Proof. We have that $\mathrm{Prim}(A)$ is the disjoint union of $\mathrm{Prim}(I)$ and $\mathrm{Prim}(A/I)$ (Remark 2.4). Since we have $\bigcup_{\phi \in \mathcal{F}_I} \mathrm{supp}(\phi) \subset \mathrm{Prim}(I)$ and $\bigcup_{\phi \in \mathcal{F}_{A/I}} \mathrm{supp}(\phi) \subset \mathrm{Prim}(A/I)$, the result about exhaustive families follows from the definition.

Let us assume that both $\mathcal{F}_I$ and $\mathcal{F}_{A/I}$ are strictly norming and let $a \in A$. By replacing A with $\widetilde{A}$, if necessary, we may assume that A is unital. We want to show that $\mathcal{F}$ is also strictly norming, that is, that there exists $\phi \in \mathcal{F}_I \cup \mathcal{F}_{A/I}$ such that $\|a\| = \|\phi(a)\|$. By replacing a with a^*a, we can assume that $a \geqslant 0$. Since $\mathcal{F}_{A/I}$ is strictly norming, there is $\phi \in \mathcal{F}_{A/I}$ such that $\|a + I\|_{A/I} = \|\phi(a)\|$. If $\|a + I\|_{A/I} = \|a\|$, we are done. Otherwise, let ψ be a continuous function on $\mathrm{Spec}(a)$ that is zero on $\mathrm{Spec}_{A/I}(a + I)$ and such that $\psi(\|a\|) = \|a\|$ and $\psi(t) \leqslant t$ for $t \geqslant 0$. Then $\psi(a) \in I$ and $\|\psi(a)\| = \|a\|$. Since the family $\mathcal{F}_I$ is strictly norming, there exists $\phi \in \mathcal{F}_I$ such that

$$\|a\| = \|\psi(a)\| = \|\phi(\psi(a))\| = \|\psi(\phi(a))\| \leqslant \|\phi(a)\|.$$

This shows that the family $\mathcal{F}$ is strictly norming. ∎

We have the following consequence that is sometimes useful in applications.

COROLLARY 4.16. *Let $I \subset A$ be a two-sided ideal in a C^*-algebra A. Let $\mathcal{F}$ be an invertibility preserving family of representations of I. Then $a \in A$ is invertible if and only if a is invertible in A/I and $\phi(a)$ is invertible for all $\phi \in \mathcal{F}$.*

Proof. Since $\mathcal{F}$ is an invertibility preserving set of representations of I, it consists of non-degenerate representations, which will hence extend uniquely to

A. Let π be an isometric representation of A/I. The result then follows from Proposition 4.15 applied to $\mathcal{F}_I := \mathcal{F}$ and $\mathcal{F}_{A/I} := \{\pi\}$. ∎

Results closely related to Proposition 4.15 and Corollary 4.16 were obtained in [49] under the name of "lifting theorems". See especially Section 6.3 of that book. The results in that book were typically obtained in a more general setting: often using ideals in a Banach algebra and sometimes using even general ideals (and morphisms). The interested reader should consult that book as well.

4.3. GROUPOID ALGEBRAS AND THE EFFROS–HAHN CONJECTURE. We show in this subsection how one can check in the framework of locally compact groupoids (with additional properties) that certain families of representations are exhaustive, thus generalizing some results of [23].

We refer to the Example 3.8 and, especially, to the references quoted before that example, for notations and results pertaining to groupoids. In particular, we shall denote by d and r the domain and range maps of a groupoid $\mathcal{G}$ and by $\mathcal{G}_x^x := d^{-1}(x) \cap r^{-1}(x)$ the *isotropy group* of x. This is the group of arrows (or morphisms) of $\mathcal{G}$ that have domain and range equal to the unit x. Also, we continue to denote by $\mathcal{R} := \{\pi_y : y \in M\}$ the set of regular representations of a locally compact groupoid $\mathcal{G}$ with Haar system and units M. Recall that we denote $\mathcal{G}_A := d^{-1}(A)$, $A \subset M$, and $\mathcal{G}_x := d^{-1}(x)$, $x \in M$.

We shall say that a locally compact groupoid $\mathcal{G}$ with a Haar system *has the generalized Effros–Hahn property* if every primitive ideal of $C^*(\mathcal{G})$ is induced from an isotropy subgroup $\mathcal{G}_y^y$ of $\mathcal{G}$ [26], [46]. (This should not be confused with the various "EH *induction* properties" introduced in [21].) We shall write $\mathrm{Ind}_y^{\mathcal{G}}(\sigma)$ for the induced representation of $C^*(\mathcal{G})$ from the representation σ of $\mathcal{G}_y^y$. If $\mathcal{G}$ has the generalized Effros–Hahn property and all the isotropy groups $\mathcal{G}_y^y$, $y \in M$ are amenable, we say that $\mathcal{G}$ is *EH-amenable*.

THEOREM 4.17. *Let $\mathcal{G}$ be a locally compact groupoid with a Haar system and units M. If $\mathcal{G}$ is* EH*-amenable, then the family $\mathcal{R} := \{\pi_y : y \in M\}$ of regular representations of $C^*(\mathcal{G})$ is exhaustive. In particular, the family $\mathcal{R}$ is strictly spectral and the canonical map $C^*(\mathcal{G}) \to C_{\mathrm{r}}^*(\mathcal{G})$ is an isomorphism.*

Proof. Let I be any primitive ideal of $C^*(\mathcal{G})$. Then I is induced from the isotropy group $\mathcal{G}_y^y$, $y \in M$, by the assumption that $\mathcal{G}$ has the generalized Effros–Hahn property. Since $\mathcal{G}_y^y$ is amenable, every irreducible representation of $\mathcal{G}_y^y$ is weakly contained in the regular representation ρ_y of $\mathcal{G}_y^y$. But $\mathrm{Ind}_y^{\mathcal{G}}(\rho_y)$ is the regular representation π_y of $C^*(\mathcal{G})$ on $L^2(\mathcal{G}_y)$. Since induction preserves the weak containment of representations (see Proposition 6.26 of [47]), we obtain that I contains $\ker(\pi_y)$. This proves that the family $\mathcal{R} := \{\pi_y : y \in M\}$ is exhaustive. Therefore $\mathcal{R}$ is also faithful, and hence $C^*(\mathcal{G}) \simeq C_{\mathrm{r}}^*(\mathcal{G})$ (see Example 3.8). The family $\mathcal{R}$ is strictly spectral since it is exhaustive (see Proposition 4.12). ∎

We then obtain the following consequence.

THEOREM 4.18. *Let $\mathcal{G}$ be a locally compact groupoid with a Haar system and units M. If $\mathcal{G}$ is Hausdorff, second countable, and (topologically) amenable, then the family $\mathcal{R} := \{\pi_y : y \in M\}$ is exhaustive.*

Proof. Since $\mathcal{G}$ is an amenable, Hausdorff, second countable, locally compact groupoid with a Haar system, we have that $\mathcal{G}$ satisfies the Effros–Hahn conjecture by the main result in [26], that is, it has the generalized Effros–Hahn property. Since $\mathcal{G}$ is amenable, all its isotropy groups $\mathcal{G}_x^x$ are amenable [2]. The result then follows from Theorem 4.17. ∎

This result extends a result of [23], who considered the special case of étale groupoids. Let $\mathcal{G}$ be a locally compact groupoid with a Haar system and units M. We notice, however, that the family $\mathcal{R} := \{\pi_y : y \in M\}$ of regular representations of the reduced C^*-algebra $C^*_{\mathrm{r}}(\mathcal{G})$ of $\mathcal{G}$ is not exhaustive in general, as can be seen from the following example.

REMARK 4.19. Let G be the free group on two generators and let $K_n \subset G$, $n \in \mathbb{N}$, be a decreasing sequence of normal subgroups of G of finite index with $\bigcap_{n=1}^{\infty} K_n = \{1\}$. Let us consider the family of groups $\mathcal{G} := \bigcup_n \{n\} \times G/K_n$, with $n \in \mathbb{N} \cup \{\infty\}$ and $K_\infty := \{1\}$. It is a groupoid with units $\mathbb{N} \cup \{\infty\}$. Its domain and range map are equal and equal to the projection onto the first component. The topology on $\mathcal{G}_{\mathbb{N}} := d^{-1}(\mathbb{N})$, the restriction of $\mathcal{G}$ to $\mathbb{N}$, is discrete. A basis of the system of neighborhoods of (∞, g) is given by the sets $\{(n, gK_n) : n \geqslant N\}$, where $N \geqslant 1$ is arbitrary ($g \in G$). We have that the trivial representation of G defines a representation χ of $C^*(\mathcal{G})$ supported at $\{\infty\}$. The trivial representation of G is the limit of the trivial representations of G/K_n, so it descends to a representation of $C^*_{\mathrm{r}}(\mathcal{G})$. However, the trivial representation of G is not contained in the support of any of the representations λ_n, $n \in \mathbb{N} \cup \{\infty\}$, since G is not amenable. Thus the family of regular representations λ_n, $n \in \mathbb{N} \cup \{\infty\}$ is not exhaustive. This example is due to Voiculescu and it answers (in the negative) a question of Exel [23].

We are ready to prove now that the class of EH-amenable groupoids is closed under extensions and that suitable ideals and quotients of EH-amenable groupoids are also EH-amenable.

PROPOSITION 4.20. *Let $\mathcal{G}$ be a locally compact groupoid with a Haar system and units M. Let $U \subset M$ be an open $\mathcal{G}$-invariant subset and $F := M \setminus U$. We have that $\mathcal{G}$ is* EH-*amenable if and only if $\mathcal{G}_F$ and $\mathcal{G}_U$ are* EH-*amenable.*

Proof. It is clear that the isotropy groups $\mathcal{G}_x^x$ of $\mathcal{G}$ are given by the isotropy groups of the restrictions $\mathcal{G}_F$ and $\mathcal{G}_U$. This gives that all the isotropy groups of $\mathcal{G}$ are amenable if and only if the same property is shared by all the isotropy groups of the restrictions $\mathcal{G}_F$ and $\mathcal{G}_U$.

Let us turn now to proving the induction property for the primitive ideals. We shall use th correspondence of Remark 2.4 as follows. Let I be primitive ideal of $C^*(\mathcal{G})$. Recall that $C^*(\mathcal{G}_U)$ is an ideal of $C^*(\mathcal{G})$ and $C^*(\mathcal{G})/C^*(\mathcal{G}_U) \simeq C^*(\mathcal{G}_F)$, by a result of Renault [45], [46]. We thus have that I corresponds uniquely to either a primitive ideal of $C^*(\mathcal{G}_F)$ or to a primitive ideal of $C^*(\mathcal{G}_U)$. We shall consider these two cases separately. Anticipating, the first case will correspond to induced representations from isotropy groups $\mathcal{G}_y^y$ with $y \in F := M \setminus U$ and the second case will correspond to induced representations from isotropy groups $\mathcal{G}_y^y$ with $y \in U$. We first notice that the restriction of the induced representation $\mathrm{Ind}_y^{\mathcal{G}}(\sigma)$ of $C^*(\mathcal{G})$ (induced from the representation σ of $\mathcal{G}_y^y$) restricts to a non-zero representation of $C^*(\mathcal{G}_U)$ if and only if $y \in U$.

Let us then consider a primitive ideal $I \supset C^*(\mathcal{G}_U)$ of $C^*(\mathcal{G})$ and $I/C^*(\mathcal{G}_U)$ the corresponding ideal of $C^*(\mathcal{G}_F) \simeq C^*(\mathcal{G})/C^*(\mathcal{G}_U)$. Then I is induced from the irreducible representation σ of $\mathcal{G}_y^y$ if and only if $y \in F$ and $I/C^*(\mathcal{G}_U)$ is induced from the irreducible representation σ of $\mathcal{G}_y^y$. This follows directly from the definition of induced representations [47]; in fact, the inducing module is the same for both ideals.

On the other hand, if the primitive ideal I of $C^*(\mathcal{G})$ does not contain $C^*(\mathcal{G}_U)$, then again we notice that I is induced from the irreducible representation σ of $\mathcal{G}_y^y$ if and only if $y \in U$ and $I \cap C^*(\mathcal{G}_U)$ is induced from the irreducible representation σ of $\mathcal{G}_y^y$. This again follows from the results in [47], more precisely, from induction in stages Theorem 5.9 of that paper. Indeed, extending non-degenerate representations of an ideal to the whole algebra is a particular case of induction in stages (see the Remark 3.2). The inductions modules are again the same. ∎

5. TOPOLOGY ON THE SPECTRUM AND STRICTLY NORMING FAMILIES

Let us discuss now in more detail the relation between the concept of strictly spectral family and the simpler (to check) concept of an exhaustive family. The following theorem studies C^*-algebras with the property that every strictly spectral family is also exhaustive. It explains Example 4.13 and Remark 4.14.

LEMMA 5.1. *Let A be a C^*-algebra, J a two-sided ideal, and π a representation of A such that π is nondegenerate on J. Also let $a \in A$, $0 \leqslant a \leqslant 1$, such that $\|\pi(a)\| = 1$ and choose $\eta > 0$. Then there exists $c \in J$, $c \geqslant 0$, $\|c\| \leqslant \eta$ such that $\|\pi(a+c)\| \geqslant 1 + \eta/2$.*

Proof. For any fixed $\varepsilon > 0$ there exists a unit vector ξ such that $\langle \pi(a)\xi, \xi\rangle \geqslant 1 - \varepsilon$. Let us consider then the positive linear form $\varphi : A \to \mathbb{C}$ defined by $\varphi(b) : = \langle \pi(b)\xi, \xi\rangle$. If (u_λ) is an approximate unit in J, then

$$\|\varphi\| \geqslant \|\varphi|_J\| = \lim \varphi(u_\lambda) = \|\xi\| = 1.$$

So $\|\varphi|_J\| = \|\varphi\| = 1$. Hence there exists $c_0 \in J$, $c_0 \geqslant 0$, $\|c_0\| = 1$, such that $\varphi(c_0) \geqslant 1 - \varepsilon$. We then set $c = \eta c_0$ and indeed, for ε small enough

$$\|a + c\| \geqslant \varphi(a + c) \geqslant 1 - \varepsilon + \eta(1 - \varepsilon) \geqslant 1 + \frac{\eta}{2}.$$

This completes the proof. ∎

We shall use the above lemma in the form of the following corollary.

COROLLARY 5.2. *Let π_0 be an irreducible representation of a C^*-algebra A and let $I_0 := \ker(\pi_0) \in \mathrm{Prim}(A)$. We assume that we are given a decreasing sequence $V_0 \supset \cdots \supset V_n \supset V_{n+1} \cdots$ of open neighborhoods of I_0 in $\mathrm{Prim}(A)$. Then there exists $a \in I_0$ such that $\|a\| = \|\pi_0(a)\| = 1$ and $\|\pi(a)\| \leqslant 1 - 2^k$ for any irreducible representation π such that $\ker(\pi) \notin V_k$.*

Proof. To construct $a \in A$ with the desired properties, let us consider the ideals J_n defining the sets V_n, that is, $V_n = \mathrm{Prim}(J_n)$, $n \geqslant 0$. Since $V_n \subset V_{n-1}$ for all n, we have that $J_n \subset J_{n-1}$ for all n.

The element a we are looking for will be the limit of a sequence (a_n), $a_n \in A$, where the a_n are defined inductively to satisfy the following properties:

(i) $0 \leqslant a_n \leqslant 1$;
(ii) $\|\pi_0(a_n)\| = 1$;
(iii) $\|\pi(a_n)\| \leqslant 1 - 2^{-k}$ for all irreducible representations π such that $\ker(\pi) \in \mathrm{Prim}(A) \setminus \mathrm{Prim}(J_k)$ for $k = 0, 1, \dots, n$;
(iv) $\|a_n - a_{n-1}\| \leqslant 2^{2-n}$ for $n \geqslant 1$.

We define the initial term a_0 as follows. We first choose $b_0 \in J_0$ such that $0 \leqslant b_0$, and $\pi_0(b_0) \neq 0$. By rescaling b_0 with a positive factor, we can assume that $\|\pi_0(b_0)\| = 1$. Let then $\chi_0 : [0, \infty) \to [0, 1]$ be the continuous function defined by $\chi_0(t) = t$ for $t \leqslant 1$ and $\chi_0(t) = 1$ for $t \geqslant 1$. Then we define $a_0 = \chi_0(b_0)$. Conditions (i)–(iv) are then satisfied.

Next, a_n is defined in terms of a_{n-1}. In order to do that, we first define auxiliary elements c_n and $b_n = a_{n-1} + c_n$ as follows. By Lemma 5.1, there exists $c_n \in J_n$, $c_n \geqslant 0$, $\|c_n\| \leqslant 2^{1-n}$, such that $\|\pi_0(b_n)\| \geqslant 1 + 2^{-n}$. Let then $\chi_n : [0, \infty) \to [0, 1]$ be the continuous function defined by $\chi_n(t) = t$ for $t \leqslant 1 - 2^{1-n}$, χ_n linear on $[1 - 2^{1-n}, 1]$ and on $[1, 1 + 2^{-n}]$, $\chi_n(1) = 1 - 2^{-n}$, and $\chi_n(t) = 1$ for $t \geqslant 1 + 2^{-n}$. Then we define $a_n = \chi_n(b_n)$.

Claim. The sequence $a_n \in A$ just constructed satisfies conditions (i)–(iv).

Indeed, we have checked our conditions for $n = 0$, so let us assume $n \geqslant 1$ and check our conditions for $a_n \in A$ one by one.

(i) We have that $a_{n-1}, c_n \geqslant 0$, hence $b_n := a_{n-1} + c_n \geqslant 0$. Since $0 \leqslant \chi_n \leqslant 1$, we obtain that $0 \leqslant a_n := \chi_n(b_n) \leqslant 1$.

(ii) Since $0 \leqslant \chi_n \leqslant 1$, $\chi_n(t) = 1$ for $t \geqslant 1 + 2^{-n}$, and $\|\pi_0(b_n)\| \geqslant 1 + 2^{-n}$, we have that $\|\pi_0(a_n)\| = \|\pi_0(\chi_n(b_n))\| = \|\chi_n(\pi_0(b_n))\| = 1$.

(iii) Let $\pi \in \widehat{A}$ be such that $\ker(\pi) \in \mathrm{Prim}(J_k)^c := \mathrm{Prim}(A) \setminus \mathrm{Prim}(J_k)$, for some k, $0 \leqslant k \leqslant n$. We need to check that $\|\pi(a_n)\| \leqslant 1 - 2^{-k}$.

We have that π vanishes on J_k, and hence $\pi(c_n) = 0$ since $c_n \in J_n \subset J_k$, $k \leqslant n$. Therefore,

$$\pi(a_n) = \pi(\chi_n(b_n)) = \chi_n(\pi(b_n)) = \chi_n(\pi(a_{n-1})).$$

We shall consider now two cases: $k < n$ and $k = n$.

Case 1. If $k < n$, then $\|\pi(a_{n-1})\| \leqslant 1 - 2^{-k} \leqslant 1 - 2^{1-n}$, by the induction hypothesis. Since $\chi_n(t) = t$ for $t \leqslant 1 - 2^{1-n}$, we obtain $\pi(a_n) = \chi_n(\pi(a_{n-1})) = \pi(a_{n-1})$, and hence $\|\pi(a_n)\| = \|\pi(a_{n-1})\| \leqslant 1 - 2^{-k}$ for $k < n$.

Case 2. If $k = n$, we have $\|\pi(a_n)\| = \|\chi_n(\pi(a_{n-1}))\| \leqslant 1 - 2^{-n} = 1 - 2^{-k}$, since $\pi(a_n) = \chi_n(\pi(a_{n-1}))$, $\chi_n(t) \leqslant 1 - 2^{-n}$ for $t \leqslant 1$, and $0 \leqslant a_{n-1} \leqslant 1$.

(iv) We have $\|b_n\| \leqslant \|a_{n-1}\| + \|c_n\| \leqslant 1 + 2^{1-n}$. Since $|\chi_n(t) - t| \leqslant 2^{1-n}$ for all $t \leqslant 1 + 2^{1-n}$, we have $\|a_n - b_n\| \leqslant 2^{1-n}$. Hence

$$\|a_n - a_{n-1}\| \leqslant \|a_n - b_n\| + \|b_n - a_{n-1}\| \leqslant 2^{1-n} + \|c_n\| \leqslant 2^{2-n}.$$

This completes the proof of our Claim, and hence the sequence a_n constructed above satisfies conditions (i)–(iv).

Let us now show how to use the fact that the sequence $a_n \in A$ satisfies conditions (i)–(iv) to construct a as in the statement of this corollary. First of all, condition (iv) allows us to define $a := \lim_{n\to\infty} a_n$. Let us show that $a \in A$ satisfied the desired conditions. Since conditions (i)–(iii) are compatible with limits, we have:

(i) $0 \leqslant a \leqslant 1$;

(ii) $\|\pi_0(a)\| = 1$;

(iii) $\|\pi(a)\| \leqslant 1 - 2^{-k}$ for all irreducible representations π such that $\ker(\pi) \in \mathrm{Prim}(A) \setminus \mathrm{Prim}(J_k)$ for $k \geqslant 0$.

Thus a has the desired properties, which completes the proof. ∎

PROPOSITION 5.3. *Let A be a unital C^*-algebra. Let us assume that every $I \in \mathrm{Prim}(A)$ has a countable base for its system of neighborhoods. Then every strictly norming family $\mathcal{F}$ of representations of A is also exhaustive.*

Let us assume that $\mathrm{Prim}(A)$ is a T_1 space. Then the converse is also true, that is, if every strictly norming family $\mathcal{F}$ of representations of A is also exhaustive, then every $I \in \mathrm{Prim}(A)$ has a countable base for its system of neighborhoods.

We think that the condition that $\mathrm{Prim}(A)$ be T_1 is not necessary. However, as noticed by Roch, the proof below requires this assumption.

Proof. Let us prove first the first part of the statement, so let us assume that every primitive ideal $I \in \mathrm{Prim}(A)$ has a countable base for its system of neighborhoods and let $\mathcal{F}$ be a strictly norming family of representations of A. We need to show that $\mathcal{F}$ is exhaustive. We shall proceed by contradiction. Thus, let us assume that the family $\mathcal{F}$ is not exhaustive. Then there exists a primitive ideal $I_0 = \ker(\pi_0) \in \mathrm{Prim}(A) \setminus \bigcup_{\phi\in\mathcal{F}} \mathrm{supp}(\phi)$. Let

$$V_0 \supset \cdots \supset V_n \supset V_{n+1} \supset \cdots \supset \{I_0\} = \bigcap_k V_k$$

be a basis for the system of neighborhoods of I_0 in $\mathrm{Prim}(A)$. We may assume without loss of generality that the neighborhoods V_n consist of open sets. Corollary 5.2 then yields $a \in A$ such that $\|a\| = \|\pi_0(a)\| = 1$, but $\|\pi(a)\| \leqslant 1 - 2^k$ for any irreducible representation π of A such that $\ker(\pi) \in \mathrm{Prim}(A) \setminus V_k$. Then, for every $\phi \in \mathcal{F}$, we have that

$$\mathrm{Prim}(A) \setminus \mathrm{supp}(\phi) = \{I \in \mathrm{Prim}(A) : \ker(\phi) \not\subset I\} = \mathrm{Prim}(\ker(\phi))$$

is an open subset of $\mathrm{Prim}(A)$ containing I_0, and hence it is a neighborhood of I_0 in $\mathrm{Prim}(A)$. Therefore there exists n such that $V_n \subset \mathrm{Prim}(A) \setminus \mathrm{supp}(\phi)$ and hence $\|\pi(a)\| \leqslant 1 - 2^{-n}$ for all π such that $\ker(\pi) \in \mathrm{supp}(\phi)$. This gives in particular for $\pi = \phi$ that $\|\phi(a)\| \leqslant 1 - 2^{-n} < 1$, thus contradicting the fact that $\mathcal{F}$ is strictly norming. This proves the first half of the statement.

Let us prove the converse, that is, the second half of the statement, which is easier. Thus let us assume that every strictly norming family of representations of A is also exhaustive and let us prove that every primitive ideal $I_0 := \ker(\pi_0) \in \mathrm{Prim}(A)$ has a countable basis for its system of neighborhoods. Let us fix then $I_0 := \ker(\pi_0) \in \mathrm{Prim}(A)$ arbitrarily and show that it has a countable basis for its system of neighborhoods. Also, we associate to each primitive ideal $I \in \mathrm{Prim}(A)$ an irreducible representation ϕ_I with kernel I. By Remark 4.9, we have that the family of representations $\mathcal{F} := \{\phi_I : I \in \mathrm{Prim}(A), I \neq I_0\}$ is not exhaustive, since $\mathrm{Prim}(A)$ is a T_1 space (and hence its points are closed) and hence $\mathrm{supp}\,\phi_I = I$. By our assumption, the family $\mathcal{F}$ is hence also not strictly norming. Therefore, by the definition of a strictly norming family of representations, there exists $a \in A$, such that $\|\pi(a)\| < \|a\|$ for all irreducible π with $\ker(\pi) \neq I_0$. Note that since the family $\widehat{A}$ is strictly norming (see Example 4.2), we have that $\|a\| = \max_{\pi\in\widehat{A}} \|\pi(a)\|$, and hence $\|a\| = \|\pi_0(a)\|$. By rescaling, we can assume $\|a\| = \|\pi_0(a)\| = 1$. Then the sets

$$V_n := \{\ker(\pi) \in \mathrm{Prim}(A) : \|\pi(a)\| > 1 - 2^{-n}\}$$

are open neighborhoods of $I_0 := \ker(\pi_0)$ in $\mathrm{Prim}(A)$ by Lemma 2.2. Let us show that they form a basis for the system of neighborhoods of I_0. Indeed, let G be an arbitrary open subset of $\mathrm{Prim}(A)$ containing I_0. Then there exists a two-sided ideal $J \subset A$ such that $G = \mathrm{Prim}(J)$. The set of irreducible representations of A/J identifies with $\mathrm{Prim}(J)^{\mathrm{c}} := \mathrm{Prim}(A) \setminus \mathrm{Prim}(J)$, and hence it does not contain π_0. Hence $\|\pi(a)\| < 1$ for all $\pi \in \mathrm{Prim}(A/J)$. Since $\widehat{A/J}$ is a strictly norming family of representations of A/J, we obtain that $\|a + J\|_{A/J} < 1$ (the norm is in A/J). Let n be such that $\|a + J\|_{A/J} \leqslant 1 - 2^{-n}$. Then $V_n \subset \mathrm{Prim}(J) = G$, which completes the proof of the second half of this theorem. The proof is now complete. ∎

Clearly, there are C^*-algebras for which the spectrum is not T_1, but for which every strictly norming family of representations is also exhaustive. We do not know, however, if the converse result is true in full generality (that is, for every C^*-algebra). It is easy to show that separable C^*-algebras satisfy the assumptions of Proposition 5.3.

THEOREM 5.4. *Let A be a separable C^*-algebra. Then every primitive ideal $I \in \mathrm{Prim}(A)$ has a countable base for its system of neighborhoods. Consequently, if $\mathcal{F}$ is a strictly norming set of representations of A, then $\mathcal{F}$ is exhaustive.*

Proof. It is known [20] that $\mathrm{Prim}(A)$ is second countable. This gives the result in view of Proposition 5.3. For the benefit of the reader, we now provide a quick proof that every point in $\mathrm{Prim}(A)$, for A separable, has a countable base for its system of neighborhoods. Indeed, we can replace A with $\widetilde{A}$ and thus assume that A is unital. Let $\{a_n\}$ be a dense subset of A and fix $I_0 := \ker(\pi_0) \in \mathrm{Prim}(A)$. Define

$$V_n := \Big\{ \ker(\pi) \in \mathrm{Prim}(A) : \|\pi(a_n)\| > \frac{\|\pi_0(a_n)\|}{2} \Big\}.$$

Then each V_n is open by Lemma 2.2. We claim that V_n is a basis of the system of neighborhoods of $I_0 := \ker(\pi_0)$ in $\mathrm{Prim}(A)$. Indeed, let $G \subset \mathrm{Prim}(A)$ be an open set containing I_0. Then $G = \mathrm{Prim}(J)$ for some two-sided ideal of A such that $\pi_0 \neq 0$ on J. Let $a \in J$ such that $\pi_0(a) \neq 0$. By the density of the sequence a_n in A, we can find n such that $\|a - a_n\| < \|\pi_0(a)\|/4$. Then $\|\pi'(a) - \pi'(a_n)\| < \|\pi_0(a)\|/4$ for any irreducible representation π', and hence

$$\|\pi'(a)\| - \frac{\|\pi_0(a)\|}{4} < \|\pi'(a_n)\| < \|\pi'(a)\| + \frac{\|\pi_0(a)\|}{4}, \quad \forall \pi' \in \widehat{A}. \tag{5.1}$$

To show that $V_n \subset G$, it is enough to show that $V_n \cap G^c = V_n \cap \mathrm{Prim}(J)^c = \emptyset$. Suppose the contrary and let $\pi \in \widehat{A}$ be such that $\ker(\pi) \in V_n \cap \mathrm{Prim}(J)^c$. Then $\|\pi(a_n)\| > \|\pi_0(a_n)\|/2$, by the definition of V_n. Moreover, $\pi(a) = 0$ since $a \in J$ and π vanishes on J. Let us show that this is not possible. Indeed, using equation (5.1) twice, for $\pi' = \pi_0$ and for $\pi' = \pi$, we obtain

$$\frac{3}{8}\|\pi_0(a)\| < \frac{1}{2}\|\pi_0(a_n)\| < \|\pi(a_n)\| < \frac{1}{4}\|\pi_0(a)\|,$$

which is a contradiction. Consequently $V_n \subset G$ and hence $\{V_n\}$ is a basis for the system of neighborhoods of π_0 in $\mathrm{Prim}(A)$, as claimed. The last part follows from the first part of Proposition 5.3. ∎

Let $\mathcal{K}$ denote (as usual) the algebra of compact operators on some Hilbert space.

COROLLARY 5.5. *Let A be a separable C^*-algebra and $\mathcal{F}$ be a strictly spectral family of representations of A. Then $\mathcal{F} \otimes 1 := \{\pi \otimes 1\}$ is a strictly spectral family of representations of $A \otimes \mathcal{K}$.*

Proof. The ideals of A and $A \otimes \mathcal{K}$ correspond to each other by induction in such a way that primitive ideals correspond to primitive ideals and $\mathrm{Prim}(A)$ is homeomorphic to $\mathrm{Prim}(A \otimes \mathcal{K})$. Hence $\mathcal{F}$ is exhaustive if and only if $\mathcal{F} \otimes 1$ is exhaustive. The result then follows from Theorem 5.4. ∎

The next two basic examples illustrate the differences between the notions of faithful and strictly norming families.

EXAMPLE 5.6. Let in this example A be the C^*-algebra of continuous functions f on $[0,1]$ with values in $M_2(\mathbb{C})$ such that $f(1)$ is diagonal, which is a type I C^*-algebra, and thus we identify $\widehat{A}$ and $\mathrm{Prim}(A)$. Then the maps $\mathrm{ev}_t \colon f \mapsto f(t) \in M_2(\mathbb{C})$, for $t < 1$, together with the maps $\mathrm{ev}_1^i \colon f \mapsto f(1)_{ii}$ $(i = 0, 1)$ provide all the irreducible representations of A (up to equivalence). The family

$$\mathcal{F} = \{\mathrm{ev}_t : t < 1\} \cup \{\mathrm{ev}_1^1\}$$

is a faithful but not exhaustive family. In fact the function $t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1-t \end{pmatrix}$ is not invertible in A but $\pi(f)$ is invertible for all $\pi \in \mathcal{F}$. Of course, in this example, every $\pi \in \widehat{A} = \mathrm{Prim}(A)$ has a countable base for its system of neighborhoods, so every strictly norming family of representations $\mathcal{F}$ of A is also exhaustive.

We now consider the Toeplitz algebra.

EXAMPLE 5.7. Let $\mathcal{T}$ be the Toeplitz algebra, which is again a type I C^*-algebra, and thus we again identify $\widehat{\mathcal{T}}$ and $\mathrm{Prim}(\mathcal{T})$. The Toeplitz algebra $\mathcal{T}$ is defined as the C^*-algebra generated by the operator defined by the unilateral shift S. (Recall that S acts on the Hilbert space $L^2(\mathbb{N})$ by $S \colon \varepsilon_k \mapsto \varepsilon_{k+1}$.) As $S^*S = 1$ and $SS^* - 1$ is a rank 1 operator, one can prove that the following is an exact sequence

$$0 \to \mathcal{K} \to \mathcal{T} \to \mathcal{C}(S^1) \to 0,$$

where $\mathcal{K}$ is the algebra of compact operators. Extend the unique irreducible representation π of $\mathcal{K}$ to $\mathcal{T}$ as in [20]. Also, the irreducible characters χ_θ of S^1 pull-back to irreducible characters of $\mathcal{T}$ vanishing on $\mathcal{K}$. Then the spectrum of $\mathcal{T}$ is

$$\widehat{\mathcal{T}} = \{\pi\} \cup \{\chi_\theta : \theta \in S^1\},$$

with S^1 embedded as a closed subset. A subset $V \subset \mathrm{Prim}(\mathcal{T})$ will be open if and only if it contains π and its intersection with S^1 is open. We thus see that the single element set $\{\pi\}$ defines an exhaustive family. In other words $\widehat{\mathcal{T}} = \overline{\{\pi\}} = \mathrm{supp}(\pi)$. Since every exhaustive family is also strictly norming, by Proposition 4.12, the family $\{\pi\}$ consisting of a single representation is also strictly norming. We can see also directly that the family $\mathcal{F} = \{\pi\}$ (consisting of π alone) is strictly norming. Indeed, it suffices to notice that $\|x\| = \|\pi(x)\|$ for all x since π is injective. In this example again every $\pi' \in \widehat{\mathcal{T}} = \mathrm{Prim}(\mathcal{T})$ has a countable base for its system of neighborhoods, so every strictly norming family of representations $\mathcal{F}$ of $\mathcal{T}$ is also exhaustive.

Here is one more example that shows that the condition that A be separable is not necessary for the classes of exhaustive families of representations and strictly norming families of representations to coincide.

EXAMPLE 5.8. Let I be an infinite *uncountable* set. We endow it with the discrete topology. Then the algebras $A_0 := \mathcal{C}_0(I)$ and $A_1 := \mathcal{K}(\ell^2(I))$ (the algebra of compact operators on $\ell^2(I)$) are not separable. Fix $i \in \{0,1\}$ and let $\mathcal{F}$ be a strictly norming family of representations of A_i, then $\mathcal{F}$ is also an exhaustive family of representations of A_i.

6. UNBOUNDED OPERATORS

The results of the previous sections are relevant often in applications to unbounded operators, so we now extend Theorem 4.4 to (possibly) unbounded observables affiliated to C^*-algebras. We begin with an abstract setting. Recall that our convention is that all morphisms (respectively, representations) of C^*-algebras are $*$-morphisms (respectively, $*$-representations).

6.1. ABSTRACT AFFILIATED OPERATORS. The notion of an affiliated self-adjoint operator to a C^*-algebra has been extensively and successfully studied due to its connections with (non-compact) quantum groups [6], [7], [9], [28], [33], [57], [58] and to its connections to the C^*-algebra approach to spectral theory [15], [24]. See also [27], [42] for results on unbounded operators on Hilbert modules [13], [29], [34], which are quite useful in defining the product in bivariant K-theory; see [8], for instance. Although our use of affiliated operators is strictly technical and our goals concerning them are very limited, let us briefly recall that in [57], an *unbounded operator affiliated to a C^*-algebra A* is defined as a suitable unbounded operator acting on A, by analogy with the similar concept in von Neumann algebra theory. See also [5] for a thorough discussion of affiliated operators in the framework of C^*-algebras.

In this paper, we shall use the related notion of an unbounded *observable* affiliated to a C^*-algebra due to Damak and Georgescu (see Remark 6.8 for a brief comparison of the two definitions).

DEFINITION 6.1 (Damak–Georgescu). Let A be a C^*-algebra. A self-adjoint *observable T affiliated to A* is a morphism $\theta_T : \mathcal{C}_0(\mathbb{R}) \to A$ of C^*-algebras. The observable T is said to be *strictly affiliated to A* if the space generated by elements of the form $\theta_T(h)a$, with $a \in A$ and $h \in \mathcal{C}_0(\mathbb{R})$, is dense in A.

Definition 6.1 provides an "integrated" form of the unbounded operators considered by Damak and Georgescu, in the sense that we rather consider the bounded functions that vanish at infinity of the operator, instead of the operator itself, as in [24]. We however drop any reference to a particular Hilbert space and or to any actual operator. The reason for using the term "affiliated observable" instead of "affiliated operator" is to better distinguish between the concepts in [24] and [57].

As in the classical case, we found it useful to consider the *Cayley transform* of an affiliated observable, which we introduce below. We first notice that an observable affiliated to A extends to a morphism $\widetilde{\theta}_T \colon \widetilde{\mathcal{C}_0(\mathbb{R})} \to \widetilde{A}$ (the algebra obtained from A by adjunction of a unit). If moreover T is strictly affiliated to A, then θ_T extends to a morphism from $C_{\mathrm{b}}(\mathbb{R})$ to the multiplier algebra of A [15], but we shall not need this fact, except in Remark 6.8. See also [5], [6].

DEFINITION 6.2. Let T be an observable affiliated to A. The *Cayley transform* $u_T \in \widetilde{A}$ of T is

$$u_T := \widetilde{\theta}_T(h_0), \quad \text{where } h_0(z) := (z+\mathrm{i})(z-\mathrm{i})^{-1}. \tag{6.1}$$

The Cayley transform allows us to reduce questions about the spectrum of an observable to questions about the spectrum of its Cayley transform. Let us first introduce, however, the *spectrum of an affiliated observable.* Let thus $\theta_T : \mathcal{C}_0(\mathbb{R}) \to A$ be a self-adjoint observable affiliated to a C^*-algebra A. The kernel of θ_T is then of the form $\mathcal{C}_0(U)$, for some open subset of $\mathbb{R}$. We define the spectrum $\mathrm{Spec}_A(T)$ as the complement of U in $\mathbb{R}$. Explicitly,

$$\mathrm{Spec}_A(T) = \{\lambda \in \mathbb{R} : h(\lambda) = 0, \forall h \in \mathcal{C}_0(\mathbb{R}) \text{ such that } \theta_T(h) = 0\}. \tag{6.2}$$

We allow the case $\mathrm{Spec}_A(T) = \emptyset$, which corresponds to the case $\theta_T = 0$, in which case we shall write $T = \infty$, by definition. See also the discussion following Theorem 1.2 in [24]. If $\sigma : A \to B$ is a morphism of C^*-algebras, then $\sigma \circ \theta_T : \mathcal{C}_0(\mathbb{R}) \to A$ is an observable $\sigma(T)$ affiliated to the C^*-algebra B and

$$\mathrm{Spec}_B(\sigma(T)) \subset \mathrm{Spec}_A(T). \tag{6.3}$$

If σ is injective, then $\mathrm{Spec}_B(\sigma(T)) = \mathrm{Spec}_A(T)$, which shows that the spectrum is preserved by increasing the C^*-algebra A. Note that

$$\sigma(u_T) = u_{\sigma(T)}. \tag{6.4}$$

Let $h_0(z) := (z+\mathrm{i})(z-\mathrm{i})^{-1}$, as before.

LEMMA 6.3. *The spectrum* $\mathrm{Spec}(T)$ *of the self-adjoint observable* $\theta_T : \mathcal{C}_0(\mathbb{R}) \to A$ *affiliated to the* C^**-algebra* A *and the spectrum* $\mathrm{Spec}(u_T)$ *of its Cayley transform are related by*

$$\mathrm{Spec}(T) = h_0^{-1}(\mathrm{Spec}(u_T)).$$

Proof. This follows from the fact that h_0 is a homeomorphism of $\mathbb{R}$ onto its image in $S^1 := \{|z| = 1\}$ and from the properties of the functional calculus. ∎

Let us notice that the above lemma is valid also in the case when

$$T = \infty \Leftrightarrow \theta_T = 0 \Leftrightarrow \mathrm{Spec}(T) = \emptyset \Leftrightarrow u_T = 1 \Leftrightarrow \sigma(u_T) = \{1\}.$$

One can make the relation in the above lemma more precise by saying that, for bounded T, we have $h_0(\mathrm{Spec}(T)) = \mathrm{Spec}(u_T)$, whereas for unbounded T we have

$$\overline{h_0(\mathrm{Spec}(T))} = h_0(\mathrm{Spec}(T)) \cup \{1\} = \mathrm{Spec}(u_T), \tag{6.5}$$

where $h_0(z) := (z+\mathrm{i})(z-\mathrm{i})^{-1}$, as before.

Here is our main result on (possibly unbounded) self-adjoint operators affiliated to C^*-algebras.

THEOREM 6.4. *Let A be a unital C^*-algebra and T a self-adjoint observable affiliated to A. Let $\mathcal{F}$ be a set of representations of A.*

(i) *If $\mathcal{F}$ is strictly norming, then*

$$\mathrm{Spec}(T) = \bigcup_{\phi\in\mathcal{F}} \mathrm{Spec}(\phi(T)).$$

(ii) *If $\mathcal{F}$ is faithful, then*

$$\mathrm{Spec}(T) = \overline{\bigcup_{\phi\in\mathcal{F}}} \mathrm{Spec}(\phi(T)).$$

Proof. The proofs of (i) and (ii) are similar, using the relation $\mathrm{Spec}(T) = h_0^{-1}(\mathrm{Spec}(u_T))$ of Lemma 6.3. We begin with (i), which is slightly easier. Since $\mathcal{F}$ is strictly norming, we can then apply Theorem 4.6 to $u_T \in \widetilde{A}$ and the family $\sigma \in \mathcal{F}$. We obtain

$$\begin{aligned}\mathrm{Spec}(T) &= h_0^{-1}[\mathrm{Spec}(u_T)] = h_0^{-1}\Big[\bigcup_{\sigma\in\mathcal{F}} \mathrm{Spec}(\sigma(u_T))\Big]\\ &= h_0^{-1}\Big[\bigcup_{\sigma\in\mathcal{F}} \mathrm{Spec}(u_{\sigma(T)})\Big] = \bigcup_{\sigma\in\mathcal{F}} h_0^{-1}[\mathrm{Spec}(u_{\sigma(T)})] = \bigcup_{\sigma\in\mathcal{F}} \mathrm{Spec}(\sigma(T)).\end{aligned}$$

If, on the other hand, $\mathcal{F}$ is faithful, we apply Proposition 3.7 after noting that h_0 is a homeomorphism of $\mathbb{R}$ onto its image in $S^1 := \{|z| = 1\}$ and hence $h_0^{-1}(\overline{S}) = \overline{h_0^{-1}(S)}$ for any $S \subset S^1$. The same argument then gives

$$\begin{aligned}\mathrm{Spec}(T) &= h_0^{-1}[\mathrm{Spec}(u_T)] = h_0^{-1}\Big[\overline{\bigcup_{\sigma\in\mathcal{F}}} \mathrm{Spec}(\sigma(u_T))\Big]\\ &= h_0^{-1}\Big[\overline{\bigcup_{\sigma\in\mathcal{F}}} \mathrm{Spec}(u_{\sigma(T)})\Big] = \overline{\bigcup_{\sigma\in\mathcal{F}}} h_0^{-1}[\mathrm{Spec}(u_{\sigma(T)})] = \overline{\bigcup_{\sigma\in\mathcal{F}}} \mathrm{Spec}(\sigma(T)).\end{aligned}$$

The proof is now complete. ∎

6.2. THE CASE OF "TRUE" OPERATORS. We now look at concrete (true) operators. The following remark discusses the relation between affiliated observables and unbounded operators.

REMARK 6.5. Let $A \subset \mathcal{L}(\mathcal{H})$ be a sub-C^*-algebra of $\mathcal{L}(\mathcal{H})$. A (possibly unbounded) self-adjoint operator $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}$ is then *an observable affiliated* *to A if, for every continuous functions h on the spectrum of T vanishing at infinity, we have $h(T) \in A$.* (This is the exact definition in [15].) The reason for considering our slight generalization is to make it easier to deal with homomorphic images. We then have that T is affiliated to A if and only if $(T-\lambda)^{-1} \in A$ for one $\lambda \notin \mathrm{Spec}(T)$ (equivalently for all such λ) [15]. We thus see that a self-adjoint operator T affiliated to A defines a morphism $\theta_T : \mathcal{C}_0(\mathbb{R}) \to A$, $\theta_T(h) := h(T)$ such that $\mathrm{Spec}(T) = \mathrm{Spec}(\theta_T)$. By classical results, if $(u_T - 1)$ is injective, then we can define a true self-adjoint operator $T := \mathrm{i}(u_T+1)(u_T-1)^{-1} \in A$ such that $\theta_T(h) = h(T)$, $h \in \mathcal{C}_0(\mathbb{R})$ [19]. This is the case, for instance, if $\mathrm{Spec}(T)$ is a bounded subset of $\mathbb{R}$, in which case we shall say that T is *bounded*. If θ_T is non-degenerate, then $u_T - 1$ is injective, and hence θ_T yields a true unbounded operator, which can also be defined by $T := \theta_T(h_n)$, where h_n are continuous, bounded, $|h_n(t)| \leqslant |t|$, and $h_n(t) \to t$ for all $t \in \mathbb{R}$.

Recall that a (possibly unbounded) operator T is invertible if and only if it is bijective and T^{-1} is bounded. This is also equivalent to $0 \notin \mathrm{Spec}(T)$. The following remark is then also useful.

REMARK 6.6. In any case, if T is a self-adjoint operator, then the definition of $\mathrm{Spec}(T)$ in terms of θ_T coincides with the classical spectrum of T defined using the resolvent, whether T is bounded or not: $\mathrm{Spec}(\theta_T) = \mathrm{Spec}(T)$. In view of the remarks preceding it, Theorem 6.4 remains valid for true self-adjoint operators T.

We have the following analog of Proposition 3.4 and Theorem 4.4

THEOREM 6.7. *Let $A \subset \mathcal{L}(\mathcal{H})$ be a unital C^*-algebra and T a self-adjoint operator that is an observable affiliated to A. Let $\mathcal{F}$ be a set of representations of A.*

(i) *Let $\mathcal{F}$ be strictly norming. Then T is invertible if and only if $\phi(T)$ is invertible for all $\phi \in \mathcal{F}$.*

(ii) *Let $\mathcal{F}$ be faithful. Then T is invertible if and only if $\phi(T)$ is invertible for all $\phi \in \mathcal{F}$ and the set $\{\|\phi(T)^{-1}\| : \phi \in \mathcal{F}\}$ is bounded.*

Proof. This follows from Theorem 6.4 as follows. First of all, we have that T is invertible if and only if $0 \notin \mathrm{Spec}(T)$. Now, if $\mathcal{F}$ is strictly norming, we have

$$0 \notin \mathrm{Spec}(T) \Leftrightarrow 0 \notin \bigcup_{\phi \in \mathcal{F}} \mathrm{Spec}(\phi(T)) \Leftrightarrow 0 \notin \mathrm{Spec}(\phi(T)) \quad \text{for all } \phi \in \mathcal{F}.$$

This proves (i). (Note that $\phi(T)$ may not be a true operator, but only an affiliated observable.) To prove (ii), we proceed similarly, noticing also that the distance from 0 to $\mathrm{Spec}(T)$ is exactly $\|T^{-1}\|$. ∎

We now compare the notions of "affiliation" from [15] and [57]. It turns out that neither concept implies in an obvious way the other.

REMARK 6.8. In [57], an *operator affiliated* to A was defined as an unbounded map L on A such that there exists $z \in A$ with the property that $\|z\| \leqslant 1$ and $Lx = y$ if and only if $x = (1 - z^*z)^{1/2}a$ and $y = za$ for some $a \in A$. So, formally, one could think of L as being the operator on A induced by left multiplication by $z(1-z^*z)^{-1/2}$. Similarly, if θ_T is an observable strictly affiliated to A, let $\chi : \mathbb{R} \to \mathbb{R}$ be given by $\chi(t) = t(1+t^2)^{-1/2}$ and let $z = \theta_T(\chi) \in M(A)$, the multiplier algebra of A. Then we can formally think of $\theta_T(\mathrm{id})$ as the operator by left multiplication with $z(1-z^2)^{-1/2}$, by the properties of the functional calculus. This is similar to Woronowicz's definition, *except that* in the case of an affiliated observable T one has $z = z^*$. Another restriction in the case of an affiliated observable T is that its Cayley transform $u_T \in \widetilde{A} = A + \mathbb{C}1$, whereas for self-adjoint operators affiliated to a C^*-algebra one allows the Cayley transform to be in the multiplier algebra $M(A)$. In particular, if θ_T is bounded, then $T := \theta_T(\mathrm{id}) = z(1-z^2)^{-1/2}$ is defined and is self-adjoint (whereas T in Woronowicz's definition T does not have to be self-adjoint). This seems to imply that the definition in [57] is more general than the one in [15], however, if A is unital and L is an operator affiliated to A, then $L \in A$, by Proposition 1.3 in [57]. This is very far from being the case in [15] (which is our framework as well).

7. PARAMETRIC PSEUDODIFFERENTIAL OPERATORS

Let M be a compact smooth Riemannian manifold and G be a Lie group (finite dimensional) with Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$. We let G act by left translations on $M \times G$. We denote by $\Psi^0(M \times G)^G$ the algebra of order 0, G-invariant pseudodifferential operators on $M \times G$ and $\overline{\Psi^0(M \times G)^G}$ be its norm closure acting on $L^2(M \times G)$. For any vector bundle E, we denote by S^*E the set of directions in its dual E^*. If E is endowed with a metric, then S^*E can be identified with the set of unit vectors in E^*. We shall be interested the quotient

$$S^*(T(M \times G))/G = S^*(TM \times TG)/G = S^*(TM \times \mathfrak{g}).$$

We have that $\overline{\Psi^0(M \times G)^G} \simeq C^*_{\mathrm{r}}(G) \otimes \mathcal{K}$ and then obtain the exact sequence

$$0 \to C^*_{\mathrm{r}}(G) \otimes \mathcal{K} \to \overline{\Psi^0(M \times G)^G} \to \mathcal{C}(S^*(M \times \mathfrak{g})) \to 0, \tag{7.1}$$

[30], [31], [36], [54]. Note that the kernel of the symbol map will now have irreducible representations parametrized by $\widehat{G}_{\mathrm{r}}$ the temperate unitary irreducible representations of G. Let $T \in \Psi^m(M \times G)^G$ and denote by $T^\sharp \in \Psi^m(M \times G)^G$ its formal adjoint (defined using the calculus of pseudodifferential operators). All operators considered below are closed with minimal domain (the closure of the operators defined on $\mathcal{C}^\infty_{\mathrm{c}}(M \times G)$). We denote by T^* the Hilbert space adjoint of a (possibly unbounded) densely defined operator.

LEMMA 7.1. *Let $T \in \Psi^m(M \times G)^G$ be elliptic. Then $T^* = T^\sharp$. Thus, if also $T = T^\sharp$, then T is self-adjoint and $(T + \mathrm{i})^{-1} \in C^*_{\mathrm{r}}(G)$, and hence it is affiliated to $C^*_{\mathrm{r}}(G)$.*

Proof. This is a consequence of the fact that $\Psi^\infty(M \times G)^G$ is closed under multiplication and formal adjoints. See [30], [31], [54] for details, where the calculus of pseudodifferential operators on groupoids is used, for the groupoid $M \times M \times G$, which is the product of the pair groupoid $M \times M$ and of the group G. ∎

In other words, any elliptic, formally self-adjoint $T \in \Psi^m(M \times G)^G$, $m > 0$, is actually self-adjoint.

Let us assume $G = \mathbb{R}^n$, regarded as an abelian Lie group. Then our exact sequence (7.1) becomes

$$0 \to \mathcal{C}_0(\mathbb{R}^n) \otimes \mathcal{K} \to \overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}} \to \mathcal{C}(S^*(TM \times \mathbb{R}^n)) \to 0. \tag{7.2}$$

This shows that $A := \overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}$ is a type I C^*-algebra, and hence we can identify $\widehat{A}$ and $\operatorname{Prim}(A)$. Then we use that, to each $\lambda \in \mathbb{R}^n$, there corresponds an irreducible representation ϕ_λ of $\mathcal{C}_0(\mathbb{R}^n) \otimes \mathcal{K}$. Recalling that every irreducible (bounded, $*$) representation of an ideal I in a C^*-algebra A extends uniquely to a representation of A, we obtain that ϕ_λ extends uniquely to an irreducible representation of $\overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}$ denoted with the same letter. It is customary to denote by $\widehat{T}(\lambda) := \phi_\lambda(T)$ for T a pseudodifferential operator in $\Psi^m(M \times \mathbb{R}^n)^{\mathbb{R}^n}$, $m \geqslant 0$. To define $\widehat{T}(\lambda)$ for $m > 0$, we can either use the Fourier transform or, notice that Δ is affiliated to the closure of $\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}$. This allows us to define $\widehat{\Delta}(\lambda)$. In general, we write $T = (1-\Delta)^k S$, with $S \in \Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}$ and define $\widehat{T}(\lambda)q = \widehat{(1-\Delta)}(\lambda)^k \widehat{S}(\lambda)$. (We consider the "analyst's" Laplacian, so $\Delta \leqslant 0$.)

LEMMA 7.2. *Let $A := \overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}$. Then the primitive ideal spectrum of A,* $\operatorname{Prim}(A)$, *is in a canonical bijection with the disjoint union $\mathbb{R}^n \cup S^*(TM \times \mathbb{R}^n)$, where the copy of $\mathbb{R}^n$ corresponds to the open subset $\{\phi_\lambda : \lambda \in \mathbb{R}^n\}$ and the copy of $S^*(TM \times \mathbb{R}^n)$ corresponds to the closed subset $\{e_p : p \in S^*(TM \times \mathbb{R}^n)\}$. The induced topologies on $\mathbb{R}^n$ and $S^*(TM \times \mathbb{R}^n)$ are the standard ones. Let $S^*M := S^*(TM) \subset S^*(TM \times \mathbb{R}^n)$ correspond to $T^*M \subset T^*M \times \mathbb{R}^n$. Then the closure of $\{\phi_\lambda\}$ in* $\operatorname{Prim}(A)$ *is $\{\phi_\lambda\} \cup S^*M$.*

Proof. By standard properties of C^*-algebras (the definition of the Jacobson topology), the ideal $\mathcal{C}_0(\mathbb{R}^n) \otimes \mathcal{K} \subset A$ defines an open subset of $\operatorname{Prim}(A)$ with complement $\operatorname{Prim}(A/I)$ with the induced topologies. This proves the first part of the statement.

In order to determine the closure of $\{\phi_\lambda\}$ in $\operatorname{Prim}(A)$, let us notice that the principal symbol of $\widehat{T}(\lambda)$ can be calculated in local coordinate carts on M (more precisely, on sets of the form $U \times \mathbb{R}^n$, with U a coordinate chart in M). This gives that the principal symbol of $\widehat{T}(\lambda)$ is given by the restriction of the principal symbol of T to S^*M.

Indeed, let $U = \mathbb{R}^k$. A translation invariant pseudodifferential P operator on $U \times \mathbb{R}^n = \mathbb{R}^{k+n}$ is of the form $P = a(x, y, D_x, D_y)$ with a independent of y: $a(x, y, \xi, \eta) = \tilde{a}(x, \xi, \eta)$. With this notation, we have $\widehat{P}(\lambda) = \tilde{a}(x, D_x, \lambda)$. The principal symbol of $\widehat{P}(\lambda)$ is then the principal symbol of the (global) symbol $\mathbb{R}^k \ni (x, \xi) \to \widetilde{a}(x, \xi, \lambda)$, and is seen to be independent of the (finite) value of $\lambda \in \mathbb{R}^n$ and is the restriction from $S^*(TU \times \mathbb{R}^n)$ to $S^*(TU \times \{0\})$ of the principal symbol of $\widetilde{a}$.

Returning to the general case, the same reasoning gives that the image of ϕ_λ is $\overline{\Psi^0(M)}$. The primitive ideal spectrum of this algebra is canonically homeomorphic to the closure of $\{\phi_\lambda\}$, and this is enough to complete the proof. ∎

By the exact sequence (7.1), in addition to the irreducible representations ϕ_λ, $\lambda \in \mathbb{R}^n$ (or, more precisely, their kernels), $\mathrm{Prim}(A)$ contains also (the kernels of) the irreducible representations $e_p(T) = \sigma_0(T)(p)$, $p \in S^*(TM \times \mathbb{R}^n)$.

PROPOSITION 7.3. *Let* $\mathcal{F} := \{\phi_\lambda : \lambda \in \mathbb{R}^n\} \cup \{e_p : p \in S^*(TM \times \mathbb{R}^n) \setminus S^*M\}$.

(i) *The family* $\mathcal{F}$ *is a strictly norming family of representations of* $\overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}$.

(ii) *Let* $P \in \Psi^m(M \times \mathbb{R}^n)^{\mathbb{R}^n}$*, then* $P : H^s(M \times \mathbb{R}^n) \to H^{s-m}(M \times \mathbb{R}^n)$ *is invertible if and only if* $\widehat{P}(\lambda) : H^s(M) \to H^{s-m}(M)$ *is invertible for all* $\lambda \in \mathbb{R}^n$ *and the principal symbol of* P *is non-zero on all rays not intersecting* S^*M.

(iii) *If* $T \in \Psi^m(M \times \mathbb{R}^n)^{\mathbb{R}^n}$, $m > 0$, *is formally self-adjoint and elliptic, then we have* $\mathrm{Spec}(e_p(T)) = \emptyset$, *and hence*

$$\mathrm{Spec}(T) = \bigcup_{\lambda \in \mathbb{R}^n} \mathrm{Spec}(\widehat{T}(\lambda)).$$

Proof. (i) follows from Lemma 7.2. To prove (ii), let us denote by $\Delta_M \leqslant 0$ the (non-positive) Laplace operator on M. Then the Laplace operator Δ on $M \times \mathbb{R}^n$ is $\Delta = \Delta_{\mathbb{R}^n} + \Delta_M$. Note that $(1-\Delta)^{m/2} : H^s(M \times \mathbb{R}^n) \to H^{s-m}(M \times \mathbb{R}^n)$ and $(c - \Delta_M)^{m/2} : H^s(M) \to H^{s-m}(M)$, $c > 0$, are isomorphisms. By [31], we have that

$$P_1 := (1-\Delta)^{(s-m)/2} P (1-\Delta)^{-s/2} \in A := \overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}.$$

It is then enough to prove that P_1 is invertible on $L^2(M \times \mathbb{R}^n)$. Moreover from part (i) we have just proved and Theorem 4.4 we know that P_1 is invertible on $L^2(M \times \mathbb{R}^n)$ if and only if $\widehat{P_1}(\lambda) := \phi_\lambda(P_1)$ is invertible on $L^2(M)$ for all $\lambda \in \mathbb{R}^n$ and the principal symbol of P_1 is non-zero on all rays not intersecting S^*M. But, using also $\widehat{1-\Delta}(\lambda) = (1 + |\lambda|^2 - \Delta_M)$, we have

$$\widehat{P_1}(\lambda) = (1 + |\lambda|^2 - \Delta_M)^{(s-m)/2} \widehat{P}(\lambda) (1 + |\lambda|^2 - \Delta_M)^{-s/2},$$

which is invertible by assumption.

To prove (iii), we recall that T is affiliated to A, by Lemma 7.1. The result then follows from Theorem 6.4(i). ∎

Operators of the kind considered in this subsection were used also in [1], [12], [16], [32], [37], [51], [52]. They turn out to be useful also for general topological index theorems [22], [41]. A more class of operators than the ones considered in this subsection were introduced in [3], [4]. The above result has turned out to be useful for the study of layer potentials [43]. There are, of course, many other relevant examples, but developing them would require too much additional materials, so we plan to discuss these other examples somewhere else.

Acknowledgements. We thank V. Georgescu for useful discussions and for providing us copies of his papers. We also thank D. and I. Beltiţă, S. Baaj, M. Măntoiu, J. Renault, and G. Skandalis for useful comments. The first named author would like also thank the Max Planck Institute for Mathematics in Bonn, where part of this work was completed, for its hospitality. After a first version of this work has been circulated, we have learned of the nice paper of Exel [23], which has also prompted us to change some of the terminology used in this paper. The authors were partially supported by ANR-14-CE25-0012-01 (SINGSTAR).

REFERENCES

[1] B. AMMANN, R. LAUTER, V. NISTOR, Pseudodifferential operators on manifolds with a Lie structure at infinity, *Ann. of Math. (2)* **165**(2007), 717–747.

[2] C. ANANTHARAMAN-DELAROCHE, J. RENAULT, *Amenable Groupoids*, Monogr. Enseignement Math., vol. 36, L'Enseignement Math., Geneva 2000.

[3] I. ANDROULIDAKIS, G. SKANDALIS, The holonomy groupoid of a singular foliation, *J. Reine Angew. Math.* **626**(2009), 1–37.

[4] I. ANDROULIDAKIS, G. SKANDALIS Pseudodifferential calculus on a singular foliation, *J. Noncommut. Geom.* **5**(2011), 125–152.

[5] S. BAAJ, Multiplicateurs nonbornés, *Thèse 3ème cycle*, 1980. Paris VI.

[6] S. BAAJ, E. BLANCHARD, G. SKANDALIS, Unitaires multiplicatifs en dimension finie et leurs sous-objets, *Ann. Inst. Fourier (Grenoble)* **49**(1999), 1305–1344.

[7] S. BAAJ, P. JULG, Théorie bivariante de Kasparov et opérateurs non bornés dans les C^*-modules hilbertiens, *C.R. Acad. Sci. Paris* **296**(1983), 875–878.

[8] S. BAAJ, G. SKANDALIS, C^*-algèbres de Hopf et théorie de Kasparov équivariante, *K-Theory* **2**(1989), 683–721.

[9] S. BAAJ, G. SKANDALIS, S. VAES, Non-semi-regular quantum groups coming from number theory, *Comm. Math. Phys.* **235**(2003), 139–167.

[10] A. BÖTTCHER, B. SILBERMANN, *Analysis of Toeplitz Operators*, second edition, Springer Monogr. Math., Springer-Verlag, Berlin 2006.

[11] M. BUNECI, Groupoid C^*-algebras, *Surv. Math. Appl.* **1**(2006), 71–98 (electronic).

[12] A. CONNES, Sur la théorie non commutative de l'intégration, in *Algèbres d'opérateurs (Sém., Les Plans-sur-Bex, 1978)*, Lecture Notes in Math., vol. 725, Springer, Berlin 1979, pp. 19–143.

[13] A. CONNES, *Noncommutative Geometry*, Academic Press, San Diego 1994.

[14] M. DĂDÂRLAT, Continuous fields of C^*-algebras over finite dimensional spaces, *Adv. Math.* **222**(2009), 1850–1881.

[15] M. DAMAK, V. GEORGESCU, Self-adjoint operators affiliated to C^*-algebras, *Rev. Math. Phys.* **16**(2004), 257–280.

[16] C. DEBORD, J.-M. LESCURE, K-duality for stratified pseudomanifolds, *Geom. Topol.* **13**(2009), 49–86.

[17] C. DEBORD, J.-M. LESCURE, F. ROCHON, Pseudodifferential operators on manifolds with fibred corners, *Ann. Inst. Fourier*, to appear, arXiv:1112.4575.

[18] C. DEBORD, G. SKANDALIS, Adiabatic groupoid, crossed product by $\mathbb{R}_+^*$ and pseudodifferential calculus, *Adv. Math.* **257**(2014), 66–91.

[19] J. DEREZIŃSKI, C. GÉRARD, *Scattering Theory of Classical and Quantum N-particle Systems*, Texts Monogr. Phys., Springer-Verlag, Berlin 1997.

[20] J. DIXMIER, *Les C^*-algèbres et leurs représentations*, second edition, Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics], Éditions Jacques Gabay, Paris 1996.

[21] S. ECHTERHOFF, D. WILLIAMS, Inducing primitive ideals, *Trans. Amer. Math. Soc.* **360**(2008), 6113–6129.

[22] H. EMERSON, R. MEYER, Equivariant embedding theorems and topological index maps, *Adv. Math.* **225**(2010), 2840–2882.

[23] R. EXEL, Invertibility in groupoid C^*-algebras, in *Operator Theory, Operator Algebras and Applications*, Oper. Theory Adv. Appl., vol. 242, Birkhäuser, Basel 2014, pp. 173–183.

[24] V. GEORGESCU, A. IFTIMOVICI, C^*-algebras of quantum Hamiltonians, in *Operator Algebras and Mathematical Physics (Constanţa, 2001)*, Theta, Bucharest 2003, pp. 123–167.

[25] V. GEORGESCU, V. NISTOR, On the essential spectrum of N-body hamiltonians with asymptotically homogeneous interactions, arxiv.org/abs/1506.03267 [math.SP].

[26] M. IONESCU, D. WILLIAMS, The generalized Effros–Hahn conjecture for groupoids, *Indiana Univ. Math. J.* **58**(2009), 2489–2508.

[27] J. KAAD, M. LESCH, A local global principle for regular operators in Hilbert C^*-modules, *J. Funct. Anal.* **262**(2012), 4540–4569.

[28] CH. KASSEL, *Quantum Groups*, Grad. Texts Math., vol. 155, Springer-Verlag, New York 1995.

[29] E.C. LANCE, *Hilbert C^*-Modules*, London Math. Soc. Lecture Note Ser., vol. 210, Cambridge Univ. Press, Cambridge 1995.

[30] R. LAUTER, B. MONTHUBERT, V. NISTOR, Pseudodifferential analysis on continuous family groupoids, *Doc. Math.* **5**(2000), 625–655 (electronic).

[31] R. LAUTER, V. NISTOR, Analysis of geometric operators on open manifolds: a groupoid approach, in *Quantization of Singular Symplectic Quotients*, Progr. Math., vol. 198, Birkhäuser, Basel 2001, pp. 181–229.

[32] M. LESCH, M. PFLAUM, Traces on algebras of parameter dependent pseudodifferential operators and the η-invariant, *Trans. Amer. Math. Soc.* **352**(2000), 4911–4936.

[33] Y. MANIN, *Topics in Noncommutative Geometry*, Princeton Univ. Press, Princeton, New Jersey 1991.

[34] V.M. MANUILOV, E.V. TROITSKY, *Hilbert C^*-Modules*, Transl. Math. Monogr., vol. 226, Amer. Math. Soc., Providence, RI 2005.

[35] M. MĂNTOIU, C^*-algebras, dynamical systems at infinity and the essential spectrum of generalized Schrödinger operators, *J. Reine Angew. Math.* **550**(2002), 211–229.

[36] B. MONTHUBERT, Pseudodifferential calculus on manifolds with corners and groupoids, *Proc. Amer. Math. Soc.* **127**(1999), 2871–2881.

[37] B. MONTHUBERT, F. PIERROT, Indice analytique et groupoïdes de Lie, *C.R. Acad. Sci. Paris* **325**(1997), 193–198.

[38] S.A. NAZAROV, B.A. PLAMENEVSKY, *Elliptic Problems in Domains with Piecewise Smooth Boundaries*, de Gruyter Exp. Math., vol. 13, Walter de Gruyter, Berlin 1994.

[39] V. NISTOR, Analysis on singular spaces: Lie manifolds and operator algebras, arxiv.org/abs/1512.06575.

[40] V. NISTOR, Desingularization of Lie groupoids and pseudodifferential operators on singular spaces, arxiv.org/abs/1512.08613 [math.DG].

[41] V. NISTOR, E. TROITSKY, The Thom isomorphism in gauge-equivariant K-theory, in *C^*-Algebras and Elliptic Theory*, Trends Math., Birkhäuser, Basel 2006, pp. 213–245.

[42] A. PAL, Regular operators on Hilbert C^*-modules, *J. Operator Theory* **42**(1999), 331–350.

[43] Y. QIAO, V. NISTOR, Single and double layer potentials on domains with conical points. I: Straight cones, *Integral Equations Operator Theory* **72**(2012), 419–448.

[44] V. RABINOVICH, S. ROCH, B. SILBERMANN, *Limit Operators and their Applications in Operator Theory*, Oper. Theory Adv. Appl., vol. 150, Birkhäuser, Basel 2004.

[45] J. RENAULT, *A Groupoid Approach to C^*-Algebras*, Lecture Notes Math., vol. 793, Springer, Boston 1980.

[46] J. RENAULT, The ideal structure of groupoid crossed product C^*-algebras, *J. Operator Theory* **25**(1991), 3–36.

[47] M. RIEFFEL, Induced representations of C^*-algebras, *Adv. in Math.* **13**(1974), 176–257.

[48] S. ROCH, Algebras of approximation sequences: structure of fractal algebras, in *Singular Integral Operators, Factorization and Applications*, Oper. Theory Adv. Appl., vol. 142, Birkhäuser, pp. 287–310.

[49] S. ROCH, P. SANTOS, B. SILBERMANN, *Non-commutative Gelfand Theories*, Universitext, Springer-Verlag London, Ltd., London 2011.

[50] M. RØRDAM, F. LARSEN, N. LAUSTSEN, *An Introduction to K-Theory for C^*-Algebras*, London Math. Soc. Stud. Texts, vol. 49, Cambridge Univ. Press, Cambridge 2000.

[51] E. SCHROHE, Fréchet algebra techniques for boundary value problems on noncompact manifolds: Fredholm criteria and functional calculus via spectral invariance, *Math. Nachr.* **199**(1999), 145–185.

[52] E. SCHROHE, B.-W. SCHULZE, Boundary value problems in Boutet de Monvel's algebra for manifolds with conical singularities. I, in *Pseudo-differential Calculus and Mathematical Physics*, Math. Top., vol. 5, Akademie Verlag, Berlin 1994, pp. 97–209.

[53] B.-W. SCHULZE, *Pseudo-differential Operators on Manifolds with Singularities*, Stud. Math. Appl., vol. 24, North-Holland Publ. Co., Amsterdam 1991.

[54] S. VASSOUT, Unbounded pseudodifferential calculus on Lie groupoids, *J. Funct. Anal.* **236**(2006), 161–200.

[55] D.V. VOICULESCU, K.J. DYKEMA, A. NICA, *Free Random Variables*, CRM Monogr. Ser., vol. 1, Amer. Math. Soc., Providence, RI 1992. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups.

[56] N.E. WEGGE-OLSEN, *K-Theory and C^*-Algebras*, Oxford Univ. Press, Oxford 1993.

[57] S.L. WORONOWICZ, Unbounded elements affiliated with C^*-algebras and noncompact quantum groups, *Comm. Math. Phys.* **136**(1991), 399–432.

[58] S.L. WORONOWICZ, C^*-algebras generated by unbounded elements, *Rev. Math. Phys.* **7**(1995), 481–521.

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Received July 26, 2016.