J. OPERATOR THEORY
79:1(2018), 213–223
doi: 10.7900/jot.2017feb14.2150



# AMBARZUMIAN THEOREM FOR NON-SELFADJOINT BOUNDARY VALUE PROBLEMS

OLGA BOYKO, OLGA MARTYNYUK, and VYACHESLAV PIVOVARCHIK

*Communicated by Florian-Horia Vasilescu*

ABSTRACT. We prove analogues of Ambarzumian's theorem for the cases of: (1) the boundary value problem with dissipative conditions dependent on the spectral parameter at both ends, (2) the boundary value problem generated by the Sturm–Liouville equation with the potential linearly dependent of the spectral parameter.

KEYWORDS: *Sturm–Liouville equation, von Neumann boundary conditions.*

MSC (2010): 2000 MSC: 34B45

## 1. INTRODUCTION

The history of the Sturm–Liouville inverse problem began with Ambarzumian's theorem [1]. This theorem is as follows:

THEOREM 1.1 (Ambarzumian's theorem). *Let the eigenvalues of the spectral problem*

$$-y'' + q(x)y = zy, \tag{1.1}$$

$$y'(0) = y'(a) = 0, \tag{1.2}$$

*with real* $q \in C[0,a]$ *be* $z_k = \pi^2 k^2 / a^2$ *where* $k = 0, 1, 2, \dots$.

*Then* $q(x) \equiv 0$.

It appeared later that the case of von Neumann conditions (1.2) at both ends of the interval and of the unperturbed spectrum is exceptional and in most cases one needs to know two spectra of boundary value problems to find the potential $q$ (see, e.g. [17], [18]). However, there exist generalizations of Ambarzumian's theorem (see [6]–[8], [12]–[14], [23], [25], [26]).

A correct proof of Ambarzumian's theorem was given in [3], [4] while a simple proof was proposed in [16]. The idea of this simple proof lies in proving that $\int_0^a q(x)\mathrm{d}x = 0$ and then to use the minimax principle according to which the lowest eigenvalue

$$0 = \min_{y \in D(A), \|y\|=1} \Big( -\int_0^a y''\overline{y} + \int_0^a q(x)|y|^2 \mathrm{d}x \Big)$$

where $D(A)$ is the domain of the corresponding operator $A$ acting in $L_2(0, a)$:

$$Ay = -y'' + q(x)y, \tag{1.3}$$

$$D(A) = \{y \in W_2^2(0, a),\ y'(0) = y'(a) = 0\}. \tag{1.4}$$

This minimum is attained at $y = C = \mathrm{const}$. Substituting $z = 0$ and $y = C \neq 0$ into (1.1) one obtains $q(x) \equiv 0$.

In this paper the same idea is used to treat certain boundary value problems describing damped vibrations of strings. In Section 2 we consider a spectral problem generated by the equation of small transverse vibrations of a smooth string damped at both ends and in Section 3 we consider a spectral problem generated by the Sturm–Liouville equation with the potential linear in the spectral parameter. In both cases we prove analogues of Ambarzumian's theorem.

## 2. BOUNDARY VALUE PROBLEM WITH POINTWISE DAMPING

There exists a vast literature on direct and inverse spectral problems describing vibrations of mechanical systems with point-wise damping (viscous friction) at an endpoint (see [15], [20], [21] for the corresponding inverse problems).

There are also some results on spectral problems for string vibrations with point-wise damping at the midpoint (see [2], [22] and for finite dimensional case [5]). We know only one publication [24] which deals with a spectral problem related to a mechanical problem with point-wise damping at both ends of the interval:

$$\frac{\partial^2 u}{\partial s^2} - \rho(s)\frac{\partial^2 u}{\partial t^2} = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial s}\Big|_{s=0} - \beta \frac{\partial u}{\partial t}\Big|_{s=0} = 0, \tag{2.2}$$

$$\frac{\partial u}{\partial s}\Big|_{s=l} + \alpha \frac{\partial u}{\partial t}\Big|_{s=l} = 0, \tag{2.3}$$

where the density of the string $\rho(s) \in C[0, l]$, $\rho(s) > 0$ for $x \in [0, l]$, the coefficient of damping $\alpha > 0$ at the right end and the coefficient of damping at the left end $\beta > 0$.

Substituting $u(s,t) = v(\lambda, s)\mathrm{e}^{\mathrm{i}\lambda t}$ into (2.1)–(2.3) we obtain

$$\frac{\partial^2 v}{\partial s^2} + \rho(s)\lambda^2 v = 0, \tag{2.4}$$

$$\left.\frac{\partial v}{\partial s}\right|_{s=0} - \mathrm{i}\beta\lambda v(0) = 0, \tag{2.5}$$

$$\left.\frac{\partial v}{\partial s}\right|_{s=l} + \mathrm{i}\alpha\lambda v(l) = 0. \tag{2.6}$$

We consider the operators acting in the Hilbert space $L_2(0,l) \oplus \mathbb{C} \oplus \mathbb{C}$

$$A_1 \begin{pmatrix} v(s) \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -v''(s) \\ -v'(0) \\ v'(l) \end{pmatrix},$$

$$D(A_1) = \left\{ \begin{pmatrix} v(x) \\ c_1 \\ c_2 \end{pmatrix} : \ v(s) \in W_2^2(0,l)),\ c_1 = v(0),\ c_2 = v(l) \right\},$$

$K_1 = \mathrm{diag}\{0, \beta, \alpha\}$, $\quad M_1 = \mathrm{diag}\{\rho(s), 0, 0\}$, $\quad D(K_1) = D(M_1) = L_2(0,l) \oplus \mathbb{C} \oplus \mathbb{C}$.

LEMMA 2.1. *The operator $A_1$ is self-adjoint and nonnegative.*

*Proof.* We apply Theorem 10.3.4 of [19] with

$$V = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and conclude that $A_1$ is selfadjoint. It is clear that

$$(A_1 Y, Y) = \int_0^l |v'(s)|^2 \mathrm{d}s \geqslant 0. \quad \blacksquare$$

LEMMA 2.2. *The spectrum of problem* (2.4)–(2.6) *consists of normal (isolated Fredholm) eigenvalues and lies in $\mathbb{C}^+ \cup \{0\}$ where $\mathbb{C}^+$ is the open upper half-plane.*

*Proof.* By Lemma 1.2.1 in [19] the spectrum of problem (2.4)–(2.6), i.e. the spectrum of the quadratic operator pencil

$$L(\lambda) = \lambda^2 M_1 - \mathrm{i}\lambda K_1 - A_1, \quad D(L) = D(M_1) \cap D(K_1) \cap D(A_1) = D(A_1),$$

consists of normal eigenvalues. Since $A_1 \geqslant 0$, $M_1 \geqslant 0$, $K_1 \geqslant 0$ and $M_1 + K_1 \gg 0$ we conclude that the spectrum of $L$ lies in the closed upper half-plane (see Lemma 1.2.4 in [19]). If $\lambda_0 \in \mathbb{R}\backslash\{0\}$ is an eigenvalue of $L(\lambda)$ and $v \in D(A_1)$ is the corresponding eigenvector then

$$\lambda_0^2 (M_1 v, v) - \mathrm{i}\lambda_0 (K_1 v, v) - (A_1 v, v) = 0$$

and consequently $(K_1 v, v) = 0$. Since $K_1$ is selfadjoint and $K_1 \geqslant 0$ we arrive at $K_1 v = 0$. This means that $v(0) = v(l) = 0$ and

$$\lambda_0^2 M_1 v - A_1 v = 0.$$

Consequently, $v'(0) = v'(l) = 0$ what together with $v(0) = v(l) = 0$ implies $v(s) = 0$ identically. A contradiction. ∎

Let the string be smooth enough, namely $\rho(s) \in W_2^2(0,l), \rho(s) \geqslant \varepsilon > 0$. Then substituting $u(s,t) = v(\lambda, s)e^{i\lambda t}$ and applying the Liouville transform (see e.g. p. 292 of [10], or p. 47 of [19])

$$x(s) = \int_0^s \rho^{1/2}(s')ds', \tag{2.7}$$

$$y(x) = \rho^{1/4}(s(x))v(\lambda, s(x)), \tag{2.8}$$

we reduce the above problem to the following Sturm–Liouville problem

$$y'' + (\lambda^2 - q(x))y = 0, \tag{2.9}$$

$$y'(0) - (i\widetilde{\beta}\lambda + \gamma_1)y(0) = y'(a) + (i\widetilde{\alpha}\lambda + \gamma_2)y(a) = 0, \tag{2.10}$$

where

$$a = \int_0^l \rho^{1/2}(s)ds,$$

$$q(x) = \rho^{-1/4}(s(x))\frac{d^2}{dx^2}\rho^{1/4}(s(x)),$$

$$\widetilde{\alpha} = \rho^{-1/2}(s(a))\alpha, \quad \widetilde{\beta} = \rho^{-1/2}(0)\beta,$$

$$\gamma_1 = \rho^{-1/2}(0)\frac{d\rho^{1/2}(s(x))}{dx}\Big|_{x=0}, \quad \gamma_2 = -\rho^{-1/2}(s(a))\frac{d\rho^{1/2}(s(x))}{dx}\Big|_{x=a}.$$

To have von Neumann conditions we consider the case where $\gamma_1 = \gamma_2 = 0$ what corresponds to $\frac{d\rho(s)}{ds}\Big|_{s=0} = \frac{d\rho(s)}{ds}\Big|_{s=l} = 0$. Then problem (2.9)–(2.10) attains the form

$$y'' + (\lambda^2 - q(x))y = 0, \tag{2.11}$$

$$y'(0) - i\widetilde{\beta}\lambda y(0) = y'(a) + i\widetilde{\alpha}\lambda y(a) = 0, \tag{2.12}$$

which can be written as the spectral problem for the operator pencil

$$\mathcal{L}(\lambda) = \lambda^2 M_2 - i\lambda K_2 - A_2, \quad D(\mathcal{L}) = D(M_2) \cap D(K_2) \cap D(A_2) = D(A_2),$$

acting in $L_2(0,a) \oplus \mathbb{C} \oplus \mathbb{C}$ where

$$A_2 \begin{pmatrix} y(x) \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -y''(x) + q(x)y(x) \\ -y'(0) \\ y'(a) \end{pmatrix},$$

$$D(A_2) = \left\{ \begin{pmatrix} y(x) \\ c_1 \\ c_2 \end{pmatrix} : \ y(x) \in W_2^2(0,a)),\ c_1 = y(0),\ c_2 = y(a) \right\},$$

$$K_2 = \operatorname{diag}\{0, \widetilde{\beta}, \widetilde{\alpha}\}, \quad M_2 = \operatorname{diag}\{I, 0, 0\}, \quad D(K_2) = D(M_2) = L_2(0,a) \oplus \mathbb{C} \oplus \mathbb{C}.$$

LEMMA 2.3. $M_2 \geqslant 0$, $K_2 \geqslant 0$, $A_2 \geqslant 0$.

*Proof.* First two statements are obvious because $\widetilde{\alpha} > 0$ and $\widetilde{\beta} > 0$. By the same reasons as $A_1$ the operator $A_2$ is self-adjoint. The spectrum of the linear operator pencil $zM_2 - A_2$ coincides with the spectrum of the linear operator pencil $zM_1 - A_1$ which is nonnegative due to $M_1 \geqslant 0$ and $A_1 \geqslant 0$.

To prove that all the eigenvalues of $A_2$ are nonnegative we consider the auxiliary pencil $T(z,\eta) = zM_2 + z\eta(I - M_2) - A_2$. By Theorem 9.2.4 in [19] the eigenvalues of $T(z,\eta)$ are continuous and piecewise analytic functions of $\eta$. They may loose analyticity only when colliding. Thus we find for an eigenvalue $z(\eta)$ and a corresponding eigenvector $y(\eta)$:

$$\frac{\mathrm{d}z}{\mathrm{d}\eta}((1-\eta)M_2 + \eta I)y + z(I - M_2)y + (zM_2 + z\eta(I - M_2) - A)\frac{\mathrm{d}y}{\mathrm{d}\eta} = 0$$

which implies

$$\frac{\mathrm{d}z}{\mathrm{d}\eta}(((1-\eta)M_2+\eta I)y,y)+z((I-M_2)y,y)+\Big((zM_2+z\eta(I-M_2)-A)\frac{\mathrm{d}y}{\mathrm{d}\eta},y\Big)=0.$$

Since $M_2$ and $A_2$ are selfadjoint we obtain

$$\Big((zM_2 + z\eta(I - M_2) - A)\frac{\mathrm{d}y}{\mathrm{d}\eta}, y\Big) = \Big((zM_2 + z\eta(I - M_2) - A)y, \frac{\mathrm{d}y}{\mathrm{d}\eta}\Big) = 0$$

and consequently

$$\frac{\mathrm{d}z}{\mathrm{d}\eta} = \frac{-z((I - M_2)y, y)}{(((1-\eta)M_2 + \eta I)y, y)}.$$

Since $(((1-\eta)M_2 + \eta I)y, y) > 0$ and $((I - M_2)y, y) \geqslant 0$ for $\eta \in (0,1)$ we obtain

$$\frac{\mathrm{d}z}{\mathrm{d}\eta} \geqslant 0 \quad \text{for } z < 0. \tag{2.13}$$

As we have seen the eigenvalues of the pencil $T(z,0) = zM_2 - A_2$ are nonnegative. The eigenvalues of $T(z,\eta)$ cannot cross the origin because they cannot move to the left on the negative semi-axis due to (2.13). Thus, all the eigenvalues of $T(z,1)$, i.e. all the eigenvalues of the operator $A_2$ are nonnegative. We conclude that $A_2$ is a selfadjoint operator with nonnegative spectrum, and consequently $A_2 \geqslant 0$. ∎

Denote by $\{\lambda_k\}_{k=-\infty}^{\infty}$ the spectrum of problem (2.11), (2.12). We enumerate the eigenvalues in the following way: $\operatorname{Re}\lambda_{-k} = -\operatorname{Re}\lambda_k$, $\operatorname{Re}\lambda_{k+1} \geqslant \operatorname{Re}\lambda_k$, $k \in N \cup \{0\}$. We call unperturbed the spectrum of the problem corresponding to $q(x) \overset{\text{a.e.}}{=} 0$.

Introduce the solution $s(\lambda,x)$ to equation (2.11) which satisfies the conditions $s(\lambda,0)=s'(\lambda,0)-1=0$ and the solution $c(\lambda,x)$ which satisfies $c(\lambda,0)-1=c'(\lambda,0)=0$. It is known (see e.g. Corollary 12.2.10 in [19]) that

$$s(\lambda,a)=\frac{\sin\lambda a}{\lambda}-B\frac{\cos\lambda a}{\lambda^2}+\frac{\psi_1(\lambda)}{\lambda^2}, \tag{2.14}$$

$$c(\lambda,a)=\cos\lambda a+B\frac{\sin\lambda a}{\lambda}+\frac{\psi_2(\lambda)}{\lambda}, \tag{2.15}$$

$$s'(\lambda,a)=\cos\lambda a+B\frac{\sin\lambda a}{\lambda}+\frac{\psi_3(\lambda)}{\lambda}, \tag{2.16}$$

$$c'(\lambda,a)=-\lambda\sin\lambda a+B\cos\lambda a+\psi_4(\lambda), \tag{2.17}$$

where $B=(1/2)\int\limits_0^a q(x)\mathrm{d}x$, $\psi_j\in\mathcal{L}^a$ and $\mathcal{L}^a$ is the Paley–Wiener class, i.e. the class of entire functions of exponential type $\leqslant a$ which belongs to $L_2(-\infty,\infty)$ for $\lambda\in\mathbb{R}$ (see, e.g. Definition 12.2.2 in [19]).

LEMMA 2.4. (i) *If* $(1-\widetilde{\alpha})(1-\widetilde{\beta})>0$ *and* $q\equiv 0$ *then the spectrum of problem* (2.11), (2.12) *is*

$$\Big\{0,\ \frac{\pi k}{a}+\frac{\mathrm{i}}{2a}\log\frac{(1+\widetilde{\alpha})(1+\widetilde{\beta})}{(1-\widetilde{\alpha})(1-\widetilde{\beta})}, k\in\mathbb{Z}\Big\}. \tag{2.18}$$

(i) *If* $(1-\widetilde{\alpha})(1-\widetilde{\beta})<0$ *and* $q\equiv 0$ *then the spectrum of problem* (2.11), (2.12) *is*

$$\Big\{0,\ \frac{\pi}{a}\Big(|k|-\frac{1}{2}\Big)\mathrm{sgn}\ k+\frac{\mathrm{i}}{2a}\log\frac{(1+\widetilde{\alpha})(1+\widetilde{\beta})}{|(1-\widetilde{\alpha})(1-\widetilde{\beta})|}, k\in\mathbb{Z}\backslash\{0\}\Big\}. \tag{2.19}$$

*Proof.* Substituting $y=C_1((\sin\lambda x)/\lambda)+C_2\cos\lambda x$ into (2.12) we obtain

$$\phi_0(\lambda)=-\lambda(1+\widetilde{\alpha}\widetilde{\beta})\sin\lambda a+\mathrm{i}\lambda(\widetilde{\alpha}+\widetilde{\beta})\cos\lambda a=0.$$

Now using Propositions 7.1.2 in [19] we arrive at (2.18) and (2.19). ∎

LEMMA 2.5. (i) *If* $(1-\widetilde{\alpha})(1-\widetilde{\beta})>0$ *then the spectrum* $\{\lambda_k\}_{k=-\infty,k\neq 0}^{\infty}$ *of problem* (2.11), (2.12) *behaves asymptotically as follows:*

$$\lambda_k=\frac{\pi(k-1)}{a}+\frac{\mathrm{i}}{2a}\log\frac{(1+\widetilde{\alpha})(1+\widetilde{\beta})}{(1-\widetilde{\alpha})(1-\widetilde{\beta})}+\frac{B}{\pi k}+\frac{\beta_k}{k} \tag{2.20}$$

$$\text{for } k\geqslant 2,\ \lambda_{-k}=-\overline{\lambda}_k,\ \lambda_{-1}=0,\ \lambda_1=\mathrm{i}c,$$

*where* $\{\beta_k\}_{-\infty,k\neq 0,\pm 1}^{\infty}\in l_2$, $c\in\mathbb{R}$.

(ii) *If* $(1-\widetilde{\alpha})(1-\widetilde{\beta})<0$ *then the spectrum* $\{\lambda_k\}_{k=-\infty}^{\infty}$ *of problem* (2.11), (2.12) *behaves asymptotically as follows:*

$$\lambda_k=\frac{\pi}{a}\Big(k-\frac{1}{2}\Big)+\frac{\mathrm{i}}{2a}\log\frac{(1+\widetilde{\alpha})(1+\widetilde{\beta})}{|(1-\widetilde{\alpha})(1-\widetilde{\beta})|}+\frac{B}{\pi k}+\frac{\beta_k}{k} \tag{2.21}$$

$$\text{for } k\geqslant 1,\ \lambda_{-k}=-\overline{\lambda}_k,\ \lambda_0=0,$$

*where* $\{\beta_k\}_{-\infty,k\neq 0}^{\infty} \in l_2$.

*Proof.* Substituting $y = C_1 s(\lambda, x) + C_2 c(\lambda, x)$ into (2.12) we obtain

$$\phi(\lambda) = c'(\lambda, a) - \lambda^2 \widetilde{\alpha}\widetilde{\beta} s(\lambda, a) + \mathrm{i}\lambda(\widetilde{\alpha} s'(\lambda, a) + \widetilde{\beta} c(\lambda, a)) = 0. \tag{2.22}$$

Substituting (2.14)–(2.17) into (2.22) we obtain

$$\lambda(1+\widetilde{\alpha}\widetilde{\beta}) \sin \lambda a - \mathrm{i}\lambda(\widetilde{\alpha}+\widetilde{\beta}) \cos \lambda a - (1+\widetilde{\alpha}\widetilde{\beta}) B \cos \lambda a - \mathrm{i}(\widetilde{\alpha}+\widetilde{\beta}) B \sin \lambda a + \psi_5(\lambda) = 0$$

where $\psi_5 \in \mathcal{L}^a$. Now using Propositions 7.1.3 in [19] we arrive at (2.20) and (2.21). ∎

THEOREM 2.6. (i) *Let the sequence* $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ *be given such that* $\lambda_{-1} = 0$ *and* $\lambda_1 = \mathrm{i}c$*:*

$$\lambda_k = \frac{\pi(|k| - 1)}{a} \operatorname{sgn} k + \mathrm{i}c \tag{2.23}$$

*where* $c > 0$ *is a constant. Then there exist* $\{\widetilde{\alpha}, \widetilde{\beta}\}$ *such that* $\widetilde{\alpha} > 0$, $\widetilde{\beta} > 0$, $(1 - \widetilde{\alpha})(1 - \widetilde{\beta}) > 0$*:*

$$c = \frac{1}{2a} \log \frac{(1 + \widetilde{\alpha})(1 + \widetilde{\beta})}{(1 - \widetilde{\alpha})(1 - \widetilde{\beta})}, \tag{2.24}$$

$\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ *is the spectrum of problem* (2.11)–(2.12) *with* $q(x) \overset{\text{a.e.}}{=} 0$ *and this* $q$ *is uniquely determined by this spectrum.*

(ii) *Let the sequence* $\{\lambda_k\}_{-\infty}^{\infty}$ *be given such that* $\lambda_0 = 0$*:*

$$\lambda_k = \frac{\pi(|k| - 1/2)}{a} \operatorname{sgn} k + \mathrm{i}c$$

*where* $c > 0$ *is a constant. Then there exist* $\{\widetilde{\alpha}, \widetilde{\beta}\}$ *such that* $\widetilde{\alpha} > 0$, $\widetilde{\beta} > 0$, $(1 - \widetilde{\alpha})(1 - \widetilde{\beta}) < 0$, $\widetilde{\alpha}$, $\widetilde{\beta}$ *and* $c$ *satisfy* (2.24), $\{\lambda_k\}_{-\infty}^{\infty}$ *is the spectrum of problem* (2.11)–(2.12) *with* $q(x) \overset{\text{a.e.}}{=} 0$ *and this* $q$ *is uniquely determined by this spectrum.*

*Proof.* Let us prove statement (i). It is clear that one can find $\widetilde{\alpha} > 0$ and $\widetilde{\beta} > 0$ such that (2.24) is true. The spectrum of problem (2.11)–(2.12) with any real $q \in L_2(0, a)$ satisfies (2.20). Comparing (2.20) with (2.23) we obtain

$$B = 0. \tag{2.25}$$

Now let us consider problem (1.1), (1.2) with the same $q$. This is the spectral problem for the operator $A$ defined by (1.3), (1.4).

Since $\lambda_0 = 0$ is an eigenvalue of problem (2.11)–(2.12) it is also an eigenvalue of problem (1.1)–(1.2), i.e. an eigenvalue of the operator $A$. Let us show that 0 is the lowest eigenvalue of $A$. To this end we use Theorem 1.3.3 in [19] according to which the number of eigenvalues of $\mathcal{L}$ in the open lower half-plane which is zero in our case equals the number of negative eigenvalues of $A_2$. Since $\lambda_{-1} = 0$ is an eigenvalue of $\mathcal{L}(\lambda)$, there exists a nonzero vector $y_{-1}$ such that $\mathcal{L}(0)y_{-1} =$

$-A_2 y_{-1} = 0$. Thus, 0 is the lowest eigenvalue of $A_2$. If $A_2 y_{-1} = 0$ then also $(zM_2 - A_2)|_{z=0} y_{-1} = 0$ and 0 is an eigenvalue of the pencil $zM_2 - A_2$.

Let us prove that this eigenvalue is the lowest for $zM_2 - A_2$. For any eigenvalue $z$ and its eigenvector $y$ we have $(M_2 y, y) \geqslant 0$, $(A_2 y, y) \geqslant 0$ and $(zM_2 y - A_2 y, y) = z(M_2 y, y) - (A_2 y, y) = 0$. If $z < 0$ we arrive at $(My, y) = (Ay, y) = 0$ what is impossible.

The spectra of $zM_2 - A_2$ and of the operator $A$ defined by (1.3), (1.4) coincide and we see that the lowest eigenvalue of $A$ is also 0. Since the lowest eigenvalue is 0, using the minimax principle we obtain

$$0 = \min_{\|y\|=1} (Ay, y) = \min_{\|y\|^2=1} \Big( - \int_0^a y'' \overline{y} \mathrm{d}x + \int_0^a q(x)|y|^2 \mathrm{d}x \Big), \tag{2.26}$$

where $\|y\| = \Big( \int_0^a |y|^2 \mathrm{d}x \Big)^{1/2}$. Substituting $y = C = \mathrm{const} \neq 0$ into (2.26) and using (2.25) we obtain that the vector $y = C$ satisfies the equation $Ay = 0$. Consequently, $q \overset{\text{a.e.}}{=} 0$. The proof of statement (ii) is analogous. ∎

## 3. BOUNDARY VALUE PROBLEM WITH DISTRIBUTED DAMPING

In this section we consider the following boundary value problem

$$\frac{\partial^2 u}{\partial s^2} - \mu(s) \frac{\partial u}{\partial t} - \rho(s) \frac{\partial^2 u}{\partial t^2} = 0, \quad \frac{\partial u}{\partial s}\Big|_{s=0} = 0, \quad \frac{\partial u}{\partial s}\Big|_{s=l} = 0,$$

which describes small transverse vibrations of a non-homogeneous string of density $\rho(x)$ subject to distributed damping of the coefficient $\mu \in C[0, l]$, $\mu(x) \geqslant 0$. Substituting $u(s, t) = v(\lambda, s) \mathrm{e}^{\mathrm{i}\lambda t}$ we obtain

$$\frac{\partial^2 v}{\partial s^2} - \mathrm{i}\mu(s)\lambda v - \rho(s)\lambda^2 v = 0, \quad \frac{\partial v}{\partial s}\Big|_{s=0} = 0, \quad \frac{\partial v}{\partial s}\Big|_{s=l} = 0.$$

Applying the Liouville transform (2.7), (2.8) we obtain

$$\begin{gathered} -y'' + 2\mathrm{i}\lambda p(x) y + q(x) y = \lambda^2 y, \\ y'(0) - \gamma_1 y(0) = y'(a) + \gamma_2 y(a) = 0, \end{gathered} \tag{3.1}$$

where $p(x) = \mu(s(x))/\rho(s(x))$. We are interested in the case of von Neumann conditions, i.e. in the case of $\gamma_1 = \gamma_2 = 0$:

$$y'(0) = y'(a) = 0. \tag{3.2}$$

We assume that $p(x) \in C[0, a]$ and $q(x) \in L_2(0, a)$. The spectrum of problem (3.1), (3.2) is the spectrum of the operator pencil

$$\widehat{L}(\lambda) = \lambda^2 I - \mathrm{i}\lambda \widehat{K} - A,$$

acting in $L_2(0,a)$ where $\widehat{K}y = 2p(x)y(x)$, $D(\widehat{K}) = L_2(0,a)$, $D(\widehat{L}) = D(\widehat{K}) \cap D(A) = D(A)$.

THEOREM 3.1. *Let the following conditions be satisfied:*

(i) $p(x) \geqslant 0$;

(ii) *the spectrum* $\sigma(\widehat{L}) = \{\lambda_k\}_{k=-\infty}^{\infty}$ *of problem* (3.1), (3.2) *lies in the closed upper half-plane;*

(iii) $0 \in \sigma(\widehat{L})$;

(iv) *we have*

$$\lambda_k = \frac{\pi k}{a} + \frac{\beta_k}{k} \tag{3.3}$$

*where* $\{\beta_k\}_{k=-\infty}^{\infty} \in l_2$.

*Then* $p(x) \equiv 0$ *and* $q(x) \overset{\text{a.e.}}{=} 0$.

*Proof.* The eigenvalues of problem (3.1), (3.2) behave asymptotically as follows (see [11]):

$$\lambda_k = \frac{\pi k}{a} + \mathrm{i}\int_0^a p(x)\mathrm{d}x + \frac{\int_0^a (q(x) - p^2(x))\mathrm{d}x}{k} + \frac{\beta_k}{k} \tag{3.4}$$

where $\{\beta_k\}_{k=-\infty}^{\infty} \in l_2$. Comparing (3.4) with (3.3) we obtain

$$\int_0^a p(x)\mathrm{d}x = 0 \quad \text{and} \tag{3.5}$$

$$\int_0^a q(x)\mathrm{d}x = \int_0^a p^2(x)\mathrm{d}x. \tag{3.6}$$

Condition (i) and (3.5) imply $p \equiv 0$ and then (3.6) implies

$$\int_0^a q(x)\mathrm{d}x = 0. \tag{3.7}$$

Condition (iii) implies that 0 is an eigenvalue of problem (1.1), (1.2). Let us prove that 0 is the lowest eigenvalue of the operator $A$. To this end we use Theorem 1.3.3 in [19] according to which the number of negative eigenvalues of the operator $A$ is equal to the number of eigenvalues of $\widehat{L}$ in the open lower half-plane which is zero. Thus, $A \geqslant 0$ which together with (3.7) implies $q(x) \overset{\text{a.e.}}{=} 0$ as we have already seen. ∎

## REFERENCES

[1] V.A. AMBARZUMIAN, Über eine Frage der Eigenwerttheorie, *Z. Phys.* **53**(1929), 690–695.

[2] A. BAMBERGER, J. RAUCH, M. TAYLOR, A model for harmonics on stringed instruments, *Arch. Rat. Mech. Anal.* **79**(1982), 267–290.

[3] G. BORG, Eine Umkehrung der Sturm–Liouvilleschen Eigenwertaufgabe, *Acta Math.* **78**(1946) 1–96.

[4] G. BORG, Uniqueness theorems in the spectral theory of $y'' + (\lambda - q(x))y = 0$, in *Proceding of the 11th Scandinavian Congress of Mathematicians, Johan Grundt Tanums Forlag Oslo*, Skand. Mat.-Kongr., Trondheim 1952, pp. 276–287.

[5] O. BOYKO, V. PIVOVARCHIK, The inverse three-spectral problem for a Stieltjes string and the inverse problem with one-dimensional damping, *Inverse Probl.* **24**(2008), ID 015019, 13 pp.

[6] R. CARLSON, V. PIVOVARCHIK, Ambarzumian's theorem for trees, *Electron. J. Differ. Eqns.* **2007**(2007), 1–9.

[7] N.K. CHAKRAVARTY, S.K. ACHARAYA, On an extension of the theorem of V.A. Ambarzumyan, *Proc. Royal Soc. Edinburg A* **110**(1988), 79–84.

[8] H.H. CHERN, C.K. LAW, H.J. WANG, Corrigendum to: Extensions of Ambarzumyan's theorem to general boundary conditions, *J. Math. Anal. Appl.* **309**(2005), 764–768.

[9] H.H. CHERN, C.L. SHEN, On the $n$-dimensional Ambarzumyan's theorem, *Inverse Probl.* **13**(1997), 15–18.

[10] R. COURANT, D. HILBERT, *Methods of Mathematical Physics*, Vol. 1, Interscience Publ., Inc., New York 1953.

[11] M.G. GASYMOV, G.SH. GUSEINOV, Determination of the diffusion operator by spectral data [Russian], *Dokl. AN Azerbajdzhan SSR* **37**(1981), 19–23.

[12] E.M. HARRELL, On the extension of Ambarzumian's inverse spectral theorem to compact symmetric spaces, *Amer. J. Math.* **109**(1987), 787–795.

[13] M. HORVÁTH, On a theorem of Ambarzumyan, *Proc. Roy. Soc. Edinburg A* **131**(2001), 899–907.

[14] M. KISS, An $n$-dimensional Ambarzumyan type theorem for Dirac operators, *Inverse Probl.* **20**(2004), 1593–1597.

[15] M.G. KREIN, A.A. NUDELMAN, On some spectral properties of an inhomogeneous string with dissipative boundary condition [Russian], *J. Operator Theory* **22**(1989), 369-395.

[16] N.V. KUZNETSOV, Generalization of a theorem of V.A. Ambarzumian, *Dokl. AN USSR* **146**(1962), 1259–1262.

[17] B. LEVITAN, M. GASYMOV, Determination of a differential equation by two of its spectra [Russian], *Russ. Math. Surveys* **19**(1964), 1–64.

[18] V.A. MARCHENKO, *Sturm–Liouville Operators and Applications*, Oper. Theory Adv. Appl., vol. 22, Birkhäuser-Verlag, Basel-Boston-Stuttgart 1986.

[19] M. MÖLLER, V. PIVOVARCHIK, *Spectral Theory of Operator Pencils, Hermite–Biehler Functions, and their Applications*, Oper. Theory Adv. Appl., vol. 264, Birkhäuser, Basel 2015.

[20] V. PIVOVARCHIK, Inverse problem for a smooth string with damping at one end, *J. Operator Theory* **38**(1997), 243–263.

[21] V. PIVOVARCHIK, Direct and inverse problems for a damped string, *J. Operator Theory* **42**(1999), 189–220.

[22] V. PIVOVARCHIK, Direct and inverse three-point Sturm–Liouville problem with parameter-dependent boundary conditions, *Asymptotic Anal.* **26**(2001), 219–238.

[23] V. PIVOVARCHIK, Ambarzumian's theorem for the Sturm–Liouville boundary value problem on star-shaped graph, *Funktsional. Anal. Prilozh. (Russian)* **39**(2005), 78–81; English *Funct. Anal. Appl.* **39**(2005), 148–151.

[24] I.YU. POPOV, A.V. STREPETOV, Completeness of the system of eigenfunctions of the two-sided Regge problem [Russian], *Vestn. Leningr. Univ. Mat.*, Leningrad 1983, vol. 13; *Mat. Mekh. Astron.* **3**(1983), 25–31.

[25] C.L. SHEN, On some inverse spectral problems related to the Ambarzumyan problem and the dual string of the string equation, *Inverse Probl.* **23**(2007), 2417–2436.

[26] C.F. YANG, X.P. YANG, Some Ambarzumyan-type theorems for Dirac operators, *Inverse Probl.* **25**(2009) ID 095012, 13 pp.

OLGA BOYKO, DEPARTMENT OF APPLIED MATHEMATICS AND COMPUTER SCIENCE, SOUTH UKRAINIAN NATIONAL PEDAGOGICAL UNIVERSITY, ODESA, 65020, UKRAINE

*E-mail address*: boykohelga@gmail.com

OLGA MARTYNYUK, DEPARTMENT OF HIGHER MATHEMATICS AND STATISTICS, SOUTH UKRAINIAN NATIONAL PEDAGOGICAL UNIVERSITY, ODESA, 65020, UKRAINE

*E-mail address*: superbarrakuda@yandex.ua

VYACHESLAV PIVOVARCHIK, DEPARTMENT OF HIGHER MATHEMATICS AND STATISTICS, SOUTH UKRAINIAN NATIONAL PEDAGOGICAL UNIVERSITY, ODESA, 65020, UKRAINE

*E-mail address*: vpivovarchik@gmail.com

Received February 14, 2017; revised April 26, 2017.