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TOPOLOGIES FOR WHICH EVERY NONZERO VECTOR IS HYPERCYCLIC

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ABSTRACT. An operator $T : X \to X$ is said to be hypercyclic if there exists a vector $x \in X$, called hypercyclic for T, such that the orbit $\mathrm{Orb}(T, x) = \{T^n x : n \in \mathbb{N}\}$ is dense in X. T is hereditarily hypercyclic if and only if $T \oplus T$ is hypercyclic on $X \times X$. We show that if T is a hereditarily hypercyclic operator on a Banach space X, then there exist separated locally convex topologies on X^* for which every nonzero vector $x^* \in X^*$ is hypercyclic for T^*, and thus for which T^* lacks nontrivial closed invariant subsets. We obtain in this way a link between properties of these topologies and the structure of hypercyclic vectors for T. In the same way, given that T^* is hereditarily hypercyclic, we can construct separated locally convex topologies on X where any nonzero vector $x \in X$ is hypercyclic for T. We introduce the notion of a nondegenerating hypercyclic vector manifold for an operator; such manifolds play a central role here, but these structures are also of independent interest.

KEYWORDS: *Hypercyclicity, hypercyclicity criterion, invariant subset, multiplier, nondegenerating hypercyclic vector manifold.*

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1. INTRODUCTION

Let X denote a locally convex space. An operator, that is a linear but not necessarily continuous map, $T : X \to X$ is said to be hypercyclic if there exists a vector $x \in X$, called hypercyclic for T, such that the orbit $\mathrm{Orb}(T, x) := \{T^n x : n \in \mathbb{N}\}$ is dense in X. We refer to [4], [17] for an overview of basic concepts and results in the theory of hypercyclicity. Hypercyclicity relates to the invariant subspace theory in the sense that T lacks nontrivial closed invariant subsets if (and only if when T is continuous) every nonzero vector is hypercyclic. (The trivial invariant subsets are $\{0\}$ and X.) Recall that The invariant subset problem consists in knowing whether there exists a continuous operator on a separable Hilbert space without nontrivial closed invariant subsets. Read [25] was able to show that there exist Banach spaces, e.g. ℓ_1, that support such operators. We should here also mention Grivaux's result, from [14], that any normed space with countable infinite algebraic dimension (thus not a Banach space) supports a continuous operator with all nonzero vectors hypercyclic. The constructions of these continuous operators without nontrivial closed invariant subsets are quite complicated. On the other hand, in a work [26] of Salas, the author constructed discontinuous operators, on Banach spaces, with all nonzero vectors hypercyclic, see also [27] for a similar study. In this note we continue this line of investigation, but our approach is somewhat different. Our operators are continuous in the norm topology, and we instead consider alternative topologies for which every nonzero vector is hypercyclic. More specifically, let T be a continuous operator on a Banach space X. Given that T is hereditarily hypercyclic (see below for the definition), we show how we can obtain separated locally convex topologies on the dual X^* for which every nonzero vector $x^* \in X^*$ is hypercyclic for the adjoint T^*. In the same way, provided that T^* is hereditarily hypercyclic, we can construct separated locally convex topologies on X so that every nonzero vector $x \in X$ is hypercyclic for T. All this may intuitively seem to be wrong, especially if you consider the example of the classical hereditarily hypercyclic backward shift operator $2B : (x_1, x_2, \ldots) \mapsto (2x_2, 2x_3, \ldots)$, and its adjoint $2F : (x_1, x_2, \ldots) \mapsto (0, 2x_1, 2x_2, \ldots)$, acting on ℓ_2 ($\simeq \ell_2^*$). How can every nonzero vector $x \in \ell_2$ possibly be hypercyclic for $2F$? Here it is important to point out that T^* and, respectively, T may fail to be continuous for the constructed topologies. The key, to obtain our topologies, is to work with hypercyclicity (of multipliers) in the algebra of continuous operators, provided with a suitable topology. Chan initiated the study of hypercyclity in operator algebras in [10], and his work has been pursued in e.g. [8], [11], [24], [28], [29].

If nothing else is specified, X denotes a separable real or complex infinite dimensional Banach space, and X^* denotes the Banach space of continuous linear forms on X. The set of continuous operators on X is denoted by $L(X)$, that we shall equip with different topologies.

DEFINITION 1.1 (Hypercyclicity criterion). An operator $T \in L(X)$ is said to satisfy the hypercyclicity criterion (HC) if there exist dense subsets $Z, Y \subseteq X$ and an increasing sequence $(n_k) \subseteq \mathbb{N}$ such that:

(i) for any $z \in Z$ we have that $T^{n_k} z \to 0$;

(ii) for any $y \in Y$ we have that $T^{n_k} x_k \to y$, for some nullsequence $(x_k) \subset X$.

Any operator $T \in L(X)$ that satisfies the HC is hereditarily hypercyclic, i.e., there exists a sequence $(n_k) \subseteq \mathbb{N}$ such that for every subsequence $(m_k) \subseteq (n_k)$ we have that $\{T^{m_k} x : k \in \mathbb{N}\}$ is dense for some $x \in X$. Indeed, we recall the following results from [5], [7].

PROPOSITION 1.2 (Bès, Peris, Bernal-González, Grosse-Erdmann). *The following are equivalent for any operator $T \in L(X)$:*

(i) *T satisfies the* HC*;*

(ii) *T is hereditarily hypercyclic;*
(iii) *$T \oplus T \in L(X \times X)$ is hypercyclic;*
(iv) *$\bigoplus_{1}^{n} T \in L(X^n)$ is hypercyclic for all $n \geqslant 1$;*
(v) *$\bigoplus_{1}^{\infty} T \in L(X^{\mathbb{N}})$ is hypercyclic.*

Here, and below, product spaces are tacitly assumed to be provided with the product topology, thus $X^{\mathbb{N}} = X \times X \times \cdots$ is for example a Fréchet space (complete metrizable locally convex space).

The hypothesis that every hypercyclic operator $T \in L(X)$ is in fact hereditarily hypercyclic, was studied intensively. However, Read and De La Rosa answered this question of Domingo Herrero, by providing an example of a continuous hypercyclic operator on a Banach space that is not hereditarily hypercyclic [13]. Most hypercyclic operators in $L(X)$ are however hereditarily hypercyclic, see e.g. [15], and another characterization of these operators is the following proposition from [8].

PROPOSITION 1.3 (Bonet, Martínez-Giménez, Peris). *Any of the equivalent conditions* (i)–(v) *in Proposition* 1.2 *is equivalent to that the left-multiplier $L_T : S \mapsto TS$ is* SOT*-hypercyclic on $L(X)$.*

SOT refers here to the strong operator topology on $L(X)$, i.e. the topology generated by the seminorms $\|T\|_x := \|Tx\|$, $x \in X$, and "SOT-hypercyclic" means of course hypercyclic with respect to this topology. Noteworthy is that $L(X)$ is not separable with respect to the operator norm topology, so there is no chance to find any hypercyclic operator on $L(X)$ for this topology. We shall also apply (and pursue) the following proposition from [8].

PROPOSITION 1.4 (Bonet, Martínez-Giménez, Peris). *If T^* is hereditarily hypercyclic, where $T \in L(X)$, the right-multiplier $R_T : S \mapsto ST$ is* SOT*-hypercyclic on $L(X)$.*

(The converse does not hold, R_T may be SOT-hypercyclic even if the adjoint of $T \in L(X)$ is not hereditarily hypercyclic, see Remark 3.4 in [8] for an example. Note that the condition that T^* is (hereditarily) hypercyclic requires that X^* is separable, and X^* beeing separable implies that X is separable.)

A key-idea in this note is to complement Proposition 1.3 with an analogous statement but for a slightly different topology, that we introduce here.

DEFINITION 1.5. The *dual strong operator topology* on $L(X)$ (DSOT) is the topology generated by the seminorms $\|V\|^*_{x^*} := \|V^* x^*\|$, $x^* \in X^*$. ($\|\cdot\|$ denotes here the dual norm on X^*.)

The DSOT on a product of $L(X)$ is referring to the product topology with respect to the DSOT (the same convention is used for other topologies on $L(X)$). It follows that DSOT (and SOT) is a separated topology. Note also that, since $(L(X), \text{DSOT}) \ni T \mapsto T^* x^* \in (X^*, \|\cdot\|)$ is continuous and surjective for any

$x^* \neq 0$, it is necessary and (as we shall see in the proof of Theorem 3.1) sufficient that X^* (and thus X) is separable in order that $L(X)$ is separable for the DSOT. Noteworthy is also that if $T \in L(X)$, then L_T and R_T are both continuous with respect to the DSOT (as well as the SOT). We refer to Subsection 6.1 for some more comments on the SOT and the DSOT.

Now, based on our "DSOT-analogue" of Proposition 1.3, see Theorem 3.1 in Section 3, we can in Section 4 define separated locally convex topologies on the dual X^* for which every nonzero vector of X^* is hypercyclic for T^*, provided $T \in L(X)$ is hereditarily hypercyclic and X^* is separable. We obtain in this way an interesting link between the properties of these topologies and the hypercyclic properties of T. In the same way, based on Proposition 1.4 we show in Section 5 how we can obtain separated locally convex topologies on X where every nonzero vector $x \in X$ is hypercyclic for T, whenever T^* is hereditarily hypercyclic.

The paper is organized as follows. In Section 2 we establish some general results, related to properties and the structure of hypercyclic vectors. In particular we introduce the notion of a nondegenerating hypercyclic vector manifold for an operator (Definition 2.3).

In Section 3 we study left- and right-multipliers, in particular we investigate the structure of their hypercyclic vectors.

In Section 4 and 5, we apply the results in Section 3, and define topologies on X^* and X, respectively, for which every nonzero vector is hypercyclic for T^* and T, respectively. The topologies are defined out of hypercyclic vectors for multipliers. We pose several problems throughout our work, in particular in the last section, Section 6.

2. PRELIMINARY RESULTS

Recall that a hypercyclic vector manifold for an operator T is an infinite dimensional subspace of, except for zero, hypercyclic vectors (for T). By a result of Bès and Bourdon, every continuous hypercyclic operator T on a locally convex space admits a hypercyclic vector manifold, in fact, a dense and invariant such a manifold (indeed, such a manifold is given by $\{p(T)x : p \text{ polynomial}\}$, where x is any given hypercyclic vector, see [6] and [9] for details).

PROPOSITION 2.1. *An operator $V \in L(X)$ is* SOT*-hypercyclic for L_T ($T \in L(X)$) if and only if for all $n \geqslant 1$ and any linearly independent set $\{x_1, \dots, x_n\} \subset X$, $(Vx_1, \dots, Vx_n)$ is hypercyclic for $\bigoplus_1^n T$. In particular, any such V is one-to-one and $\mathcal{V} := \operatorname{Im} V$ forms a hypercyclic vector manifold for T such that, for all $n \geqslant 1$, every linearly independent n-tuple from $\mathcal{V}$ is hypercyclic for $\bigoplus_1^n T$.*

Proof. A fundamental family of seminorms on $L(X)$, with respect to the SOT, is given by $\|T\|_F \equiv \sum_i \|Tx_i\|$, where $F = \{x_i\}$ runs through all finite subsets of X. We note first that an equivalent family of seminorms is obtained by letting the sets F be formed by linearly independent vectors only. Indeed, if $F = \{x_0, x_1, \ldots, x_n\}$ is any finite set and $x_0 = \sum_1^n \alpha_i x_i \in \operatorname{span}\{x_1, \ldots, x_n\}$, then

$$\|T\|_F \leqslant \sum |\alpha_i| \|Tx_i\| + \|T\|_{F_0} \leqslant (1 + A)\|T\|_{F_0},$$

where $F_0 := \{x_1, \ldots, x_n\}$ and $A := \max |\alpha_i|$. Hence, for some constant C and linearly independent set $G \subseteq F$, $\|\cdot\|_F \leqslant C\|\cdot\|_G$ which proves our claim. The topology on X^n is generated by the norm $\|(x_i)\|_n := \sum_i \|x_i\|$.

Assume now that V is SOT-hypercyclic for L_T, and let $F = \{x_i\}_1^n$ be any linearly independent set. We must prove that (Vx_i) is hypercyclic for $\bigoplus_1^n T$. So let $\varepsilon > 0$ and $(y_i) \in X^n$ be arbitrary. Since F is linearly independent, we can find an $R \in L(X)$ such that $Rx_i = y_i$ for all i. Consequently, for some m,

$$\varepsilon > \|L_T^m(V) - R\|_F = \sum \|T^m V x_i - Rx_i\| = \Big\| \Big(\bigoplus_1^n T \Big)^m (Vx_i) - (y_i) \Big\|_n,$$

and so (Vx_i) is hypercyclic for $\bigoplus_1^n T$. Conversely, assume (Vx_i) is hypercyclic for $\bigoplus_1^n T$ for any linearly independent set $F = \{x_i\}_1^n$. Pick $R \in L(X)$ and $\varepsilon > 0$ arbitrarily. Put $y_i := Rx_i$. Then, for some m,

$$\varepsilon > \Big\| \Big(\bigoplus_1^n T \Big)^m (Vx_i) - (y_i) \Big\|_n = \sum \|T^m V x_i - Rx_i\| = \|L_T^m(V) - R\|_F,$$

and so V is SOT-hypercyclic for L_T.

The last part follows by the fact that if $Vx_1, \ldots, Vx_n$ are linearly independent vectors in $\operatorname{Im} V$, then $x_1, \ldots, x_n$ must be linearly independent. ∎

Here one may reflect on the converse.

PROBLEM 2.2. Assume $T \in L(X)$ is hereditarily hypercyclic. Is every hypercyclic vector for $\bigoplus_1^\infty T$ of the form (Vx_i), where V is an SOT-hypercyclic vector for L_T?

Proposition 2.1 motivates the following definition.

DEFINITION 2.3. A *nondegenerating hypercyclic vector manifold* for an operator T on a locally convex space X is an infinite dimensional subspace $\mathcal{V} \subseteq X$ such that for every $n \geqslant 1$, any linearly independent n-tuple $(x_1, \ldots, x_n)$ from $\mathcal{V}$ is hypercyclic for $\bigoplus_1^n T$.

Accordingly, by Propositions 1.3 and 2.1, any hereditarily hypercyclic operator $T \in L(X)$ supports a nondegenerating hypercyclic vector manifold. Of course, any nondegenerating hypercyclic vector manifold is a hypercyclic vector manifold, but the converse is not true, in fact we have the following proposition.

PROPOSITION 2.4. *A nondegenerating hypercyclic vector manifold $\mathcal{V}$ for a continuous operator T on a locally convex space X cannot be invariant under T.*

Proof. Assume, by way of contradiction, that $\mathcal{V}$ is invariant under T. Let x be any nonzero vector of $\mathcal{V}$. Then x is hypercyclic for T, and thus x and Tx are linearly independent vectors of $\mathcal{V}$. But (x, Tx) belongs to the graph of T, which is a closed $(T \oplus T)$-invariant proper subset of $X \times X$ ($(x, x) \in X \times X$ does not belong to the graph). Thus (x, Tx) is not a hypercyclic vector for $T \oplus T$, which contradicts that $\mathcal{V}$ is a nondegenerating hypercyclic vector manifold. ∎

In particular the proposition shows that the full space X is never a nondegenerating hypercyclic vector manifold for some continuous operator, but it can indeed be a hypercyclic vector manifold for an operator $T \in L(X)$, in view of Read's examples. Moreover, we conclude that the construction $\{p(T)x : p$ polynomial$\}$ cannot be a nondegenerating hypercyclic vector manifold.

From Proposition 1.2 we know that $\bigoplus_1^\infty T$ admits a hypercyclic vector whenever $T \in L(X)$ is hereditarily hypercyclic. In fact, we can say a little bit more:

PROPOSITION 2.5. *Assume $T \in L(X)$ is hereditarily hypercyclic. Then we can find a hypercyclic vector $(x_i) \in X^{\mathbb{N}}$ for $\bigoplus_1^\infty T$ such that $\{x_n : n \in \mathbb{N}\}$ is dense in X.*

Proof. Let $\{G_i : i \in \mathbb{N}\}$ be any countable open base for X. We must thus find a hypercyclic vector (x_i) such that $\{x_i : i \in \mathbb{N}\}$ meets every G_i. But let (z_i) be any hypercyclic vector for $\bigoplus_1^\infty T$, that is, $(z_1, \ldots, z_n)$ is hypercyclic for $\bigoplus_1^n T$ for all n. We claim that for any family $\{p_n : n \in \mathbb{N}\}$ of nonzero one variable polynomials, $(x_i := p_i(T)z_i) \in X^{\mathbb{N}}$ is another hypercyclic vector for $\bigoplus_1^\infty T$. Indeed, just note that for arbitrary n, $p_1(T) \oplus \cdots \oplus p_n(T) \in L(X^n)$ has dense range (because $\operatorname{Im} p_i(T)$ is dense [6]), and $\bigoplus_1^n p_i(T)$ commutes with $\bigoplus_1^n T$. Since every z_i is hypercyclic for T, we can for each i find a monomial p_i with $p_i(T)z_i \in G_i$, and we are done. ∎

PROPOSITION 2.6. *Let T be a continuous operator on a separated locally convex space X. Any countable linearly independent subset $\{x_i\}$ of a nondegenerating hypercyclic vector manifold $\mathcal{V}$ for T forms a hypercyclic vector $\mathbb{V} = (x_i)$ for $\bigoplus_1^\infty T$. Conversely, any hypercyclic vector $\mathbb{V}$ for $\bigoplus_1^\infty T$ defines a nondegenerating hypercyclic vector manifold for T by $\mathcal{V} := \operatorname{span} \mathbb{V}$.*

Proof. The first part is elementary, for we have that (x_i) is hypercyclic for $\bigoplus_1^\infty T$ if and only if $(x_1, \dots, x_n)$ is hypercyclic for $\bigoplus_1^n T$ for all $n \geqslant 1$.

So assume now that $\mathbb{V} = (x_i)$ is a hypercyclic vector for $\bigoplus_1^\infty T$, and let

$$y_i = \sum_{j=1}^m \alpha_{ij} x_j, \quad i = 1, \dots, n,$$

be any linearly independent family of vectors from span $\mathbb{V}$ (note that $m \geqslant n$ must hold here). We must prove that $(y_1, \dots, y_n)$ is a hypercyclic vector for $\bigoplus_1^n T$. Let p be any continuous seminorm on X and let $(z_1, \dots, z_n)$ be any n-tuple in X^n. Since $\{y_i\}$ is a linearly independent set, we can find vectors $u_1, \dots, u_m$ (in fact, in span$\{z_i\}$) such that

$$z_i = \sum_{j=1}^m \alpha_{ij} u_j, \quad i = 1, \dots, n.$$

By the fact that $(x_1, \dots, x_m)$ is hypercyclic for $\bigoplus_1^m T$, there exists an $r \in \mathbb{N}$ so that

$$p(T^r x_i - u_i) \leqslant \frac{1}{\max \left\{ \sum_{j=1}^m |\alpha_{kj}| \right\}_{k=1}^n} \quad \text{for all } i = 1, \dots, m.$$

Hence

$$p(T^r y_i - z_i) \leqslant \sum_{j=1}^m |\alpha_{ij}| p(T^r x_j - u_j) \leqslant 1, \quad i = 1, \dots, n,$$

and so (y_i) is hypercyclic for $\bigoplus_1^n T$.

Finally we must prove that span $\mathbb{V}$ is infinite dimensional, when $\mathbb{V} = (x_i)$ is a hypercyclic vector for $\bigoplus_1^\infty T$. We prove that $\{x_i\}$ must be a linearly independent set (cf. the proof of Proposition 2.4). Assume this is not the case, then

$$x_m = \sum_1^{m-1} \alpha_i x_i$$

for some $m \geqslant 2$ and constants α_i. Consider now the proper subspace

$$M = \left\{ (y_1, y_2, \dots, y_m) : y_m = \sum_{i=1}^{m-1} \alpha_i y_i \right\}$$

of X^m. It follows that M is closed and invariant under $\bigoplus_1^m T$. Since $(x_1, \dots, x_m) \in M$, $(x_1, \dots, x_m)$ cannot be hypercyclic for $\bigoplus_1^m T$. This contradicts that $\mathbb{V} = (x_i)$ is a hypercyclic vector for $\bigoplus_1^\infty T$, so span $\mathbb{V}$ is indeed infinite dimensional. ∎

Thus we know two different ways how to construct a nondegenerating hypercyclic vector manifold $\mathcal{V}$ for, say, a hereditarily hypercyclic operator $T \in L(X)$:

(i) by the image $\mathcal{V} := \operatorname{Im} V$ of any SOT-hypercyclic vector $V \in L(X)$ for L_T and

(ii) by $\mathcal{V} := \operatorname{span} \mathbb{V}$, where $\mathbb{V} = (x_i)$ is any hypercyclic vector for $\bigoplus_1^\infty T$.

In particular, by Proposition 2.5, Proposition 2.6 gives the next corollary.

COROLLARY 2.7. *Every hereditarily hypercyclic operator $T \in L(X)$ supports a dense nondegenerating hypercyclic vector manifold.*

Another consequence of Proposition 2.6 is the following.

COROLLARY 2.8. *An operator $T \in L(X)$ is hereditarily hypercyclic if and only if it supports a nondegenerating hypercyclic vector manifold.*

3. HYPERCYCLIC MULTIPLIERS

In this section we establish some hypercyclic properties of left-multipliers $L_T : S \mapsto ST$ and right-multipliers $R_T : S \mapsto TS$, acting on $L(X)$.

THEOREM 3.1. *Assume that $T \in L(X)$ is hereditarily hypercyclic and that X^* is separable. Then L_T is* DSOT-*hypercyclic on $L(X)$, in fact, $\bigoplus_1^\infty L_T$ (and thus any $\bigoplus_1^n L_T$, $n \geqslant 1$) is hypercyclic with respect to the* DSOT.

Further,

(i) *$\bigoplus_1^\infty L_T$ admits a* DSOT-*hypercyclic vector $\mathbb{V} = (V_i) \in L(X)^{\mathbb{N}}$ such that the set $\{V_n : n \in \mathbb{N}\}$ is* DSOT-*dense in $L(X)$;*

(ii) *L_T supports a dense nondegenerating hypercyclic vector manifold $\mathcal{V} \subseteq L(X)$, all this with respect to the* DSOT.

Proof. Let $A(X)$ denote the set of finite rank operators in $L(X)$, and let $A_0(X)$ denote the closure of $A(X)$ in $L(X)$ with respect to the operator norm topology. Thus $A_0(X)$ is a Banach space, and we claim that $A(X)$, and thus $A_0(X)$, is DSOT-dense in $L(X)$. Indeed, let $x_1^*, \dots, x_n^*$ be arbitrary linearly independent vectors in X^*. Then we can find a biorthogonal system $(x_i) \subset X$ to (x_i^*). Consider now the operator $P = \sum \langle \cdot, x_i^* \rangle x_i \in A(X)$. The adjoint $P^* = \sum \langle x_i, \cdot \rangle x_i^*$ is now a projector onto the span of $\{x_i^*\}$. We conclude that $PT \in A(X)$ and $(PT - T)^* = 0$ on $\operatorname{span}\{x_i^*\}$, which proves our claim since $\|PT - T\|_{x_i^*}^* = 0$ for all i.

Next we prove that L_T is hypercyclic on $A_0(X)$ (note that $A(X)$, and by continuity thus $A_0(X)$, is invariant under L_T). In fact, we prove that this restriction $L_T : A_0(X) \to A_0(X)$ of L_T satisfies the HC. In order to apply the HC, we must ensure that $A_0(X)$ is separable. It suffices to prove that $A(X)$ is separable. A proof of this can be found in Theorem 2 of [11].

Next, let $Z, Y \subseteq X$ be dense subsets and (n_k) a sequence in $\mathbb{N}$ for which T satisfies (i) and (ii) in the HC (Definition 1.1). We may assume that Z and Y are subspaces. Let Z_0 denote the subset of $A_0(X)$ formed by the operators $S \in A(X)$ of the form $S = \sum\langle \cdot, x_i^* \rangle z_i$, where $x_i^* \in X^*$ and $z_i \in Z$. In the same way we define Y_0 (the elements z_i are replaced by elements y_i in Y). It is a routine work to show that Y_0 and Z_0 both are dense in $A(X)$, and hence in $A_0(X)$. We evidently have that $L_T^{n_k} \to 0$ pointwise on Z_0. Moreover, if $S = \sum\langle \cdot, x_i^* \rangle y_i \in Y_0$ we can for each i find a nullsequence (x_k^i) in X such that $T^{n_k} x_k^i \to y_i$ $(k \to \infty)$. Hence the elements $S_k := \sum_i \langle \cdot, x_i^* \rangle x_k^i$ form a nullsequence (S_k) in $A_0(X)$ such that $L_T^{n_k}(S_k) = T^{n_k} S_k \to S$ in $A_0(X)$.

Accordingly, $\bigoplus_1^\infty L_T : A_0(X)^{\mathbb{N}} \to A_0(X)^{\mathbb{N}}$ is hypercyclic (Proposition 1.2).

Consider now the simple, and often applied, comparison principle:

Comparison principle. If S, T are operators on Y and Z, respectively, and $S\varphi = \varphi T$ for some continuous densely ranged map $\varphi : Z \to Y$, then $\varphi(x)$ is a hypercyclic vector for S for any such vector x for T.

Applied to the imbedding $\varphi : A_0(X)^{\mathbb{N}} \to L(X)^{\mathbb{N}}$, we deduce that $\bigoplus_1^\infty L_T$ is DSOT-hypercyclic on the countable product $L(X)^{\mathbb{N}}$. In fact, any hypercyclic vector (V_i) for $\bigoplus_1^\infty L_T : A_0(X)^{\mathbb{N}} \to A_0(X)^{\mathbb{N}}$ is also hypercyclic for $\bigoplus_1^\infty L_T : L(X)^{\mathbb{N}} \to L(X)^{\mathbb{N}}$. In particular, by Proposition 2.5 we may find (V_i) so that $\{V_i\}$ is dense in $A_0(X)$, and thus DSOT-dense in $L(X)$. Hence (i), and (ii) follows by Proposition 2.6. ∎

REMARK 3.2. Note that $A_0(X)$ is formed by compact operators, and if X has the approximation property, $A_0(X)$ is precisely the ideal $K(X)$ of compact operators, see Theorem 1.e.4 in [19]. Thus the proof shows that, in this case, L_T is hypercyclic on $K(X)$ with respect to the operator norm topology, and this result was also obtained in Theorem 2.1 of [8].

Note that the topologies SOT and DSOT are not comparable, which, in view of Proposition 1.3, leads to the following problem.

PROBLEM 3.3. Are the SOT- and DSOT-hypercyclic vectors for L_T the same, when $T \in L(X)$ is hereditarily hypercyclic and X^* is separable?

It follows for example that any SOT-hypercyclic vector (and any DSOT-hypercyclic vector) for L_T, must be injective (see Remark 3.8 below). We may here also point out that $V \in L(X)$ is DSOT-hypercyclic for L_T if and only if V^* is SOT-hypercyclic for R_{T^*}. Just note that $\|L_T^n(V) - S\|_{x^*}^* = \|R_{T^*}^n(V^*) - S^*\|_{x^*}$, and it is easily checked that the operators of the form S^*, $S \in L(X)$, form an SOT-dense set in $L(X^*)$. See Remark 3.7 for more on this. In any case we have (see also Theorem 6.3 and its remarks) the following corollary.

COROLLARY 3.4. *Assume T is a hereditarily hypercyclic operator on X, where X^* is separable. Then there exists a vector $(V_i) \in L(X)^{\mathbb{N}}$ that is both* SOT- *and* DSOT-*hypercyclic for $\bigoplus_1^\infty L_T$ and where $\{V_n : n \in \mathbb{N}\}$ is dense in $L(X)$ with respect to both the* SOT *and the* DSOT. *In particular, there exists a vector manifold $\mathcal{V} \subseteq L(X)$ which is a dense nondegenerating hypercyclic vector manifold for L_T with respect to both the* SOT *and the* DSOT.

Proof. For the first part, just note that the imbedding $A_0(X) \to L(X)$, and thus $A_0(X)^{\mathbb{N}} \to L(X)^{\mathbb{N}}$, is continuous and has dense range when $L(X)$ is provided with any of the topologies SOT and DSOT. From this point the arguments in the last part of the proof of Theorem 3.1 give the statement. The last part follows now by Proposition 2.6. ∎

Another problem, partially related to Problem 3.3, is the following one.

PROBLEM 3.5. Does DSOT-hypercyclicity of L_T imply that $T \in L(X)$ is hereditarily hypercyclic?

Our next goal is to establish a similar result (Theorem 3.1) for right-multipliers $R_T : L(X) \to L(X)$, where $L(X)$ is endowed with the SOT. We already know from Proposition 1.4 that R_T admits an SOT-hypercyclic vector $V \in L(X)$ whenever T^* is hereditarily hypercyclic, but in view of our purposes we need the following theorem.

THEOREM 3.6. *Assume T^* is hereditarily hypercyclic where $T \in L(X)$. Then $\bigoplus_1^\infty R_T$ (and hence any $\bigoplus_1^n R_T$, $n \geqslant 1$) is hypercyclic with respect to the* SOT. *Further,*

(i) *$\bigoplus_1^\infty R_T$ admits an* SOT-*hypercylic vector $\mathbb{V} = (V_i) \in L(X)^{\mathbb{N}}$ such that the set $\{V_n : n \in \mathbb{N}\}$ is* SOT-*dense in $L(X)$;*

(ii) *R_T supports a dense nondegenerating hypercyclic vector manifold $\mathcal{V} \subseteq L(X)$, all this with respect to the* SOT.

Proof. The proof follows that of Theorem 3.1. That is, we prove that R_T defines a hereditarily hypercyclic operator on the Banach space $A_0(X)$ (note that $A(X)$, and thus $A_0(X)$, is invariant under R_T). Recall that $A_0(X)$ is the closure in $L(X)$, with respect to the operator norm topology, of the space $A(X)$ of finite rank operators. Since X^* is separable, by our hypothesis, X is separable. Hence, as in the proof of Theorem 3.1, we conclude that $A_0(X)$ is separable. Moreover, $A_0(X)$ is SOT-dense in $L(X)$. Indeed, let $x_1, \dots, x_n$ be any linearly independent family of vectors in X. Then $TP \in A(X)$ and $TP - T$ vanishes on $\{x_1, \dots, x_n\}$, here $P := \sum \langle \cdot, x_i^* \rangle x_i$ where $(x_1^*, \dots, x_n^*)$ is a biorthogonal n-tuple with respect to the ordered set $(x_1, \dots, x_n)$. By the comparison principle in the proof of Theorem 3.1, together with Propositions 2.5 and 2.6, we only have to prove that $R_T : A_0(X) \to A_0(X)$ satisfies the HC.

Let Z and Y denote dense subsets (we can assume subspaces) of X^* so that (i) and (ii) in the HC, see Definition 1.1, hold for T^* and some sequence (n_k).

Let Z_0 be the space formed by all operators $S \in A(X)$ of the form $S = \sum\langle \cdot, z_i^* \rangle x_i$, where $z_i^* \in Z$ and $x_i \in X$. Thus the range of any S^*, $S \in Z_0$, is contained in Z. In the same way, Y_0 is formed by all finite rank operators $S = \sum\langle \cdot, y_i^* \rangle x_i$, where $y_i^* \in Y$ and $x_i \in X$. Hence $\operatorname{Im} S^* \subseteq Y$ whenever $S \in Y_0$. Note that for any element S of Z_0 or Y_0, S^* is a finite rank operator. It is straight-forward to prove that Y_0 and Z_0 are dense in $A(X)$, and therefore also in $A_0(X)$.

Let $S = \sum\langle \cdot, z_i^* \rangle x_i \in Z_0$. We prove that $R_T^{n_k}(S) \to 0$ in $A_0(X)$. But

$$\|R_T^{n_k}(S)\| = \|ST^{n_k}\| = \|(ST^{n_k})^*\| = \|T^{*n_k}S^*\| = \Big\| \sum\langle x_i, \cdot \rangle T^{*n_k} z_i^* \Big\|,$$

and since the sum is finite it follows that $R_T^{n_k}(S) \to 0$. We conclude that $R_T^{n_k} \to 0$ pointwise on Z_0. Next, let $S = \sum\langle \cdot, y_i^* \rangle x_i \in Y_0$. By our assumptions, there exists for every i a nullsequence (x_{ik}^*) so that $T^{*n_k} x_{ik}^* \to y_i^*$ when $k \to \infty$. Put

$$S_k := \sum_i \langle \cdot, x_{ik}^* \rangle x_i,$$

it is then easily checked that (S_k) is a nullsequence and $R_T^{n_k}(S_k) \to S$ as $k \to \infty$. We conclude that $R_T : A_0(X) \to A_0(X)$ satisfies the HC. ∎

REMARK 3.7. Recall that it is not true that R_T being SOT-hypercyclic implies that T^* is hereditarily hypercyclic (or equivalently, satisfies the HC). However, it follows that R_T is DSOT-hypercyclic if and only if T^* is hereditarily hypercyclic. Let us give some details for a proof of this. It is easy to prove that $V \in L(X)$ is DSOT-hypercyclic for R_T if and only if V^* is SOT-hypercyclic for L_{T^*}. Since L_{T^*} is SOT-hypercyclic if and only if T^* is hereditarily hypercyclic, Proposition 1.3, we only have to prove that if T^* is hereditarily hypercyclic, then L_{T^*} admits an SOT-hypercyclic vector of the form V^*, where $V \in L(X)$. Consider now the operator $L_{T^*} : A_0(X^*) \to A_0(X^*)$, where $A_0(X^*)$ is the closure of the set $A(X^*)$ of operators of the form $\sum\langle x_i, \cdot \rangle x_i^*$, where $\{x_i\}$ and $\{x_i^*\}$ are finite sets of X and X^*, respectively. With arguments as in the proof of Theorems 3.1 and 3.6 it follows that L_{T^*} admits a hypercyclic vector $U \in A_0(X^*)$. By density, $U_n \to U$ for some sequence (U_n) in $A(X^*)$. But every operator in $A(X^*)$ is the adjoint of some operator in $L(X)$. Hence $U_n = V_n^* \to U$, $V_n \in L(X)$. Finally we only have to apply the fact that the map $S \to S^*$ between $L(X)$ and $L(X^*)$ is continuous and has closed range, in the operator norm topologies ($\|S\| = \|S^*\|$), and so $U = V^*$ for some $V \in L(X)$. Thus, L_{T^*} admits an SOT-hypercyclic vector of the form V^*, $V \in L(X)$, when T^* is hereditarily hypercyclic.

We have also pointed out that V is DSOT-hypercyclic for L_T if and only if V^* is SOT-hypercyclic for R_{T^*}, and so we may summarize some conclusions in the following way. If X^*, and thus X, is separable, then:

(i) T satisfies the HC $\Rightarrow$ L_T is DSOT-hypercyclic $\Rightarrow$ R_{T^*} is SOT-hypercyclic;

(ii) T^* satisfies the HC $\Leftrightarrow$ L_{T^*} is SOT-hypercyclic $\Leftrightarrow$ R_T is DSOT-hypercyclic.

We do not know if the implications in (i) in fact may be replaced by equivalences. (Problem 3.5 concerns the converse of the first implication.)

REMARK 3.8. We know, from Proposition 2.1, that any SOT-hypercyclic vector V for L_T must be one-to-one, in other words, V^* must have weak* dense range. It follows that any DSOT-hypercyclic vector V for L_T also must be injective. Indeed, for any $x^* \neq 0$ the map $u_{x^*} : L(X) \to X^*$ defined by $T \mapsto T^* x^*$ is surjective and continuous, where X^* carries the norm topology and $L(X)$ the DSOT. Thus $u_{x^*} \operatorname{Orb}(L_T, V) = V^* \operatorname{Orb}(T^*, x^*)$ must be norm dense in X^*. Consequently, $\operatorname{Im} V^*$ must be weak* dense, which is equivalent to that V is injective.

A hypercyclic vector V for right-multipliers R_T, for any of the topologies SOT and DSOT, must have dense range (V^* is injective). If, for example, V is DSOT-hypercyclic for R_T, then $u_{x^*} \operatorname{Orb}(R_T, V) = \{T^{*n} V^* x^* : n \in \mathbb{N}\}$ is dense for any $x^* \neq 0$. Hence $V^* x^* \neq 0$ for any nonzero x^*, so V^* is injective ($\operatorname{Im} V$ is dense). Similar arguments show that $\operatorname{Im} V$ must be dense for any SOT-hypercyclic vector V for R_T.

4. TOPOLOGIES ON X^* WHERE ANY NONZERO VECTOR OF X^* IS HYPERCYCLIC

For any operator $V \in L(X)$ we define the seminorm $\|x^*\|_V^* := \|V^* x^*\|$ on X^*. Given a family $\mathcal{V} \subseteq L(X)$ of operators, we let $\mathcal{J}_{\mathcal{V}}^*$ denote the topology on X^* that is generated by the seminorms $\|\cdot\|_V^*$, $V \in \mathcal{V}$. This means that $\mathcal{J}_{\mathcal{V}}^*$ is the weakest locally convex topology τ for which all the maps $V^* : (X^*, \tau) \to (X^*, \|\cdot\|)$, $V \in \mathcal{V}$ are continuous. Note also that

$$\|x^*\|_V^* = \sup_{x \in B_X} |\langle x, V^* x^* \rangle| = \sup_{x \in B_X} |\langle Vx, x^* \rangle| = \sup_{x \in V(B_X)} |\langle x, x^* \rangle|,$$

where B_X denotes the closed unit ball in X. Hence, $\mathcal{J}_{\mathcal{V}}^*$ can be described as the (polar) topology of uniform convergence on the sets $V(B_X)$, $V \in \mathcal{V}$, see for example Chapter 3 in [18]. When $\mathcal{V}$ consists of one single operator V, we simply write $\mathcal{J}_V^*$ for the corresponding topology.

THEOREM 4.1. *Assume $T \in L(X)$ is hereditarily hypercyclic and let $V \in L(X)$ be any* DSOT-*hypercyclic vector for L_T. Then every nonzero vector $x^* \in X^*$ is $\mathcal{J}_V^*$-hypercyclic for T^*, thus, T^* lacks nontrivial $\mathcal{J}_V^*$-closed invariant subsets.*

Proof. Let $x^* \in X^*$ be any nonzero vector and choose $x \in X$ such that $\langle x, x^* \rangle = 1$. Now, for any given $y^* \in X^*$ we have

$$\|T^{*n} x^* - y^*\|_V^* = \|L_T^n(V) - S\|_{x^*},$$

where

$$S := \langle \cdot, V^* y^* \rangle x \in L(X)$$

because $S^* x^* = \langle x, x^* \rangle V^* y^* = V^* y^*$. Hence the theorem. ∎

REMARK 4.2. Note that $\|\cdot\|_V^*$ is a norm, and thus $(X^*, \mathcal{J}_V^*)$ a normed space, if and only if V^* is one-to-one. That is, if and only if V has dense range, and if and only if $(X^*, \mathcal{J}_V^*)$ is separated (cf. Proposition 4.3). In particular we have that $\mathcal{J}_V^*$ is equivalent to the norm topology on X^* if and only if V is surjective (and thus bijective, see Remark 3.8), for we recall that this is equivalent to $\|V^* x^*\| \geqslant c\|x^*\|$ for some $c > 0$. We do not know if this is ever possible. (We know that $A_0(X)$ in the proof of Theorem 3.1 contains "many" hypercyclic vectors V for L_T, but none of them can be bijective, as $A_0(X)$ is formed by compact operators only. We also recall that an SOT-hypercyclic vector $V \in L(X)$ for L_T cannot be surjective, because then $\operatorname{Im} V = X$ would be a nondegenerating hypercyclic vector manifold, which is not possible by Proposition 2.4.)

Note also that T^* may fail to be continuous with respect to the topology $\mathcal{J}_V^*$.

There are indeed examples where a DSOT-hypercyclic vector $V \in L(X)$ for L_T fails to have dense range. Indeed, consider the operator $\lambda B : c_0 \to c_0$, where $|\lambda| > 1$ and B is the backward shift $(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$. It is well-known that $T := \lambda B$ is hereditarily hypercyclic and thus, since $c_0^* \simeq \ell_1$ is separable, L_T admits a DSOT-hypercyclic vector V_0 (Theorem 3.1). But $\ker T = \operatorname{span}\{e_0\}$ is complemented, $c_0 = [e_0] \oplus [e_1, e_2, \dots]$, and so we can find a projector $\Pi : c_0 \to [e_1, \dots]$ $(:= \overline{\operatorname{span}\{e_1, \dots\}})$. It is now clear that $V := \Pi V_0$ is another hypercyclic vector for L_T, but V has evidently not a dense range. On the other hand, in general, given any DSOT-hypercyclic vector $V \in L(X)$ for L_T, $V + W$ is another (possibly densely ranged) DSOT-hypercyclic vector for L_T provided $\operatorname{Im} W \subseteq \ker T^n$ for some n (in Example 4.7 below we apply a similar observation).

Thus to any DSOT-hypercyclic vector V for L_T, we can associate a topology $\mathcal{J}_V^*$ (not necessarily separated) for which every nonzero vector is hypercyclic for T^*. The question is now if we can find families $\mathcal{V}$ of operators on X, to obtain separated topologies $\mathcal{J}_{\mathcal{V}}^*$ for which every nonzero vector is hypercyclic for T^*. Before we establish such constructions, we have the following general proposition.

PROPOSITION 4.3. *Let $\mathcal{V}$ be any family of operators in $L(X)$. Then*

(i) $(X^*, \mathcal{J}_{\mathcal{V}}^*)$ *is separated if and only if any of the following equivalent properties holds true:*

(a) $\bigcap_{V \in \mathcal{V}} \ker V^* = \{0\}$,

(b) $\bigcup_{V \in \mathcal{V}} \operatorname{Im} V$ *is total in X;*

(ii) *the dual of $(X^*, \mathcal{J}_{\mathcal{V}}^*)$ contains*

$$\operatorname{span} \bigcup_{V \in \mathcal{V}} \operatorname{Im} V^{**} (\subseteq X^{**}),$$

and if $(X^*, \mathcal{J}_{\mathcal{V}}^*)$ is separated this is precisely the dual of $(X^*, \mathcal{J}_{\mathcal{V}}^*)$.

Proof. Since $\mathcal{J}_{\mathcal{V}}^*$ is generated by the seminorms $x^* \mapsto \|V^* x^*\|$, $V \in \mathcal{V}$, it is clear that $\mathcal{J}_{\mathcal{V}}^*$ is separated if and only if $\bigcap_{\mathcal{V}} \ker V^* = \{0\}$. But, for the duality between X and X^*,

$$\bigcap_{\mathcal{V}} \ker V^* = \Big(\bigcup_{\mathcal{V}} \operatorname{Im} V\Big)^{\perp}$$

and $\Big(\bigcup_{\mathcal{V}} \operatorname{Im} V\Big)^{\perp\perp}$ is the closed linear hull of $\bigcup_{\mathcal{V}} \operatorname{Im} V$, so (i) follows.

Next, it is easily checked that any element of $\operatorname{Im} V^{**}$ is a continuous linear form with respect to $\mathcal{J}_{\mathcal{V}}^*$ whenever $V \in \mathcal{V}$, and so $\operatorname{span} \bigcup_{\mathcal{V}} \operatorname{Im} V^{**}$ is contained in the dual of $(X^*, \mathcal{J}_{\mathcal{V}})$. The last part of (ii) follows by Proposition 3.14.5(a) in [18]. ∎

COROLLARY 4.4. *$(X^*, \mathcal{J}_{\mathcal{V}}^*)$ is separated provided $\mathcal{V}$ is* DSOT*-dense in $L(X)$.*

Proof. Assume $\mathcal{V}$ is DSOT-dense but that $\bigcup_{V \in \mathcal{V}} \operatorname{Im} V$ is not total. Then there exists an $x^* \neq 0$ that is orthogonal to any Vx, $(x, V) \in X \times \mathcal{V}$. But if I_{X^*} denotes the identity operator on X^*, there exists a net (V_α) in $\mathcal{V}$ such that $V_\alpha^* x^* \to I_X^* x^* = x^*$ in norm. Hence, for any $x \in X$ we have $\langle x, x^* \rangle = \lim_\alpha \langle x, V_\alpha^* x^* \rangle = \lim_\alpha \langle V_\alpha x, x^* \rangle = 0$, and thus a contradiction. ∎

THEOREM 4.5. *Assume $T \in L(X)$ is hereditarily hypercyclic and let $\mathcal{V}$ be any nondegenerating hypercyclic vector manifold for L_T with respect to the* DSOT *(hence X^* and X are separable in norm). Then $\mathcal{J}_{\mathcal{V}}^*$ is a topology on X^* for which every nonzero vector $x^* \in X^*$ is hypercyclic for T^* (and thus for which T^* lacks nontrivial closed invariant subsets). In particular, if $\mathcal{V}$ is* DSOT*-dense (exists by Theorem* 3.1*), $(X^*, \mathcal{J}_{\mathcal{V}}^*)$ is separated.*

Proof. Let U be any neighbourhood, with respect to $\mathcal{J}_{\mathcal{V}}^*$, for an arbitrary point $y^* \in X^*$. This means that there exist nonzero vectors $V_1, \ldots, V_n \in \mathcal{V}$ and $\varepsilon > 0$ such that

$$\{z^* : \|z^* - y^*\|_{V_i}^* \leqslant \varepsilon,\ i = 1, \ldots, n\} \subseteq U.$$

In fact, we may assume that $V_1, \ldots, V_n$ are linearly independent. Indeed, just note that if $V_1 \in \operatorname{span}\{V_2, \ldots, V_n\}$, then $\|\cdot\|_{V_1}^* \leqslant C \sum_2^n \|\cdot\|_{V_i}^*$ for some constant $C > 0$ (cf. the proof of Proposition 2.1). Thus $(V_1, \ldots, V_n)$ is a DSOT-hypercyclic vector for $\bigoplus_1^n L_T$.

Let now x^* be any nonzero vector of X^*.

Pick $x \in X$ so that $\langle x, x^* \rangle = 1$. Next we define $S_i := \langle \cdot, V_i^* y^* \rangle x \in L(X)$. Then $S_i^* x^* = V_i^* y^*$ and so

$$\|T^{*m} x^* - y^*\|_{V_i}^* = \|L_T^m(V_i) - S_i\|_{x^*}^*. \tag{4.1}$$

Since $(V_1, \ldots, V_n)$ is DSOT-hypercyclic for $\bigoplus_1^n L_T$, we can find an m so that these expressions (4.1) are less than ε for all i, and thus $T^{*m} x^* \in U$.

The last statement follows by Corollary 4.4. ∎

As in Remark 4.2 it is not necessarily true that T^* is continuous with respect to the topology $\mathcal{J}_{\mathcal{V}}^*$. Note also that if $\mathcal{V}$ is DSOT-dense, then $X_{\mathcal{V}} := \operatorname{span} \bigcup_{V \in \mathcal{V}} \operatorname{Im} V^{**}$ ($\subseteq X^{**}$) is the dual of $(X^*, \mathcal{J}_{\mathcal{V}}^*)$ (Proposition 4.3). Thus the weak topology $\sigma(X^*, X_{\mathcal{V}})$ on X^* is coarser or equal to $\mathcal{J}_{\mathcal{V}}^*$, and so every $x^* \neq 0$ is $\sigma(X^*, X_{\mathcal{V}})$-hypercyclic for T^* under the hypothesis of Theorem 4.5. This motivates a study on describing $X_{\mathcal{V}}$, see Subsection 6.3 for more comments on this.

Another important class of sets $\mathcal{V} \subseteq L(X)$, in the context, is the following theorem.

THEOREM 4.6. *Assume $T \in L(X)$ is hereditarily hypercyclic and let $\mathbb{V} = (V_i) \in L(X)^{\mathbb{N}}$ be any* DSOT-*hypercyclic vector for $\bigoplus_1^\infty L_T$ (hence X^* and X are separable in norm). Then $\mathcal{J}_{\mathbb{V}}^*$ is a topology for which every nonzero vector x^* is hypercyclic for T^*. In particular, if we take $\mathbb{V}$ so that $\{V_n : n \in \mathbb{N}\}$ is* DSOT-*dense in $L(X)$ (exists by Theorem* 3.1*), $\mathcal{J}_{\mathbb{V}}^*$ is a metrizable locally convex topology on X^*.*

Proof. For the first part, just note that if we take any n-tuple $(V_{m_1}, \ldots, V_{m_n})$ of different vectors from $\{V_i\}$, $(V_{m_1}, \ldots, V_{m_n})$ is a hypercyclic vector for $\bigoplus_1^n L_T$. The proof of the fact that every nonzero vector is hypercyclic goes parallel to that of Theorem 4.5.

Next, if $\{V_n : n \in \mathbb{N}\}$ is DSOT-dense in $L(X)$, $(X, \mathcal{J}_{\mathbb{V}}^*)$ is separated by Corollary 4.4, and since the topology $\mathcal{J}_{\mathbb{V}}^*$ is generated by a countable family of seminorms $(\|\cdot\|_{V_i})$, it is metrizable, see e.g. Theorem 2.6.1 in [18]. ∎

More generally we have that $\mathcal{J}_{\mathbb{V}}^*$ is metrizable once $(X^*, \mathcal{J}_{\mathbb{V}}^*)$ is separated, i.e., once the set $\bigcup_{n \geqslant 1} \operatorname{Im} V_n$ is total (Proposition 4.3). We now show how we can force this to happen by a "small" perturbation of the given hypercyclic vector $\mathbb{V}$ for $\bigoplus_1^\infty L_T$.

EXAMPLE 4.7. Let $\mathbb{V} = (V_i) \in L(X)^{\mathbb{N}}$ be any DSOT-hypercyclic vector for $\bigoplus_1^\infty L_T$. Assume $T \in L(X)$ has dense generalized kernel $\bigcup_{n \geqslant 1} \ker T^n$ (in particular this applies to $T = \lambda B$ in Remark 4.2). We may thus find a countable dense set $\{x_n : n \in \mathbb{N}\}$ in X of vectors x_n from $\bigcup_{n \geqslant 1} \ker T^n$. The open balls B_n with centre at x_n and radius $1/n$, respectively, form an open base $\{B_n : n \in \mathbb{N}\}$ for X. Choose for each n vectors $x_n^* \in X^*$ and $y_n \in X$ so that $\|V_n y_n\| < 1/n$ and $\langle y_n, x_n^* \rangle = 1$. Put $W_n := \langle \cdot, x_n^* \rangle x_n + V_n \in L(X)$ and $\mathbb{W} := (W_i)$. Clearly, $\mathbb{W}$ is DSOT-hypercyclic for $\bigoplus_1^\infty L_T$. Indeed, for any m we have that $T^n W_i = T^n V_i$ for all $i \leqslant m$, if n is sufficiently large. Moreover, $\bigcup_{n \geqslant 1} \operatorname{Im} W_n$ is dense, and thus total, because it contains the vectors $W_n y_n = x_n + V_n y_n \in B_n$. We conclude that

$\mathcal{J}_{\mathbb{W}}^*$ is metrizable. Summing up, if T has dense generalized kernel, then we can perturbate any DSOT-hypercyclic vector $\mathbb{V} \in L(X)^{\mathbb{N}}$ for $\bigoplus_1^\infty L_T$ by a sequence of rank one operators to obtain a new DSOT-hypercyclic vector $\mathbb{W}$ such that $\mathcal{J}_{\mathbb{W}}^*$ is metrizable.

Note that since $\mathcal{J}_{\mathbb{V}}^*$ is always coarser than the norm topology, the open mapping theorem gives that $(X^*, \mathcal{J}_{\mathbb{V}}^*)$ is a Fréchet space if and only if $\mathcal{J}_{\mathbb{V}}^*$ equals the norm topology. Thus, in view of Remark 4.2, under the hypothesis that $(X^*, \mathcal{J}_{\mathbb{V}}^*)$ is separated and thus metrizable, $(X^*, \mathcal{J}_{\mathbb{V}}^*)$ is most probably not complete (i.e. not a Fréchet space).

Observe also that we never used that T is hereditarily hypercyclic in Theorems 4.1, 4.5 and 4.6, thus these statements remain true once we can find a DSOT-hypercyclic vector V, a nondegenerating DSOT-hypercyclic vector manifold $\mathcal{V}$ and a DSOT-hypercyclic vector $\mathbb{V}$, respectively. However, since hereditarily hypercyclicity of T implies the existence of these objects, we preferred to add this hypothesis.

5. TOPOLOGIES ON X WHERE ANY NONZERO VECTOR OF X IS HYPERCYCLIC

For any operator $V \in L(X)$ we define the seminorm $\|x\|_V := \|Vx\|$ on X. (Hence $\|\cdot\|_V^* = \|\cdot\|_{V^*}$ on X^*.) The topology on X that is generated by the seminorms $\|\cdot\|_V$, $V \in \mathcal{V}$, where $\mathcal{V} \subseteq L(X)$, is denoted by $\mathcal{J}_{\mathcal{V}}$. We write, for simplicity, $\mathcal{J}_V$ when $\mathcal{V} = \{V\}$. The topology $\mathcal{J}_{\mathcal{V}}$ can also be described as the weakest locally convex topology τ for which all the maps $V : (X, \tau) \to (X, \|\cdot\|)$, $V \in \mathcal{V}$, are continuous. Moreover, with similar computations as in the beginning of Section 4 we deduce that $\|x\|_V = \sup_{x^* \in V^*(B_{X^*})} |\langle x, x^* \rangle|$, and so $\mathcal{J}_{\mathcal{V}}$ is the (polar) topology of uniform convergence on the sets $V^*(B_{X^*})$, $V \in \mathcal{V}$. Here B_{X^*} denotes the closed unit ball in X^*. We have the following analogue of Proposition 4.3.

PROPOSITION 5.1. *Let $\mathcal{V}$ be any family of operators in $L(X)$. Then*

(i) *$(X, \mathcal{J}_{\mathcal{V}})$ is separated if and only if any of the following two equivalent properties holds true*:

(a) $\bigcap_{V \in \mathcal{V}} \ker V = \{0\}$,

(b) $\bigcup_{V \in \mathcal{V}} \operatorname{Im} V^*$ *is weak* total in X^*;*

(ii) *the dual of $(X, \mathcal{J}_{\mathcal{V}})$ contains* $\operatorname{span} \bigcup_{V \in \mathcal{V}} \operatorname{Im} V^*$, *and if $(X, \mathcal{J}_{\mathcal{V}})$ is separated this is precisely the dual of $(X, \mathcal{J}_{\mathcal{V}})$.*

Proof. Clearly, $(X, \mathcal{J}_{\mathcal{V}})$ is separated if and only if $\bigcap_{V \in \mathcal{V}} \ker V = \{0\}$. But we have that $\bigcap_{V \in \mathcal{V}} \ker V = \left(\bigcup_{\mathcal{V}} \operatorname{Im} V^* \right)^{\perp}$, where the orthogonal complement is taken with respect to the dual pair (X, X^*). Consequently, $\Big(\bigcap_{V\in\mathcal{V}} \ker V\Big)^{\perp}$ is the weak* closed linear hull of $\bigcup_{\mathcal{V}} \operatorname{Im} V^*$. This completes the proof of (i). The second statement follows by Proposition 3.14.5(a) in [18]. ∎

From these arguments, as in the proof of Corollary 4.4, we obtain the following corollary.

COROLLARY 5.2. *$(X, \mathcal{J}_{\mathcal{V}})$ is separated provided $\mathcal{V}$ is* SOT*-dense in $L(X)$.*

We know, from Theorem 3.6, that $\bigoplus_1^\infty R_T$ admits an SOT-hypercyclic vector (V_i) (respectively, a hypercyclic vector manifold $\mathcal{V}$) where $\{V_i\}$ (respectively, $\mathcal{V}$) is SOT-dense. And in this way we may obtain separated topologies on X where all nonzero vectors are hypercyclic for T.

THEOREM 5.3. *Assume T^* is hereditarily hypercyclic, where $T \in L(X)$. Then, any nonzero vector $x \in X$ is hypercyclic for T, for any of the following topologies on X:*

(i) *$\mathcal{J}_V$ where $V \in L(X)$ is an* SOT*-hypercyclic vector for R_T;*

(ii) *$\mathcal{J}_{\mathbb{V}}$ where $\mathbb{V} = (V_i)$ is an* SOT*-hypercyclic vector for $\bigoplus_1^\infty R_T$;*

(iii) *$\mathcal{J}_{\mathcal{V}}$ where $\mathcal{V}$ is a nondegenerating* SOT*-hypercyclic vector manifold for R_T.*

In (ii) *and* (iii) *we may take (Theorem* 3.6*) $\mathbb{V}$ and $\mathcal{V}$ as* SOT*-dense, respectively, so that the corresponding topologies $\mathcal{J}_{\mathbb{V}}$ and $\mathcal{J}_{\mathcal{V}}$ are separated (Corollary* 5.2*).*

Proof. Let $V \in L(X)$ be an SOT-hypercyclic vector for R_T. Let x be any nonzero element, and y any element, of X. In order to prove the first statement, we must show that for any given $\varepsilon > 0$ there is an n so that $\|T^n x - y\|_V < \varepsilon$. We may find an $x^* \in X^*$ so that $\langle x, x^*\rangle = 1$. Consider the operator $S := \langle \cdot, x^*\rangle Vy$ in $L(X)$. Clearly $Sx = Vy$, and so

$$\|T^n x - y\|_V = \|V(T^n x - y)\| = \|VT^n x - Sx\| = \|R_T^n(V) - S\|_x.$$

Since V is SOT-hypercyclic for R_T, $\|R_T^n(V) - S\|_x$ is, for some n, smaller than any given $\varepsilon > 0$. This completes the proof of the first part. The proofs of (ii) and (iii) go similar to those of Theorems 4.6 and 4.5, respectively. ∎

6. CONCLUDING REMARKS

6.1. With an additional assumption on the separable Banach space X, the statements above remain true for a finer topology on $L(X)$, i.e., finer than the SOT and the DSOT. Indeed, by the $*$-strong operator topology ($*$-SOT) we mean the topology on $L(X)$ that is generated by the seminorms

$$\|V\|_{x,x*} := \max(\|V\|_x, \|V\|^*_{x^*}) = \max(\|Vx\|, \|V^* x^*\|),$$

$(x, x^*) \in X \times X^*$. Evidently, the $*$-SOT is finer or equal to the SOT as well as to the DSOT. (The $*$-SOT is some sort of generalisation of the strong* topology [12], that is defined by the seminorms $V \mapsto \max(\|Vx\|, \|V^*x\|)$, $x \in X$, where X denotes a Hilbert space.) We have now the following lemma.

LEMMA 6.1. *Assume X has a shrinking Schauder basis $(e_n)_{n \geqslant 1}$. Then the set $A(X)$ of finite rank operators is dense in $L(X)$ with respect to the $*$-SOT.*

Proof. Let $(e_n^*) \subset X^*$ be the biorthogonal functionals relative to the basis (e_n). By our hypothesis that (e_n) is shrinking, (e_n^*) forms a basis of X^* (recall that this is the definition of (e_n) being shrinking [19]) and its biorthogonal system in X^{**} is the system $(e_n) \subset X \subseteq X^{**}$. Let now $P_n := \sum_1^n \langle \cdot, e_i^* \rangle e_i \in A(X)$, $n = 1, 2, \ldots$. Then $P_n \to I$ $(n \to \infty)$ uniformly on compact sets, where $I = I_X$ denotes the identity operator on X, see e.g. [19] page 30. Thus, since P_n^* is the analogue of P_n relative to the basis (e_n^*), we have in the same way that $P_n^* \to I_{X^*}$ uniformly on compact sets in X^*. From the identity $P_n T P_n - T = (P_n - I) T P_n + T(P_n - I)$ we obtain for any $x \in X$:

$$\|P_n T P_n - T\|_x \leqslant \|P_n - I\|_{TP_n x} + \|T\| \|P_n - I\|_x.$$

If now $\{x_1, \ldots, x_m\}$ is a finite set of points in X, $\{TP_i x_j : i \geqslant 0,\ j \leqslant m\}$ is a relatively compact set in X (because $\lim_i TP_i x_j$ exists for each j). Thus, for any given $\varepsilon > 0$ there is an N such that $\|P_n T P_n - T\|_{x_i} \leqslant \varepsilon$ for all i whenever $n \geqslant N$. By symmetry, given points $x_1^*, \ldots, x_r^*$ in X^*, there is an M such that $\|P_n T P_n - T\|_{x_i^*}^* \leqslant \varepsilon$ for all i provided $n \geqslant M$. This shows that $P_n T P_n \to T$ with respect to the $*$-SOT, and hence the theorem since $P_n T P_n \in A(X)$. ∎

REMARK 6.2. Recall that the canonical unit bases of c_0 and ℓ_p $(1 < p < \infty)$ are shrinking, and further examples of Banach spaces with such bases are given in [19].

It is now clear that, in the present setting, the proof of Theorem 3.1 applies to the $*$-SOT, so we have the following theorem.

THEOREM 6.3. *Assume $T \in L(X)$ is hereditarily hypercyclic where X is a Banach space with shrinking basis. Then $\bigoplus_1^\infty L_T$ (and thus any $\bigoplus_1^n L_T : L(X)^n \to L(X)^n$) is $*$-SOT-hypercyclic.*

Further, $\bigoplus_1^\infty L_T$ admits a $*$-SOT-hypercylic vector $\mathbb{V} = (V_i) \in L(X)^{\mathbb{N}}$ such that $\{V_n : n \in \mathbb{N}\}$ is $*$-SOT-dense in $L(X)$ and, accordingly, under the $*$-SOT L_T supports a dense nondegenerating hypercyclic vector manifold $\mathcal{V} \subseteq L(X)$.

In particular Theorem 6.3 applies when X is a separable Hilbert space, and a weaker result for this setting was obtained in [29].

Any $*$-SOT-hypercyclic vector $V \in L(X)$ for L_T must be both SOT and DSOT-hypercyclic for L_T. Thus, by Proposition 1.3, $T \in L(X)$ is hereditarily hypercyclic if and only if L_T is $*$-SOT hypercyclic on $L(X)$, when X is a Banach space with shrinking basis.

With the hypothesis of Theorem 6.3 we know thus that, in Theorem 4.1, we can take V as a $*$-SOT-hypercyclic vector for L_T. In the same way, in Theorems 4.5 and 4.6 we can in fact take $\mathcal{V}$ and $\mathbb{V}$ to be a nondegenerating hypercyclic vector manifold for L_T and, respectively, a hypercyclic vector for $\bigoplus_1^\infty L_T$, with respect to the $*$-SOT.

6.2. An interesting line of further investigation is to take into account results on common hypercyclic vectors (see e.g. [1], [2], [3], [16]), in the context. Indeed, consider for example a countable family $\{T_n \in L(X) : n \in \mathbb{N}\}$ of hereditarily hypercyclic operators. From the proof of Theorem 3.1 we know that each $\bigoplus_1^\infty L_{T_n} : A_0(X)^{\mathbb{N}} \to A_0(X)^{\mathbb{N}}$ ($n \in \mathbb{N}$) is hypercyclic, and thus, see e.g. Proposition 8 in [16], these operators have a common hypercyclic vector $\mathbb{V}$, which now forms a common DSOT-hypercyclic vector for the operators $\bigoplus_1^\infty L_{T_n} : L(X)^{\mathbb{N}} \to L(X)^{\mathbb{N}}$, $n \in \mathbb{N}$. Accordingly, $\mathcal{J}^*_{\mathbb{V}}$ is a topology such that every nonzero vector of X^* is hypercyclic for every T_n^*, $n \in \mathbb{N}$, and so each T_n^* lacks nontrivial $\mathcal{J}^*_{\mathbb{V}}$-closed invariant subsets.

6.3. Any topology $\mathcal{J}^*_{\mathcal{V}}$, $\mathcal{V} \subseteq L(X)$, in Section 4, is in general coarser than the norm topology on X^* (see Remark 4.2). However, an interesting problem is how such a topology $\mathcal{J}^*_{\mathcal{V}}$ relates to the weak* topology $\sigma(X^*, X)$ (and to the weak topology $\sigma(X^*, X^{**})$), where $\mathcal{V}$ is a suitable family of DSOT-hypercyclic vectors for L_T (operator families as in Theorems 4.1, 4.5 and 4.6 are, in the context, of special interest). Assume $\bigcup_{V \in \mathcal{V}} \operatorname{Im} V$ is total in X (this holds when $\mathcal{V}$ is DSOT-dense), so that $\mathcal{J}^*_{\mathcal{V}}$ is separated (Proposition 4.3). By Proposition 4.3 it then follows that $\mathcal{J}^*_{\mathcal{V}}$ is equal or finer than the weak* topology if and only if $\operatorname{span} \bigcup_{\mathcal{V}} \operatorname{Im} V^{**} \supseteq X$. Thus, in particular, if $\operatorname{Im} V$ is dense then $\mathcal{J}^*_V \geqslant \sigma(X^*, X)$ if and only if $\operatorname{Im} V^{**}$ contains X ($\subseteq X^{**}$). This may happen even if V is not surjective. The following problem is therefore relevant:

How/when can we find a DSOT-*hypercyclic vector* $V \in L(X)$ *for* L_T *with* $\overline{\operatorname{Im} V} = X$ *and* $X \subseteq \operatorname{Im} V^{**}$?

Recall that V is weakly compact if and only if $\operatorname{Im} V^{**} \subseteq X$, see Theorem 3.5.8 in [20] for a proof, and by the proof of Theorem 3.1 we can, under the hypothesis there, find a DSOT-dense nondegenerating hypercyclic vector manifold $\mathcal{V}$ of compact (hence weakly compact) operators. With the assumptions in Theorem 3.1, there also exists a DSOT-dense hypercyclic vector $\mathcal{V} = \mathbb{V}$ of (weakly) compact operators for $\bigoplus_1^\infty L_T$. For such families of operators we thus have that the dual $X_{\mathcal{V}} = \operatorname{span} \bigcup_{\mathcal{V}} \operatorname{Im} V^{**}$ of $(X^*, \mathcal{J}_{\mathcal{V}}^*)$ is contained in X.

In the same way we may ask how the topologies $\mathcal{J}_V, \mathcal{J}_{\mathcal{V}}, \mathcal{J}_{\mathbb{V}}$, in Theorem 3.6, relate to the weak topology $\sigma(X, X^*)$. We know for example, from Proposition 5.1, that if $\bigcup_1^\infty \operatorname{Im} V_i^*$ is weak* total, then $\mathcal{J}_{\mathbb{V}}$ is equal or finer than $\sigma(X, X^*)$ if and only if $\operatorname{span} \bigcup_1^\infty \operatorname{Im} V_i^* = X^*$. Here $\mathbb{V} = (V_i)$. This leads us to the problem:

How/when can we find an SOT-*hypercyclic vector* (V_i) *for* $\bigoplus_1^\infty R_T$ *so that the linear hull of* $\bigcup_1^\infty \operatorname{Im} V_i^*$ *is equal to* X^*?

6.4. We have constructed topologies on Banach spaces X so that every nonzero vector is hypercyclic for $T \in L(X)$. We suggest a study on the following. Given a suitable set $M \subseteq X$, how can we construct topologies so that every nonzero element of M is hypercyclic for T? In particular, given an infinite dimensional subspace $M \subseteq X$:

How/when can we define a topology on X so that M is a hypercyclic subspace for T?

We recall that a hypercyclic subspace for an operator is a closed infinite dimensional subspace of, except for zero, hypercyclic vectors. Such subspaces have been studied for example in [1], [8], [11], [21], [22], [23]. Note that we have constructed topologies so that the full space X forms a hypercyclic subspace, but this does not imply that any infinite dimensional subspace $M \subseteq X$ is a hypercyclic subspace for this topology.

6.5. We have pointed out that T (and T^*) may not be continuous for the topology $\mathcal{J}_{\mathcal{V}}$ (respectively $\mathcal{J}_{\mathcal{V}}^*$). We have thus questions like:

How/when can we find an SOT-*hypercyclic vector* (V_i) *for* $\bigoplus_1^\infty R_T$ *so that* T *is continuous for* $\mathcal{J}_{\mathbb{V}}$?

Note that if V is SOT-hypercyclic for R_T, then so is any operator $V_n = VT^n$. It is evident that T is continuous for $\mathcal{J}_{\mathbb{V}}$, where $\mathbb{V} = (V_n)$, but $\mathbb{V}$ cannot here be an SOT-hypercyclic vector for $\bigoplus_1^\infty R_T$ (see the proof of Proposition 2.4).

6.6. We have in this note introduced the concept *nondegenerating hypercyclic vector manifold* (Definition 2.3). We suggest a further study on this type of manifolds. In particular, what can be said about closed nondegenerating hypercyclic vector manifolds (see Subsection 6.4 above)?

6.7. Based on the arguments in the proof of Theorem 3.1 in [8], some results in this note remain presumably true if we replace X by a separable Fréchet space that admits a continuous norm. The DSOT is here assumed to be taken with respect to the strong topology on X^*, that is, the DSOT is generated by the seminorms

$$\|V\|_{B,x^*} := \sup_{x\in B} |\langle x, V^*x^*\rangle| = \sup_{x\in B} |\langle Vx, x^*\rangle|$$

where $B \subseteq X$ is bounded and $x^* \in X^*$. (Recall that Propositions 1.2 and 1.3 extend to separable Fréchet spaces with a continuous norm.)

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