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FAITHFULNESS OF THE FOCK REPRESENTATION OF THE C^*-ALGEBRA GENERATED BY q_{ij}-COMMUTING ISOMETRIES

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ABSTRACT. We consider the C^*-algebra Isom_Q, where $Q = (q_{ij})_{i,j=1}^n$ is a matrix of complex numbers. This algebra is generated by n isometries $a_1, \ldots, a_n$ satisfying the relations $a_i^* a_j = q_{ij} a_j a_i^*$, $i \neq j$ with $\max |q_{ij}| < 1$. This C^*-algebra is shown to be nuclear. We prove that the Fock representation of Isom_Q is faithful. Further we describe an ideal in Isom_Q which is isomorphic to the algebra of compact operators.

KEYWORDS: *C^*-algebra, Cuntz algebra, nuclear, q-deformation, Fock representation, operator algebras.*

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1. INTRODUCTION

Operator algebras generated by various deformations of the canonical commutation relations (CCR) have been extensively studied in the last two decades. In particular, considerable attention has been paid to the study of so-called Q-CCR introduced by M. Bozejko and R. Speicher, see [3]. Assume that $Q = (q_{ij})_{i,j=1}^n$ satisfy the following conditions:

$$|q_{ii}| < 1, \quad q_{ii} \in \mathbb{R}, i = 1, \ldots, n; \quad |q_{ij}| \leqslant 1,\ \overline{q}_{ji} = q_{ij}, \quad q_{ij} \in \mathbb{C}, i \neq j.$$

Then define Q-CCR to be a $*$-algebra generated by elements a_i, a_i^*, $i = 1, \ldots, n$, satisfying the following relations:

$$a_i^* a_i = 1 + q_{ii} a_i a_i^*; \quad a_i^* a_j = q_{ij} a_j a_i^* \quad \text{for } i \neq j.$$

It is a deformation of $*$-algebras of the classical commutation relations in the sense that in the Fock realisation, the limiting cases $q_{ij} = 1$ and $q_{ij} = -1$ correspond to $*$-algebras of the canonical commutation relations (CCR) and the canonical anticommutation relations (CAR) respectively.

Let us describe the main results on the subject.

The $*$-algebra Q-CCR possesses a particular representation π_{F} called the Fock representation. It is determined uniquely up to a unitary equivalence by the following property: there exists a cyclic vector Ω such that $\pi_{\mathrm{F}}(a_i^*)(\Omega) = 0$ for $i = 1, \dots, n$. The problem of existence and uniqueness of π_{F} was studied in [3], [6], [10] and in [19] for a more general class of Wick algebras.

It can easily be verified that in any $*$-representation π of the $*$-algebra Q-CCR by bounded operators one has

$$\|\pi(a_i)\| \leqslant \frac{1}{\sqrt{1 - |q_{ii}|}}, \quad i = 1, \dots, n.$$

Hence, there exists a universal enveloping C^*-algebra associated to Q-CCR denoted below by Q-$\mathcal{CCR}$.

When $\max |q_{ij}| < 1$ it is natural to think of the C^*-algebra Q-$\mathcal{CCR}$ as a deformation of the Cuntz–Toeplitz algebra $\mathcal{KO}_n$. Recall that $\mathcal{KO}_n$ is the universal C^*-algebra generated by s_i, s_i^*, $i = 1, \dots, n$ subject to the relations

$$s_i^* s_j = \delta_{ij} 1, \quad i, j = 1, \dots, n.$$

This point of view was justified by P.E.T. Jorgensen, L.M. Schmidt and R.F. Werner who showed that Q-$\mathcal{CCR} \simeq \mathcal{KO}_n$ whenever $\max |q_{ij}| < \sqrt{2} - 1$, see [9] for more details. It remains a conjecture that Q-$\mathcal{CCR} \simeq \mathcal{KO}_n$ whenever $\max |q_{ij}| < 1$.

Many authors were also interested in the study of the C^*-algebra generated by operators of the Fock representation of Q-CCR. Namely, K. Dykema and A. Nica in [5] proved that $\pi_{\mathrm{F}}(Q\text{-}\mathcal{CCR}) \simeq \mathcal{KO}_n$ for slightly larger value of $\max |q_{ij}|$. Also an embedding of $\mathcal{KO}_n$ into $\pi_{\mathrm{F}}(Q\text{-}\mathcal{CCR})$ was constructed for any values of deformation parameters with $\max |q_{ij}| < 1$.

Later M. Kennedy in [11] showed the existence of an embedding of $\pi_{\mathrm{F}}(Q\text{-}\mathcal{CCR})$ into $\mathcal{KO}_n$ and proved that $\pi_{\mathrm{F}}(Q\text{-}\mathcal{CCR})$ is an exact C^*-algebra.

Let us stress out that results concerning $\pi_{\mathrm{F}}(Q\text{-}\mathcal{CCR})$ cannot be automatically lifted to the universal C^*-algebra level since at the moment we do not know whether or not π_{F} is a faithful $*$-representation of Q-$\mathcal{CCR}$ for any $|q_{ij}| < 1$. However π_{F} is a faithful representation of Q-CCR, i.e. it is faithful on the $*$-algebraic level, see [7].

Some boundary cases of Q-$\mathcal{CCR}$ corresponding to $|q_{ii}| < 1$, $i = 1, \dots, n$, $|q_{ij}| = 1$, $i \neq j$ were studied in [12], [16], see also [18]. For these values of q_{ij} it was shown that Q-$\mathcal{CCR} \simeq \mathrm{Isom}_Q$ where Isom_Q is a C^*-algebra generated by isometries $a_1, \dots, a_n$ such that $a_i^* a_j = q_{ij} a_j a_i^*$, $i \neq j$. It was also shown that for the values of parameters q_{ij} specified above, the C^*-algebra Isom_Q is nuclear and its Fock representation is faithful.

Notice, that the C^*-algebras Isom_Q with $\max |q_{ij}| < 1$ have completely different structure than in the case of $|q_{ij}| = 1$, $i \neq j$. It was shown in [16] that for two generators and $|q_1| = |q_2| = 1$, $\mathrm{Isom}_{q_1} \simeq \mathrm{Isom}_{q_2}$ if and only if $q_1 = q_2$ or $q_1 = \overline{q}_2$. If $|q| < 1$ one has $\mathrm{Isom}_q \simeq \mathcal{KO}_2$ as established in [8]. For more than two generators the isomorphism problem for Isom_Q and $\mathcal{KO}_n$ with $\max|q_{ij}| < 1$ remains open.

In this paper we study the C^*-algebras Isom_Q with $\max|q_{ij}| < 1$, $i,j = 1,\ldots,n$. Notice that this C^*-algebra is the same as Q-$\mathcal{CCR}$ with $q_{ii} = 0$, $i = 1,\ldots,n$.

When a C^*-algebra admits an action of a compact group, some of its properties can be studied at the level of the fixed point subalgebra for this action. In Section 3 we discuss conditions for a compact group to define a filtration preserving action on Q-$\mathcal{CCR}$ (and in particular on Isom_Q). We find the largest admissible group acting on Q-$\mathcal{CCR}$ regardless of the values of q_{ij} — it happens to be the n-dimensional torus $\mathbb{T}^n$. Then we begin to study the fixed point subalgebra with respect to the action of $\mathbb{T}^n$ named $\mathcal{GICAR}_Q$.

From Section 4 till the end of the paper we will consider the case of Isom_Q, i.e. $q_{ii} = 0$, $i = 1,\ldots,n$. In this case $\mathcal{GICAR}_Q$ turns out to be an AF-algebra. As a consequence it will follow that Isom_Q is nuclear.

In Section 5 we describe the Bratteli diagram of $\mathcal{GICAR}_Q$ and conclude its independence of the values of q_{ij}. Hence, we get an isomorphism of the fixed point subalgebras of Isom_Q and $\mathcal{KO}_n$.

Another problem which can be reduced to the fixed point subalgebra level is faithfulness of a $*$-homomorphism. Using information on the structure of the fixed point subalgebra, we prove in Section 6 that the Fock representation of Isom_Q, $\max|q_{ij}| < 1$, is faithful. This allows us to extend results and techniques of [5], [11] to the case of Isom_Q.

In Section 7 we prove the existence of an ideal $\mathcal{K}_Q \subset \mathrm{Isom}_Q$ isomorphic to the algebra of compact operators and describe a generator of this ideal as a projection in some finite-dimensional subalgebra of Isom_Q.

2. THE DEFORMED FOCK INNER PRODUCT

As it was mentioned in the introduction, the q_{ij}-deformed Fock representation has been the subject of numerous studies. For our purpose we only need a few basic facts about the structure of the Fock representation space of Q-CCR.

Let $\mathcal{H} = \mathbb{C}^n$ and $\{\xi_1,\ldots,\xi_n\}$ be its orthonormal basis. Consider the full tensor space

$$T(\mathcal{H}) = \{\Omega\} \oplus \bigoplus_{k\geqslant 1} \mathcal{H}^{\otimes k}.$$

Put $\mathcal{H}^{\otimes k} \perp \mathcal{H}^{\otimes l}$, $k \neq l$, and supply each $\mathcal{H}^{\otimes k}$ with the inner product $\langle\cdot,\cdot\rangle_{\mathrm{Fock}}$ specified below, see [3] for more details. Namely,

$$\langle\Omega,\Omega\rangle_{\mathrm{Fock}} = 1, \quad \text{and}$$

$$\langle \xi_{i_1} \otimes \cdots \otimes \xi_{i_k}, \xi_{j_1} \otimes \cdots \otimes \xi_{j_k} \rangle_{\text{Fock}}$$
$$= \sum_{t=1}^{k} \delta_{j_1 i_t} q_{j_1 i_1} \cdots q_{j_1 i_{t-1}} \langle \xi_{i_1} \otimes \cdots \otimes \widehat{\xi_{i_t}} \otimes \cdots \otimes \xi_{i_k}, \xi_{j_2} \otimes \cdots \otimes \xi_{j_k} \rangle_{\text{Fock}},$$

where $1 \leqslant i_1, \ldots, i_k, j_1, \ldots, j_k \leqslant n$, and the hat over ξ_{i_t} means that ξ_{i_t} is deleted from the tensor. Note that the natural basis $\xi_{i_1} \otimes \cdots \otimes \xi_{i_k}$ of $\mathcal{H}^{\otimes k}$ is not orthogonal with respect to $\langle \cdot, \cdot \rangle_{\text{Fock}}$.

3. COMPACT GROUP ACTIONS ON Q-$\mathcal{CCR}$

In this part of our paper we discuss symmetries of the C^*-algebra Q-$\mathcal{CCR}$ and explain how they can be useful. First, recall the definition of a group action.

DEFINITION 3.1. Let G be a compact group, and A a C^*-algebra.
(i) An action of G on A is a homomorphism $\gamma : G \to \mathrm{Aut}(A)$ which is continuous in the point-norm topology.
(ii) The fixed point subalgebra A^γ is the subset of all $a \in A$ such that $\gamma_g(a) = a$ for all $g \in G$.

Recall that for every action of a compact group G on a C^*-algebra A one can construct a faithful conditional expectation $E_\gamma : A \to A^\gamma$ onto the fixed point subalgebra, given by

$$E_\gamma(a) = \int_G \gamma_g(a) \mathrm{d}\lambda,$$

where λ is the Haar measure on G.

The following proposition explains our interest in studying the fixed point subalgebras.

PROPOSITION 3.2 ([4] Section 4.5, Theorem 1, 2). (i) *Let γ be an action of a compact group G on a C^*-algebra A. Then A is nuclear if and only if A^γ is nuclear.*
(ii) *Let α, β be actions of a compact group G on C^*-algebras A and B respectively and $\pi : A \to B$ is a $*$-homomorphism such that*

$$\pi \circ \alpha_g = \beta_g \circ \pi \quad \textit{for any } g \in G.$$

Then π is injective on A if and only if π is injective on A^α.

Our aim is to find a way to generate suitable actions on Q-$\mathcal{CCR}$. Recall that the Cuntz–Toeplitz algebra $\mathcal{KO}_n$ admits a natural filtration preserving action for every closed subgroup of the unitary group U_n. For the reader's convenience we provide below a short proof of this well-known fact.

PROPOSITION 3.3. *Let G be a closed subgroup of U_n and s_i, $i = 1, \dots, n$ be the generators of $\mathcal{KO}_n$. Then there exists an action γ of G on $\mathcal{KO}_n$ such that*

$$\gamma_g(s_i) = \sum_{j=1}^{n} s_j g_{ji}, \quad g = (g_{ij}) \in G.$$

Proof. One has to examine that $\gamma_g(s_i)$, $i = 1, \dots, n$, satisfy the basic relations of $\mathcal{KO}_n$. Indeed,

$$\gamma_g(s_i^*)\gamma_g(s_j) = \sum_{k,l=1}^{n} s_k^* s_l \overline{g}_{ki} g_{lj} = \sum_{k,l=1}^{n} \delta_{kl} \overline{g}_{ki} g_{lj} = \sum_{k=1}^{n} \overline{g}_{ki} g_{kj} = \delta_{ij}.$$

Thus, there is a correctly defined $*$-homomorphism $\gamma_g : \mathcal{KO}_n \to \mathcal{KO}_n$. Since g is unitary, this $*$-homomorphism is an automorphism. ∎

Our goal is to verify when the construction above is applicable to Q-$\mathcal{CCR}$. Unlike for $\mathcal{KO}_n$, not every closed subgroup of U_n gives a well-defined action on Q-$\mathcal{CCR}$. In the next lemma we state the conditions for an element of U_n to define an automorphism of Q-$\mathcal{CCR}$.

LEMMA 3.4. *Suppose $u \in U_n$, $a_i, i = 1, \dots, n$ are generators of Q-$\mathcal{CCR}$ and*

$$a_i' = \sum_{j=1}^{n} a_j u_{ji}, \quad i = 1, \dots, n.$$

Then the relations

$$a_i'^* a_j' - q_{ij} a_j' a_i'^* = \delta_{ij}, \quad i, j = 1, \dots, n$$

hold if and only if

$$q_{kl} \overline{u}_{ki} u_{lj} = q_{ij} u_{lj} \overline{u}_{ki}, \quad i, j, k, l = 1, \dots, n.$$

Proof. Suppose that

$$a_i'^* a_j' - q_{ij} a_j' a_i'^* = \delta_{ij}, \quad \text{for any } i, j = 1, \dots, n.$$

Then

$$\begin{aligned}
\delta_{ij} = a_i'^* a_j' - q_{ij} a_j' a_i'^* &= \sum_{k,l} (a_k^* a_l \overline{u}_{ki} u_{lj} - q_{ij} a_l a_k^* u_{lj} \overline{u}_{ki}) \\
&= \sum_{k,l} ((\delta_{lk} 1 + q_{kl} a_l a_k^*) \overline{u}_{ki} u_{lj} - q_{ij} a_l a_k^* u_{lj} \overline{u}_{ki}) \\
&= \sum_{k} \overline{u}_{ki} u_{kj} + \sum_{k,l} (q_{kl} a_l a_k^* \overline{u}_{ki} u_{lj} - q_{ij} a_l a_k^* \overline{u}_{ki} u_{lj}) \\
&= \delta_{ij} + \sum_{k,l} a_l a_k^* \overline{u}_{ki} u_{lj} (q_{kl} - q_{ij}).
\end{aligned}$$

The linear independence of $a_l a_k^*$, $l, k = 1, \dots, n$, implies the claim. ∎

Notice that if the groups $G_1, G_2 \subset U_n$ act on Q-$\mathcal{CCR}$ as in Lemma 3.4 and $G_1 \leqslant G_2$ then the fixed point subalgebra by the action of G_2 is included into the fixed point subalgebra by the action of G_1. Hence we would like to act by the largest possible subgroup of U_n to make the fixed point subalgebra as small as possible. In the following theorem we prove that the largest subgroup exists for an arbitrary choice of the parameters $\{q_{ij}\}$.

THEOREM 3.5. *Let* $G = \{u \in U_n : q_{kl}\overline{u}_{ki}u_{lj} = q_{ij}u_{lj}\overline{u}_{ki},\ i,j,k,l = 1,\dots,n\}$. *Then* G *is a closed subgroup of* U_n.

Proof. Step 1. Let u be the identity matrix. Then

$$q_{kl}\overline{u}_{ki}u_{lj} = q_{kl}\delta_{ki}\delta_{lj} = q_{ij}\delta_{lj}\delta_{ki} = q_{ij}u_{lj}\overline{u}_{ki}.$$

Step 2. Suppose $u \in G$. We prove that $u^{-1} = u^* \in G$, where $u^*_{ij} = \overline{u}_{ji}$. If u_{ik} or u_{jl} equal to 0 then trivially

$$q_{kl}\overline{u}^*_{ki}u^*_{lj} = q_{ij}u^*_{lj}\overline{u}^*_{ki}.$$

Otherwise, for $u \in G$ we know that

$$q_{ij}\overline{u}_{ik}u_{jl} = q_{kl}u_{jl}\overline{u}_{ik},$$

so $q_{kl} = q_{ij}$ and

$$q_{kl}\overline{u}^*_{ki}u^*_{lj} = q_{ij}u^*_{lj}\overline{u}^*_{ki}.$$

Step 3. Suppose $u, v \in G$. Let $h = uv$. As $u, v \in G$ we get

$$q_{kl}\overline{u}_{k\alpha}u_{l\beta} = q_{\alpha\beta}u_{l\beta}\overline{u}_{k\alpha}, \quad q_{\alpha\beta}\overline{v}_{\alpha i}v_{\beta j} = q_{ij}v_{\beta j}\overline{v}_{\alpha i}$$

for $\alpha, \beta = 1, \dots, n$. Then

$$\begin{aligned} q_{kl}\overline{h}_{ki}h_{lj} &= \sum_{\alpha,\beta} q_{kl}\overline{u}_{k\alpha}\overline{v}_{\alpha i}u_{l\beta}v_{\beta j} = \sum_{\alpha,\beta} q_{\alpha\beta}u_{l\beta}\overline{v}_{\alpha i}\overline{u}_{k\alpha}v_{\beta j} = \sum_{\alpha,\beta} q_{ij}u_{l\beta}v_{\beta j}\overline{u}_{k\alpha}\overline{v}_{\alpha i} \\ &= q_{ij}h_{lj}\overline{h}_{ki}. \quad \blacksquare \end{aligned}$$

Consider the subgroup of diagonal matrices in U_n. It is isomorphic to $\mathbb{T}^n$.

REMARK 3.6. The subgroup $\mathbb{T}^n$ of diagonal matrices in U_n satisfies the following properties:

(i) $\mathbb{T}^n$ defines a well-defined action on Q-$\mathcal{CCR}$ for any $Q = (q_{ij})_{i,j=1}^n$. In particular, if $u \in \mathbb{T}^n$ is diagonal then

$$q_{kl}\overline{u}_{ki}u_{lj} = q_{kl}\delta_{ki}\delta_{lj}\overline{u}_{ki}u_{lj} = q_{ij}\delta_{lj}\delta_{ki}u_{lj}\overline{u}_{ki} = q_{ij}u_{lj}\overline{u}_{ki}.$$

(ii) There exists a special choice of $\{q_{ij}\}$ such that the group defined in Theorem 3.5 is isomorphic to $\mathbb{T}^n$. Indeed, take arbitrary $\{q_{ij}\}$ with $q_{ii} \neq q_{jj}$ whenever $i \neq j$. Then in particular

$$\overline{u}_{ji}u_{ji}(q_{jj} - q_{ii}) = 0.$$

Hence $u_{ji} = 0$ for $j \neq i$.

We denote the action of $\mathbb{T}^n$ defined above by Ψ. For $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{T}^n$ let $\Psi_{\mathbf{w}} : Q$-$\mathcal{CCR} \to Q$-$\mathcal{CCR}$,

$$\Psi_{\mathbf{w}}(a_i) = w_i a_i, \quad i = 1, \dots, n$$

be the corresponding automorphism.

Now we would like to describe the fixed point subalgebra for the action Ψ on Q-$\mathcal{CCR}$.

Recall that by applying the q_{ij}-relations, any monomial in a_i, a_i^* can be brought into a normal form with all starred operators to the right of all unstarred ones. For a multi-index $\mu = (\mu_1, \ldots, \mu_k)$, $\mu_i \in \{1, \ldots, n\}$ we write

$$a_\mu = a_{\mu_1} \cdots a_{\mu_k}.$$

The following result follows obviously from the Diamond lemma (see [1]).

PROPOSITION 3.7. *The monomials $a_\mu a_\sigma^*$, where μ, σ vary over all multi-indices, form a linear basis of Q-CCR.*

The following technical statement is easily obtained (see [2]).

PROPOSITION 3.8. *Let γ be an action of a compact group on a C^*-algebra $\mathcal{B}$ and $\mathcal{B}_0$ be a dense $*$-subalgebra of $\mathcal{B}$. If E_γ is a conditional expectation onto the corresponding fixed point subalgebra $\mathcal{B}^\gamma$ of $\mathcal{B}$ and $E_\gamma(\mathcal{B}_0) \subset \mathcal{B}_0$, then $\mathcal{B}_0 \cap \mathcal{B}^\gamma$ is dense in $\mathcal{B}^\gamma$.*

Put $\mathrm{occ}_i(\mu)$ to be the number of occurrences of i in a multi-index μ. For a multi-index μ let $\mathrm{occ}(\mu) = (\mathrm{occ}_1(\mu), \ldots, \mathrm{occ}_n(\mu)) \in \mathbb{Z}_+^n$. Write $\mathrm{occ}(\mu) = \mathrm{occ}(\sigma)$ if $\mathrm{occ}_i(\mu) = \mathrm{occ}_i(\sigma)$ for $i = 1, \ldots, n$.

Consider the linear space $GICAR_Q = \mathrm{span}\{a_\mu a_\sigma^* : \mathrm{occ}(\mu) = \mathrm{occ}(\sigma)\}$.

THEOREM 3.9. *If $x \in Q$-CCR then $\Psi_{\mathbf{w}}(x) = x$ for every $\mathbf{w} \in \mathbb{T}^n$ if and only if $x \in GICAR_Q$.*

Proof. Compute the action of $\Psi_{\mathbf{w}}$, $\mathbf{w} = (w_1, \ldots, w_n)$ on the basis of Q-CCR:

$$\begin{aligned}\Psi_{\mathbf{w}}(a_\mu a_\sigma^*) &= w_{\mu_1} \cdots w_{\mu_k} \overline{w}_{\sigma_1} \cdots \overline{w}_{\sigma_k} a_\mu a_\sigma^* = \\ &= w_1^{\mathrm{occ}_1(\mu)} \overline{w}_1^{\mathrm{occ}_1(\sigma)} \cdots w_n^{\mathrm{occ}_n(\mu)} \overline{w}_n^{\mathrm{occ}_n(\sigma)} a_\mu a_\sigma^*.\end{aligned}$$

If $a_\mu a_\sigma^* \in GICAR_Q$ then for $i = 1, \ldots, n$ we have $w_i^{\mathrm{occ}_i(\mu)} \overline{w}_i^{\mathrm{occ}_i(\sigma)} = 1$, so $\Psi_{\mathbf{w}}(a_\mu a_\sigma^*) = a_\mu a_\sigma^*$. Thus if $x \in GICAR_Q$ then $\Psi_{\mathbf{w}}(x) = x$.

Conversely, if $x \in Q$-CCR then we can write $x = \sum\limits_{\mu,\sigma} C_{\mu,\sigma} a_\mu a_\sigma^*$. Suppose $\Psi_{\mathbf{w}}(x) = x$ for any $\mathbf{w} \in \mathbb{T}^n$. In particular it is true for $\mathbf{w}_i = (1, \ldots, 1, z, 1, \ldots, 1)$, where z is on the i-th place. Then

$$0 = \Psi_{\mathbf{w}_i}(x) - x = \sum_{\mu,\sigma} (z^{\mathrm{occ}_i(\mu)} \overline{z}^{\mathrm{occ}_i(\sigma)} - 1) C_{\mu,\sigma} a_\mu a_\sigma^* \quad \text{for any } z \in \mathbb{T}.$$

Since $a_\mu a_\sigma^*$ are linearly independent, the above equality implies $\mathrm{occ}(\mu) = \mathrm{occ}(\sigma)$ whenever $C_{\mu,\sigma} \neq 0$. ∎

Theorem 3.9 easily implies that $GICAR_Q$ is a $*$-subalgebra of Q-$\mathcal{CCR}$. Combining Proposition 3.8 with Theorem 3.9 we obtain the following corollary.

COROLLARY 3.10. *Let $\mathcal{GICAR}_Q = \overline{GICAR}_Q$. Then the fixed point subalgebra $(Q\text{-}\mathcal{CCR})^\Psi$ coincides with $\mathcal{GICAR}_Q$.*

4. NUCLEARITY OF Isom_Q

Starting from this section we will only consider the case of Isom_Q, i.e. $Q = (q_{ij})_{i,j=1}^n$ such that $q_{ii} = 0$, $i = 1, \ldots, n$. For such Q we will prove that $\mathcal{GICAR}_Q$ is an AF-algebra.

In the following lemma we present the multiplication rules for elements of $\mathcal{GICAR}_Q$.

LEMMA 4.1. *Let $q_{ii} = 0$, $i = 1, \ldots, n$. Let $a_{\mu(1)}a^*_{\sigma(1)}, a_{\mu(2)}a^*_{\sigma(2)} \in \mathcal{GICAR}_Q$. Then $a_{\mu(1)}a^*_{\sigma(1)}a_{\mu(2)}a^*_{\sigma(2)} = \alpha a_{\mu(3)}a^*_{\sigma(3)}$ for some $\alpha \in \mathbb{C}$. Moreover,*

$$\mathrm{occ}_i(\mu(3)) = \max(\mathrm{occ}_i(\mu(1)), \mathrm{occ}_i(\mu(2)))$$

for $i = 1, \ldots, n$.

Proof. First rewrite the product $a_{\mu(1)}a^*_{\sigma(1)}$ and $a_{\mu(2)}a^*_{\sigma(2)}$ as an element of $\mathrm{span}\{a_\mu a^*_\sigma\}$:

(i) If $\sigma(1)_i \neq \mu(2)_k$ for any k, then move $a^*_{\sigma(1)_i}$ to the right of $a_{\mu(2)}$ using the q_{ij}-commutation relations.

(ii) If $\sigma(1)_i = \mu(2)_k$ for some k, then move $a^*_{\sigma(1)_i}$ right to $a_{\mu(2)_k}$ using the q_{ij}-commutation relations and annihilate them using the fact that $a_{\sigma(1)_i}$ is an isometry.

As a result we get an expression of the form $\alpha a_{\mu(3)}a^*_{\sigma(3)}$ for some $\alpha \in \mathbb{C}$.

If $\mathrm{occ}_i(\sigma(1)) > \mathrm{occ}_i(\mu(2))$, then each occurrence of a_i in $a_{\mu(2)}$ annihilates with the corresponding element of $a^*_{\sigma(1)}$. Hence, $\mathrm{occ}_i(\mu(3)) = \mathrm{occ}_i(\mu(1))$ and $\mathrm{occ}_i(\sigma(3)) = \mathrm{occ}_i(\sigma(2)) + (\mathrm{occ}_i(\sigma(1)) - \mathrm{occ}_i(\mu(2)))$. But $\mathrm{occ}_i(\mu(1)) = \mathrm{occ}_i(\sigma(1))$ and $\mathrm{occ}_i(\mu(2)) = \mathrm{occ}_i(\sigma(2))$. Hence $\mathrm{occ}_i(\sigma(3)) = \mathrm{occ}_i(\mu(3)) = \mathrm{occ}_i(\sigma(1))$.

If $\mathrm{occ}_i(\sigma(1)) \leqslant \mathrm{occ}_i(\mu(2))$, then the same arguments as in the first case lead us to $\mathrm{occ}_i(\sigma(3)) = \mathrm{occ}_i(\mu(3)) = \mathrm{occ}_i(\mu(2))$.

Therefore $\mathrm{occ}_i(\mu(3)) = \max(\mathrm{occ}_i(\mu(1)), \mathrm{occ}_i(\mu(2)))$. ∎

Put

$$\mathcal{W}_k = \mathrm{span}\Big\{a_\mu a^*_\sigma \in \mathcal{GICAR}_Q : \max_i \mathrm{occ}_i(\mu) \leqslant k\Big\}. \tag{4.1}$$

Lemma 4.1 implies the following statement.

COROLLARY 4.2. *If $q_{ii} = 0$, $i = 1, \ldots, n$ then $\mathcal{W}_k$ is a finite-dimensional $*$-subalgebra of $\mathcal{GICAR}_Q$.*

THEOREM 4.3. *If $q_{ii} = 0$, $i = 1, \ldots, n$ then $\mathcal{GICAR}_Q$ is an* AF*-algebra.*

Proof. Since any $x \in GICAR_Q$ belongs to $\mathcal{W}_k$ for sufficiently large k, we get

$$GICAR_Q = \bigcup_k \mathcal{W}_k.$$

Hence $\mathcal{GICAR}_Q$ is an AF-algebra. ∎

Every AF-algebra is nuclear (see [2]). Therefore we obtain an important corollary of Theorem 4.3 and Proposition 3.2.

THEOREM 4.4. Isom_Q *is nuclear.*

5. STABILITY OF $\mathcal{GICAR}_Q$

In this section we prove that the structure of $\mathcal{GICAR}_Q$ does not depend on $\{q_{ij}\}$, i.e. $\mathcal{GICAR}_Q \simeq \mathcal{GICAR}_0$ for any Q such that $\max|q_{ij}| < 1$, $q_{ii} = 0$, $i = 1, \dots, n$. For this purpose we compute the Bratelli diagram of $\mathcal{GICAR}_Q$.

Denote by $\mathbb{Z}_+^n$ the set of all n-tuples of non-negative integers. For $\mathbf{v} = (v_1, \dots, v_n)$, $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{Z}_+^n$ put $\mathbf{v} + \mathbf{u}$, $\mathbf{v} - \mathbf{u}$, $\max(\mathbf{v}, \mathbf{u})$ for an n-tuple which is obtained by componentwise application of the corresponding functions to the entries of $\mathbf{v}$ and $\mathbf{u}$. We write $\mathbf{v} \leqslant \mathbf{u}$ if $v_i \leqslant u_i$ and $\mathbf{v} = \mathbf{u}$ if $v_i = u_i$ for all $i = 1, \dots, n$. For $S \subset \{1, \dots, n\}$ denote by δ_S the n-tuple which has 1 in the i-th entry when $i \in S$ and 0 otherwise. For $k \geqslant 0$ write $\mathbf{k}^n$ for the n-tuple $(k, \dots, k)$.

For $\mathbf{v} \in \mathbb{Z}_+^n$ we put

$$\mathcal{W}_{\mathbf{v}} = \mathrm{span}\{a_\mu a_\sigma^* : \mathrm{occ}(\mu) = \mathrm{occ}(\sigma) = \mathbf{v}\}.$$

From Lemma 4.1 one can easily get that if $q_{ii} = 0$, $i = 1, \dots, n$ then

$$\mathcal{W}_{\mathbf{v}} \cdot \mathcal{W}_{\mathbf{u}} \subset \mathcal{W}_{\max(\mathbf{v},\mathbf{u})}. \tag{5.1}$$

As a consequence of this inclusion, we have that $\mathcal{W}_{\mathbf{v}}$ is closed under multiplication, so it is a $*$-subalgebra of $\mathcal{GICAR}_Q$. In the following two lemmas we construct some faithful $*$-representation of $\mathcal{W}_{\mathbf{v}}$.

LEMMA 5.1. *For* $\mathbf{v} \in \mathbb{Z}_+^n$ *put*

$$\mathcal{H}_{\mathbf{v}} = \mathrm{span}\{\xi_\alpha : \mathrm{occ}(\alpha) = \mathbf{v}\}.$$

Equip $\mathcal{H}_{\mathbf{v}}$ *with the Fock inner product (see Section 2). Notice that* $a_\beta^* a_\alpha = \langle a_\alpha, a_\beta \rangle_{\mathbf{v}} \cdot 1$ *for some* $\langle a_\alpha, a_\beta \rangle_{\mathbf{v}} \in \mathbb{C}$. *Then for* $\xi_\alpha, \xi_\beta \in \mathcal{H}_{\mathbf{v}}$ *one has*

$$\langle \xi_\alpha, \xi_\beta \rangle_{\mathrm{Fock}} = \langle a_\alpha, a_\beta \rangle_{\mathbf{v}}.$$

Proof. If $\mathbf{v} = 0$ then

$$\langle 1, 1 \rangle_{\mathbf{0}} = 1 = \langle \Omega, \Omega \rangle_{\mathrm{Fock}}.$$

Assume $v_1 + \dots + v_n = m + 1$ for some $m \geqslant 0$. Take $\xi_\mu, \xi_\sigma \in \mathcal{H}_{\mathbf{v}}$. Let t_0 be index of the first occurrence of σ_1 in μ. Since $q_{ii} = 0$ for $i = 1, \dots, n$, we get

$$\begin{aligned}
&\langle \xi_{\mu_1} \otimes \cdots \otimes \xi_{\mu_{m+1}}, \xi_{\sigma_1} \otimes \cdots \otimes \xi_{\sigma_{m+1}} \rangle_{\mathrm{Fock}} \\
&\quad = \sum_{t=1}^{m+1} \delta_{\sigma_1 \mu_t} q_{\sigma_1 \mu_1} \cdots q_{\sigma_1 \mu_{t-1}} \langle \xi_{\mu_1} \otimes \cdots \otimes \widehat{\xi}_{\mu_t} \otimes \cdots \otimes \xi_{\mu_{m+1}}, \xi_{\sigma_2} \otimes \cdots \otimes \xi_{\sigma_{m+1}} \rangle_{\mathrm{Fock}} \\
&\quad = q_{\sigma_1 \mu_1} \cdots q_{\sigma_1, \mu_{t_0 - 1}} \langle \xi_{\mu_1} \otimes \cdots \otimes \widehat{\xi}_{\mu_{t_0}} \otimes \cdots \otimes \xi_{\mu_{m+1}}, \xi_{\sigma_2} \otimes \cdots \otimes \xi_{\sigma_{m+1}} \rangle_{\mathrm{Fock}}
\end{aligned}$$

$$= q_{\sigma_1\mu_1}\cdots q_{\sigma_1,\mu_{t_0-1}}\langle a_{\mu_1}\cdots a_{\mu_{t_0-1}}a_{\mu_{t_0+1}}\cdots a_{\mu_{m+1}}, a_{\sigma_2}\cdots a_{\sigma_{m+1}}\rangle_{\mathbf{v}-\mathbf{ffi}_{\{\sigma_1\}}}$$
$$= q_{\sigma_1\mu_1}\cdots q_{\sigma_1,\mu_{t_0-1}}\langle a_{\mu_1}\cdots a_{\mu_{t_0-1}}a^*_{\sigma_1}a_{\mu_{t_0}}a_{\mu_{t_0+1}}\cdots a_{\mu_{m+1}}, a_{\sigma_2}\cdots a_{\sigma_{m+1}}\rangle_{\mathbf{v}-\mathbf{ffi}_{\{\sigma_1\}}}$$
$$= \langle a^*_{\sigma_1}a_{\mu_1}\cdots a_{\mu_{m+1}}, a_{\sigma_2}\cdots a_{\sigma_{m+1}}\rangle_{\mathbf{v}-\mathbf{ffi}_{\{\sigma_1\}}} = \langle a_\mu, a_\sigma\rangle_{\mathbf{v}}. \qquad \blacksquare$$

In what follows we write M_n for the C^*-algebra of n by n complex matrices.

LEMMA 5.2. *For $\mathbf{v}\in\mathbb{Z}_+^n$ define the $*$-representation $\pi_{\mathbf{v}}:\mathcal{W}_{\mathbf{v}}\to\mathcal{B}(\mathcal{H}_{\mathbf{v}})$ by*

$$\pi_{\mathbf{v}}(a_\mu a^*_\sigma)(\xi_\alpha) = \langle\xi_\alpha,\xi_\sigma\rangle_{\mathrm{Fock}}\xi_\mu.$$

Then $\pi_{\mathbf{v}}$ is a faithful and surjective $*$-representation of $\mathcal{W}_{\mathbf{v}}$ and

$$\mathcal{W}_{\mathbf{v}}\simeq\mathcal{B}(\mathcal{H}_{\mathbf{v}})\simeq M_{\frac{(v_1+\cdots+v_n)!}{v_1!\cdots v_n!}}.$$

Proof. Multiplicativity of $\pi_{\mathbf{v}}$ follows from Lemma 5.1:

$$\begin{aligned}\pi_{\mathbf{v}}(a_{\mu_1}a^*_{\sigma_1})\pi_{\mathbf{v}}(a_{\mu_2}a^*_{\sigma_2})(\xi_\alpha) &= \pi_{\mathbf{v}}(a_{\mu_1}a^*_{\sigma_1})\langle\xi_\alpha,\xi_{\sigma_2}\rangle_{\mathrm{Fock}}\xi_{\mu_2}\\ &= \langle\xi_\alpha,\xi_{\sigma_2}\rangle_{\mathrm{Fock}}\langle\xi_{\mu_2},\xi_{\sigma_1}\rangle_{\mathrm{Fock}}\xi_{\mu_1},\\ \pi_{\mathbf{v}}(a_{\mu_1}a^*_{\sigma_1}a_{\mu_2}a^*_{\sigma_2})(\xi_\alpha) &= \pi_{\mathbf{v}}(a_{\mu_1}a^*_{\sigma_2})\langle a_{\mu_2},a_{\sigma_1}\rangle_{\mathbf{v}}\xi_\alpha\\ &= \langle\xi_{\mu_2},\xi_{\sigma_1}\rangle_{\mathrm{Fock}}\langle\xi_\alpha,\xi_{\sigma_2}\rangle_{\mathrm{Fock}}\xi_{\mu_1}.\end{aligned}$$

Recall that the set of all rank one operators on a finite-dimensional Hilbert space $\mathcal{H}$ generates the whole algebra $\mathcal{B}(\mathcal{H})$. Hence surjectivity of $\pi_{\mathbf{v}}$ follows from the fact that every rank one operator on $\mathcal{H}_{\mathbf{v}}$ is in the image of $\pi_{\mathbf{v}}$.

Observe that $\dim\mathcal{W}_{\mathbf{v}} = (\frac{(v_1+\cdots+v_n)!}{v_1!\cdots v_n!})^2 = \dim\mathcal{B}(\mathcal{H}_{\mathbf{v}})$. Therefore $\pi_{\mathbf{v}}$ is injective, since it is surjective. $\blacksquare$

Notice that $\mathcal{W}_{\mathbf{v}}$ and $\mathcal{W}_{\mathbf{u}}$ are not orthogonal with respect to the multiplication for $\mathbf{v}\neq\mathbf{u}$. Hence we cannot simply take the direct sum of $\pi_{\mathbf{v}}$ to obtain a $*$-representation of the subalgebras $\mathcal{W}_k$ defined by (4.1). Nevertheless the following result holds.

THEOREM 5.3. *We have:*

$$\mathcal{W}_k\simeq\bigoplus_{\mathbf{v}\leqslant\mathbf{k}^n} M_{\frac{(v_1+\cdots+v_n)!}{v_1!\cdots v_n!}}.$$

Proof. We build $\mathcal{W}_k$ in nk steps starting from $j=1$ to $j=nk$.

Step 1. Put $\mathcal{W}_k^{(1)}=\mathcal{W}_k$. Then $\mathcal{W}_{\mathbf{k}^n}\subset\mathcal{W}_k^{(1)}$. By property (5.1) it is an ideal in $\mathcal{W}_k^{(1)}$. Lemma 5.2 implies that $\mathcal{W}_{\mathbf{k}^n}\simeq M_{\frac{(nk)!}{(k!)^n}}$. Let $\mathcal{W}_k^{(2)}=\mathcal{W}_k^{(1)}/\mathcal{W}_{\mathbf{k}^n}$. Then due to $\dim\mathcal{W}_k<\infty$ one has

$$\mathcal{W}_k\simeq M_{\frac{(nk)!}{(k!)^n}}\oplus\mathcal{W}_k^{(2)}.$$

Step 2. Suppose that $\mathcal{W}_k\simeq J_j\oplus\mathcal{W}_k^{(j)}$, where

$$J_j=\bigcup_{v_1+\cdots+v_n>nk-j+1}\mathcal{W}_{\mathbf{v}}\simeq\bigoplus_{v_1+\cdots+v_n>nk-j+1}M_{\frac{(v_1+\cdots+v_n)!}{v_1!\cdots v_n!}}$$

and $\mathcal{W}_k^{(j)} = \mathcal{W}_k/J_j$. We will from this assumption now show that $\mathcal{W}_k \simeq J_{j+1} \oplus \mathcal{W}_k^{(j+1)}$. Let $\sigma_j : \mathcal{W}_k \to \mathcal{W}_k^{(j)}$ be a projection. Let $\mathcal{W}_{\mathbf{v}}$ and $\mathcal{W}_{\mathbf{u}}$ in $\mathcal{W}_k$ be such that $\sum\limits_t v_t = \sum\limits_t u_t = nk - j + 1$. By property (5.1) if $\mathbf{v} \neq \mathbf{u}$ then $\mathcal{W}_{\mathbf{v}} \cdot \mathcal{W}_{\mathbf{u}} \subset \mathcal{W}_{\max(\mathbf{v},\mathbf{u})} \subset J_j$. Hence $\sigma_j(\mathcal{W}_{\mathbf{v}})$ and $\sigma_j(\mathcal{W}_{\mathbf{u}})$ are ideals in $\mathcal{W}_k^{(j)}$ such that $\sigma_j(\mathcal{W}_{\mathbf{v}}) \cdot \sigma_j(\mathcal{W}_{\mathbf{u}}) = 0$. By Lemma 5.2

$$\sigma_j(\mathcal{W}_{\mathbf{v}}) \simeq \mathcal{W}_{\mathbf{v}} \simeq M_{\frac{(v_1+\cdots+v_n)!}{v_1!\cdots v_n!}}\,.$$

Let

$$\mathcal{W}_k^{(j+1)} = \frac{\mathcal{W}_k^{(j)}}{\bigoplus_{v_1+\cdots+v_n=nk-j+1} \sigma_j(\mathcal{W}_{\mathbf{v}})}.$$

Then

$$\mathcal{W}_k^{(j)} = \bigoplus_{v_1+\cdots+v_n=nk-j+1} M_{\frac{(v_1+\cdots+v_n)!}{v_1!\cdots v_n!}} \oplus \mathcal{W}_k^{(j+1)}.$$

Step 3. To complete the proof it remains to note that with $j = nk$ we get $\mathcal{W}_k^{(nk)} \simeq \mathbb{C}$. ∎

Denote by $\mathcal{V}_{\mathbf{v}}^k$ the component $M_{\frac{(v_1+\cdots+v_n)!}{v_1!\cdots v_n!}}$ of $\mathcal{W}_k$ from the decomposition above. The following proposition follows from the proof of Theorem 5.3:

PROPOSITION 5.4. *Let $1_{\mathbf{v}}^k$ be a unit of $\mathcal{V}_{\mathbf{v}}^k$. Then:*

(i) *$\mathcal{V}_{\mathbf{v}}^k$ coincides with $\mathcal{W}_{\mathbf{v}} \cdot 1_{\mathbf{v}}^k$. Hence,*

$$\mathcal{W}_k = \bigoplus_{\mathbf{v}\leqslant\mathbf{k}^n} \mathcal{V}_{\mathbf{v}}^k = \bigoplus_{\mathbf{v}\leqslant\mathbf{k}^n} \mathcal{W}_{\mathbf{v}} \cdot 1_{\mathbf{v}}^k.$$

(ii) *Denote $J_{\mathbf{v}} = \operatorname{span}\Big\{ \bigcup\limits_{\mathbf{v}\leqslant\mathbf{j}\leqslant\mathbf{k}^n} \mathcal{W}_{\mathbf{j}} \Big\}$. Then:*

$$\sum_{\mathbf{v}\leqslant\mathbf{j}\leqslant\mathbf{k}^n} \mathcal{V}_{\mathbf{v}}^k = J_{\mathbf{v}}.$$

The structure of $\mathcal{W}_k$ is independent of the values of q_{ij} as it follows from Theorem 5.3. For the analysis of embedding of $\mathcal{W}_k$ into $\mathcal{W}_{k+1}$ we construct a special $*$-representation of $\mathcal{W}_k$.

In Lemma 5.2 we have constructed a bijective $*$-representation $\pi_{\mathbf{v}}$ of $\mathcal{W}_{\mathbf{v}}$. Since $\mathcal{V}_{\mathbf{v}}^k = \mathcal{W}_{\mathbf{v}} \cdot 1_{\mathbf{v}}^k$, we can define a bijective $*$-representation $\widehat{\pi}_{\mathbf{v}}^k$

$$\widehat{\pi}_{\mathbf{v}}^k : \mathcal{V}_{\mathbf{v}}^k \to \mathcal{B}(\mathcal{H}_{\mathbf{v}}), \quad \widehat{\pi}_{\mathbf{v}}^k(x 1_{\mathbf{v}}^k) = \pi_{\mathbf{v}}(x),\ x \in \mathcal{W}_{\mathbf{v}}.$$

Put

$$\mathcal{H}_k = \bigoplus_{\mathbf{v}\leqslant\mathbf{k}^n} \mathcal{H}_{\mathbf{v}}.$$

Let $x \in \mathcal{W}_k$. Then there exists $y_{\mathbf{v}} \in \mathcal{W}_{\mathbf{v}}$ such that $x1^k_{\mathbf{v}} = y_{\mathbf{v}}1^k_{\mathbf{v}}$. Define the $*$-representation $\pi_k : \mathcal{W}_k \to \mathcal{B}(\mathcal{H}_k)$ by

$$\pi_k(x) = \bigoplus_{\mathbf{v}\leqslant \mathbf{k}^n} \widehat{\pi}^k_{\mathbf{v}}(x1^k_{\mathbf{v}}) = \bigoplus_{\mathbf{v}\leqslant \mathbf{k}^n} \widehat{\pi}^k_{\mathbf{v}}(y_{\mathbf{v}}1^k_{\mathbf{v}}) = \bigoplus_{\mathbf{v}\leqslant \mathbf{k}^n} \pi_{\mathbf{v}}(y_{\mathbf{v}}).$$

π_k is obviously faithful since if $x \in \ker \pi_k$ then $\widehat{\pi}^k_{\mathbf{v}}(x1^k_{\mathbf{v}}) = 0$. Since $\widehat{\pi}^k_{\mathbf{v}}$ is faithful, $x1^k_{\mathbf{v}} = 0$ for any $\mathbf{v} \leqslant \mathbf{k}^n$.

LEMMA 5.5. *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n_+$, $\mathbf{u}, \mathbf{v} \leqslant \mathbf{k}^n$. *If* $\mathbf{v} \nleqslant \mathbf{u}$ *then* $\pi_k(\mathcal{W}_{\mathbf{v}})|_{\mathcal{H}_{\mathbf{u}}} = 0$.

Proof. Let $J_{\mathbf{v}}$ be as in Proposition 5.4. $J_{\mathbf{v}}$ is an ideal in $\mathcal{W}_k$ by property (5.1). If $\mathbf{v} \nleqslant \mathbf{u}$ then $J_{\mathbf{v}}$ has zero intersection with $\mathcal{V}^k_{\mathbf{u}}$, because $\mathcal{V}^k_{\mathbf{u}} \cap \mathcal{V}^k_{\mathbf{j}} = 0$ for $\mathbf{v} \leqslant \mathbf{j} \leqslant \mathbf{k}^n$. However $\mathcal{W}_{\mathbf{v}} \cdot 1^k_{\mathbf{u}} \subset J \cap \mathcal{V}^k_{\mathbf{u}}$, so $\mathcal{W}_{\mathbf{v}} \cdot 1^k_{\mathbf{u}} = 0$ and $\pi_k(\mathcal{W}_{\mathbf{v}})|_{\mathcal{H}_{\mathbf{u}}} = \widehat{\pi}^k_{\mathbf{u}}(\mathcal{W}_{\mathbf{v}} \cdot 1^k_{\mathbf{u}}) = 0$. ∎

Let $\mathbf{u} \geqslant \mathbf{v}$ and $P^{\mathbf{u}}_{\mathbf{v}} \in \mathcal{B}(\mathcal{H}_{\mathbf{u}})$ be the orthogonal projection onto the subspace of $\mathcal{H}_{\mathbf{u}}$ spanned by $\xi_\mu \in \mathcal{H}_{\mathbf{u}}$ such that $\operatorname{occ}(\mu_1, \dots, \mu_{v_1+\cdots+v_n}) = \mathbf{v}$. Notice that this subspace has dimension equal to $\dim \mathcal{H}_{\mathbf{v}} \cdot \dim \mathcal{H}_{\mathbf{u}-\mathbf{v}}$. Put $p^{\mathbf{u}}_{\mathbf{v}} = \pi^{-1}_{\mathbf{u}}(P^{\mathbf{u}}_{\mathbf{v}}) \in \mathcal{W}_{\mathbf{u}}$. If $\mathbf{u} \nleqslant \mathbf{w}$ then $\pi_k(p^{\mathbf{u}}_{\mathbf{v}})|_{\mathcal{H}_{\mathbf{w}}} = 0$ due to Lemma 5.5. Also observe that if $\mathbf{u_1} \leqslant \mathbf{w}$ and $\mathbf{u_2} \leqslant \mathbf{w}$ then $\pi_k(p^{\mathbf{u_1}}_{\mathbf{v}})|_{\mathcal{H}_{\mathbf{w}}} = \pi_k(p^{\mathbf{u_2}}_{\mathbf{v}})|_{\mathcal{H}_{\mathbf{w}}}$.

In the following lemma we express $1^k_{\mathbf{v}}$ as a sum of the projections $p^{\mathbf{u}}_{\mathbf{v}}$.

LEMMA 5.6. *Let* $\mathbf{v} \in \mathbb{Z}^n_+$ *such that* $\mathbf{v} \leqslant \mathbf{k}^n$. *Then*

$$1^k_{\mathbf{v}} = \sum_{S\subset\{1,\dots,n\},\, \mathbf{v}+\delta_S \leqslant \mathbf{k}^n} (-1)^{l(\delta_S)} p^{\mathbf{v}+\delta_S}_{\mathbf{v}}, \tag{5.2}$$

where $l(\delta_S) = \sum\limits_{i=1}^{n} (\delta_S)_i$.

Proof. We show that (5.2) acts as the identity operator on $\mathcal{H}_{\mathbf{v}}$ and as the zero operator on $\mathcal{H}_{\mathbf{u}}$ for $\mathbf{u} \neq \mathbf{v}$ in the representation π_k. Consider the following cases:

Case 1. If $\mathbf{v} = \mathbf{u}$ then whenever $\delta_S \neq \mathbf{0}^n$, we have

$$\pi_k(p^{\mathbf{v}+\delta_S}_{\mathbf{v}})|_{\mathcal{H}_{\mathbf{u}}} = 0.$$

Hence

$$\sum_{S\subset\{1,\dots,n\},\, \mathbf{v}+\delta_S \leqslant \mathbf{k}^n} (-1)^{l(\delta_S)} \pi_k(p^{\mathbf{v}+\delta_S}_{\mathbf{v}})|_{\mathcal{H}_{\mathbf{u}}} = \pi_k(p^{\mathbf{v}}_{\mathbf{v}})|_{\mathcal{H}_{\mathbf{u}}} = \mathrm{id}_{\mathcal{H}_{\mathbf{u}}}.$$

Case 2. If $\mathbf{v} \nleqslant \mathbf{u}$ then by Lemma 5.5 $\pi_k(p^{\mathbf{v}+\delta_S}_{\mathbf{v}})|_{\mathcal{H}_{\mathbf{u}}} = 0$ for arbitrary $S \subset \{1, \dots, n\}$.

Case 3. The other case is $\mathbf{v} < \mathbf{u} \leqslant \mathbf{k}^n$. If $\mathbf{u}_t > \mathbf{v}_t$ then for $S \subset \{1, \dots, n\}$ one has $\mathbf{v} + \delta_S \leqslant \mathbf{u}$ if and only if $\mathbf{v} + \delta_{S\setminus\{t\}} \leqslant \mathbf{u}$, so

$$\pi_k(p^{\mathbf{v}+\delta_S}_{\mathbf{v}})|_{\mathcal{H}_{\mathbf{u}}} = \pi_k(p^{\mathbf{v}+\delta_{S\setminus\{t\}}}_{\mathbf{v}})|_{\mathcal{H}_{\mathbf{u}}}.$$

Hence

$$\sum_{S\subset\{1,\dots,n\},\,\mathbf{v}+\delta_S\leqslant\mathbf{k}^n}(-1)^{l(\delta_S)}\pi_k(p_{\mathbf{v}}^{\mathbf{v}+\delta_S})|_{\mathcal{H}_{\mathbf{u}}}$$
$$=\sum_{S\subset\{1,\dots,n\}\setminus\{t\},\,\mathbf{v}+\delta_S\leqslant\mathbf{k}^n}((-1)^{l(\delta_{S\setminus\{t\}})}\pi_k(p_{\mathbf{v}}^{\mathbf{v}+\delta_S})|_{\mathcal{H}_{\mathbf{u}}}+(-1)^{1+l(\delta_{S\setminus\{t\}})}\pi_k(p_{\mathbf{v}}^{\mathbf{v}+\delta_S+\delta_{\{t\}}})|_{\mathcal{H}_{\mathbf{u}}})$$
$$=0. \quad \blacksquare$$

Recall that every $*$-homomorphism $\phi:\mathcal{B}(V_1)\to\mathcal{B}(V_2)$, where V_1 and V_2 are finite-dimensional, is determined up to a unitary equivalence by a natural number m and has the form

$$\phi(x)=\begin{bmatrix}1_m\otimes x & 0\\ 0 & 0\end{bmatrix}.$$

This number can be determined as

$$m=\frac{\dim\operatorname{Im}\phi(\mathrm{id}_{V_1})}{\dim V_1}.$$

THEOREM 5.7. *Let $\mathbf{v},\mathbf{u}\in\mathbb{Z}_+^n$ be such that $\mathbf{v}\leqslant\mathbf{k}^n$, $\mathbf{u}\leqslant(\mathbf{k}+\mathbf{1})^n$. Then $\mathcal{V}_{\mathbf{v}}^k$ is embedded into $\mathcal{V}_{\mathbf{u}}^{k+1}$ with nonzero multiplicity $m_{\mathbf{v},\mathbf{u}}$ if and only if $\mathbf{0}^n\leqslant\mathbf{u}-\mathbf{v}\leqslant\mathbf{1}^n$ with $u_t>v_t$ only in the case when $v_t=k$. If the multiplicity is nonzero then it is equal to $\frac{(\sum_t u_t-v_t)!}{\prod_t(u_t-v_t)!}$.*

Proof. Suppose $\mathbf{v}\nleqslant\mathbf{u}$. Then by Lemma 5.5 for $S\subset\{1,\dots,n\}$ one has $\pi_{k+1}(p_{\mathbf{v}}^{\mathbf{v}+\delta_S})|_{\mathcal{H}_{\mathbf{u}}}$, so

$$\pi_{k+1}(1_{\mathbf{v}}^k)=\sum_{S\subset\{1,\dots,n\},\,\mathbf{v}+\delta_S\leqslant\mathbf{k}^n}(-1)^{l(\delta_S)}\pi_{k+1}(p_{\mathbf{v}}^{\mathbf{v}+\delta_S})|_{\mathcal{H}_{\mathbf{u}}}=0.$$

Hence for the embedding to be nonzero we need $\mathbf{v}\leqslant\mathbf{u}$.

If there exists t such that $\min(u_t,k)>\mathbf{v}_t$ then using the same argument as in the Case 3 of Lemma 5.6 one has $\pi_{k+1}(1_{\mathbf{v}}^k)|_{\mathcal{H}_{\mathbf{u}}}=0$. Hence for the embedding to be nonzero we need to have $\mathbf{u}-\mathbf{v}\leqslant\mathbf{1}^n$ and $u_t=k+1$ whenever $u_t>v_t$. Hence for $t=1,\dots,n$ we have either $u_t=v_t$ or $u_t=k+1$, so if $S\subset\{1,\dots,n\}$ is non-empty then $\mathbf{v}+\delta_S\nleqslant\min(\mathbf{u},\mathbf{k}^n)$ and $\pi_{k+1}(1_{\mathbf{v}}^k)|_{\mathcal{H}_{\mathbf{u}}}=\pi_{k+1}(p_{\mathbf{v}}^{\mathbf{v}})|_{\mathcal{H}_{\mathbf{u}}}$.

Now we calculate the multiplicity $m_{\mathbf{v},\mathbf{u}}$ when the embedding is nonzero:

$$\begin{aligned}m_{\mathbf{v},\mathbf{u}}&=\frac{\dim\{\pi_{k+1}(1_{\mathbf{v}}^k)(\mathcal{H}_{\mathbf{u}})\}}{\dim\mathcal{H}_{\mathbf{v}}}=\frac{\dim\{\pi_{k+1}(p_{\mathbf{v}}^{\mathbf{v}})(\mathcal{H}_{\mathbf{u}})\}}{\dim\mathcal{H}_{\mathbf{v}}}\\&=\frac{\dim\{\pi_{k+1}(p_{\mathbf{v}}^{\mathbf{u}})(\mathcal{H}_{\mathbf{u}})\}}{\dim\mathcal{H}_{\mathbf{v}}}=\frac{\dim\mathcal{H}_{\mathbf{v}}\dim\mathcal{H}_{\mathbf{u}-\mathbf{v}}}{\dim\mathcal{H}_{\mathbf{v}}}=\frac{(\sum_t\mathbf{u}_t-\mathbf{v}_t)!}{\prod_t(\mathbf{u}_t-\mathbf{v}_t)!}.\quad\blacksquare\end{aligned}$$

Hence we proved that the multiplicities of the embeddings do not depend on q_{ij}. Since two AF-algebras are isomorphic if they have equal Bratelli diagrams, we have $\mathcal{GICAR}_Q\simeq\mathcal{GICAR}_0$.

6. FAITHFULNESS OF THE FOCK REPRESENTATION

In this section we prove that the Fock representation of Isom_Q is faithful for any Q such that $\max|q_{ij}| < 1$, $q_{ii} = 0$, $i = 1, \ldots, n$. Recall that the Fock representation π_{F} of Isom_Q is the one, determined uniquely up to a unitary equivalence by the following property: there exists a cyclic vector Ω such that $\pi_{\mathrm{F}}(a_i^*)(\Omega) = 0$ for $i = 1, \ldots, n$.

To prove faithfulness of π_{F} we will apply the second part of Proposition 3.2. In the following lemma we describe an automorphism which interwines π_{F} with the action Ψ of $\mathbb{T}^n$ defined in Section 3.

LEMMA 6.1. *For any $\mathbf{w} \in \mathbb{T}^n$ there exists an automorphism $\Psi_{\mathbf{w}}^{\mathrm{Fock}}$ of $\pi_{\mathrm{F}}(\mathrm{Isom}_Q)$ such that $\Psi_{\mathbf{w}}^{\mathrm{Fock}} \circ \pi_{\mathrm{F}} = \pi_{\mathrm{F}} \circ \Psi_{\mathbf{w}}$.*

Proof. Let $A_i' = w_i \pi_{\mathrm{F}}(a_i)$ and Ω be a cyclic vector such that $\pi_{\mathrm{F}}(a_i^*)(\Omega) = 0$, $i = 1, \ldots, n$. Then by Remark 3.6, A_i' satisfies the q_{ij}-commutation relations and Ω is a cyclic vector such that $A_i'^*(\Omega) = 0$, $i = 1, \ldots, n$. By the uniqueness of the Fock representation, there is a unitary U acting on the deformed Fock space which implements an isomorphism between $\pi_{\mathrm{F}}(\mathrm{Isom}_Q)$ and the C^*-algebra generated by A_i', i.e.

$$U^* \pi_{\mathrm{F}}(a_i) U = A_i', \quad i = 1, \ldots, n.$$

These algebras coincide because $\pi_{\mathrm{F}}(a_i) = \overline{w}_i A_i'$, so U implements an automorphism of $\pi_{\mathrm{F}}(\mathrm{Isom}_Q)$. Let $\Psi_{\mathbf{w}}^{\mathrm{Fock}}(x) = U^* x U$. Then

$$\Psi_{\mathbf{w}}^{\mathrm{Fock}}(\pi_{\mathrm{F}}(a_i)) = U^* \pi_{\mathrm{F}}(a_i) U = A_i' = w_i \pi_{\mathrm{F}}(a_i) = \pi_{\mathrm{F}}(w_i a_i) = \pi_{\mathrm{F}}(\Psi_{\mathbf{w}}(a_i)).$$

Since the relation $\Psi_{\mathbf{w}}^{\mathrm{Fock}} \circ \pi_{\mathrm{F}} = \pi_{\mathrm{F}} \circ \Psi_{\mathbf{w}}$ holds on $\pi_{\mathrm{F}}(a_i)$ and $\{\pi_{\mathrm{F}}(a_i) : i = 1, \ldots, n\}$ generates $\pi_{\mathrm{F}}(\mathrm{Isom}_Q)$, the proof is completed. ∎

Now Proposition 3.2 can be applied to the actions $\Psi_{\mathbf{w}}^{\mathrm{Fock}}$ and $\Psi_{\mathbf{w}}$ and the representation π_{F}.

THEOREM 6.2. *The Fock representation π_{F} of the C^*-algebra* Isom_Q *is faithful.*

Proof. If $x \in \bigcup_k \mathcal{W}_k$ then $x \in \mathcal{W}_k$ for a sufficiently large k. Since $1 \in \mathcal{W}_k$, $\|x\| = \|x\|_{\mathcal{W}_k}$. It follows from the main result of [7] that π_{F} is faithful on the dense $*$-subalgebra of Isom_Q generated by a_i, a_i^*, $i = 1, \ldots, n$. Hence, $\pi_{\mathrm{F}}|_{\mathcal{W}_k}$ is an injective $*$-homomorphism from the C^*-algebras $\mathcal{W}_k$ to $\pi_{\mathrm{F}}(\mathrm{Isom}_Q)$, so $\|x\| = \|x\|_{\mathcal{W}_k} = \|\pi_{\mathrm{F}}(x)\|$. Hence by Theorem 4.3 and Corollary 3.10 π_{F} is faithful on $\mathcal{GICAR}_Q$ and by Proposition 3.2 π_{F} is faithful on Isom_Q. ∎

7. DESCRIPTION OF THE IDEAL $\mathcal{K}_Q$ IN Isom_Q

Consider Q such that $\max |q_{ij}| < 1$, $q_{ii} = 0$, $i = 1, \ldots, n$. In this section we describe an ideal $\mathcal{K}_Q$ in Isom_Q which is isomorphic to the algebra of compact operators $\mathcal{K}$.

Recall that $\mathcal{K}$ can be described as the universal C^*-algebra generated by e_{ij}, $i, j \geqslant 0$ satisfying the relations $e_{ij}e_{kl} = \delta_{jk}e_{il}$, $e_{ij}^* = e_{ji}$ (see [2]).

When $q_{ij} = 0$ for $i, j = 1, \ldots, n$, we have $\mathrm{Isom}_0 \simeq \mathcal{K}\mathcal{O}_n$, which is an extension of $\mathcal{O}_n$ by an ideal isomorphic to $\mathcal{K}$. In this case it is generated by $p = 1 - \sum\limits_k a_k a_k^*$. Notice that p is a nontrivial projection such that

$$ps_i = 0, \quad i = 1, \ldots, n.$$

In the next theorem we prove that the same conditions are sufficient for an element p in Isom_Q to generate an ideal isomorphic to $\mathcal{K}$.

THEOREM 7.1. *Let $p \in \mathrm{Isom}_Q$ be a nontrivial projection such that*

$$pa_i = 0, \quad i = 1, \ldots, n.$$

Then the ideal $\mathcal{K}_Q$ generated by p is isomorphic to $\mathcal{K}$.

Proof. By Proposition 3.7 the span of all words of the form $a_{\mu_1} a_{\sigma_1}^* p a_{\mu_2} a_{\sigma_2}^*$ is dense in the ideal generated by p. If $\sigma_1 \neq 0$ or $\mu_2 \neq 0$ then this monomial is equal to 0 since $a_i^* p = 0$, $pa_i = 0$, $i = 1, \ldots, n$. Therefore the span of monomials of the form $a_\mu p a_\sigma^*$ is dense in the ideal generated by p.

Let $\mathcal{H} = \mathrm{span}\{a_\mu : \mu \in \mathbb{Z}_+^n\}$. Split $\mathcal{H}$ into the subspaces $\mathcal{H}_{\mathbf{v}}$ and equip it with $\langle a_\mu, a_\sigma \rangle_{\mathbf{v}}$ as in Lemma 5.1. Apply the orthogonalization process to the basis $\{a_\mu\}$ of $\mathcal{H}_{\mathbf{v}}$ and denote the result by $\{\widehat{a}_\mu\}$. Consider the following cases:

(i) If $\mathrm{occ}(\alpha) = \mathrm{occ}(\beta) = \mathbf{v} \in \mathbb{Z}_+^n$ then $p\widehat{a}_\beta^* \widehat{a}_\alpha p = p\langle \widehat{a}_\alpha, \widehat{a}_\beta \rangle_{\mathbf{v}} p = \delta_{\alpha\beta} p$.

(ii) If $\mathrm{occ}(\alpha) \neq \mathrm{occ}(\beta)$ then $\widehat{a}_\beta^* \widehat{a}_\alpha$ is a non-trivial monomial, so $p\widehat{a}_\beta^* \widehat{a}_\alpha p = 0$.

Put $e_{\alpha\beta} = \widehat{a}_\alpha p \widehat{a}_\beta^*$. Then

$$\begin{aligned} e_{\alpha\beta} e_{\sigma\mu} = \widehat{a}_\alpha p \widehat{a}_\beta^* \widehat{a}_\sigma p \widehat{a}_\mu^* &= \begin{cases} \widehat{a}_\alpha p \delta_{\sigma\beta} \widehat{a}_\mu^* & \text{if } \mathrm{occ}(\beta) = \mathrm{occ}(\sigma), \\ 0 & \text{otherwise}, \end{cases} \\ &= \delta_{\sigma\beta} \widehat{a}_\alpha p \widehat{a}_\mu^* = \delta_{\beta\sigma} e_{\alpha\mu}. \end{aligned}$$

The homomorphism from $\mathcal{K}$ to $\mathcal{K}_Q$ obtained from the universal property is injective because $\mathcal{K}$ is simple. Hence $\mathcal{K}_Q \simeq \mathcal{K}$. ∎

So it remains to prove the existence of $p \in \mathrm{Isom}_Q$ satisfying the conditions of Theorem 7.1.

PROPOSITION 7.2. *There exists an element $p \in \mathrm{Isom}_Q$ such that*

$$p \neq 0,\ p = p^*,\ p^2 = p,\ pa_i = 0, \quad i = 1, \ldots, n.$$

Proof. Let $\mathcal{B}$ be the C^*-subalgebra generated by $a_i a_i^*$, $i = 1, \ldots, n$. Since $\mathcal{B} \subset \mathcal{W}_1$, it is finite-dimensional. Every finite-dimensional C^*-algebra is unital. Obviously, the unit of Isom_Q does not belong to $\mathcal{B}$. Let $1_{\mathcal{B}}$ be the unit of $\mathcal{B}$. Then for any $i = 1, \ldots, n$ we observe:

$$(1 - 1_{\mathcal{B}})a_i = a_i - 1_{\mathcal{B}}a_i = a_i - 1_{\mathcal{B}}a_i(a_i^* a_i) = a_i - (1_{\mathcal{B}}a_i a_i^*)a_i = a_i - (a_i a_i^*)a_i = 0.$$

Thus $p = 1 - 1_{\mathcal{B}}$ is a nontrivial projection satisfying all required conditions. ∎

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