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TAYLOR ASYMPTOTICS OF SPECTRAL ACTION FUNCTIONALS

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ABSTRACT. We establish a Taylor asymptotic expansion of the spectral action functional on self-adjoint operators $V \mapsto \tau(f(H+V))$ with remainder $\mathcal{O}(\|f^{(n)}\|_\infty \|V\|^n)$ and derive an explicit representation for the remainder in terms of spectral shift functions. For this expansion we assume only that H has τ-compact resolvent and V is a bounded perturbation; in particular, neither summability of V nor of the resolvent of H is required.

KEYWORDS: *Spectral action, perturbation theory.*

MSC (2010): 47A55.

1. INTRODUCTION

Let $\mathcal{M}$ be a semifinite von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ and let τ be a normal faithful semifinite trace on $\mathcal{M}$. Let H be a self-adjoint operator affiliated with $\mathcal{M}$ and assume its resolvent is τ-compact. Examples of such operators include differential operators on compact Riemannian manifolds (see, e.g., Chapter 3, Section B of [2], Chapter 3, Section 6 of [8]). For f a sufficiently smooth compactly supported function and V a self-adjoint element in $\mathcal{M}$, we consider a spectral action functional $V \mapsto \tau(f(H+V))$ that was introduced in [3] to encompass different field actions in noncommutative geometry. Applications of the spectral action functional and its expansions can be found in, e.g., [5], [7], [13]; its conceptual advantages over particular quantum field actions are discussed in [4]. We establish an alternative, Taylor asymptotic expansion of the spectral action functional with an accurate estimate and description of the remainder.

We prove the asymptotic expansion

$$\tau(f(H+V)) = \sum_{k=0}^{n-1} \frac{1}{k!}\, \tau\Big(\frac{\mathrm{d}^k}{\mathrm{d}s^k} f(H+sV)|_{s=0}\Big) + \mathcal{O}(\|f^{(n)}\|_\infty\, \|V\|^n), \tag{1.1}$$

for $n \in \mathbb{N}$, $f \in C_c^{n+1}(\mathbb{R})$, and derive an explicit upper bound for $\mathcal{O}(\|f^{(n)}\|_\infty \|V\|^n)$ in Theorem 4.1. This result is a counterpart of the estimate $\mathcal{O}(\|f^{(n)}\|_\infty \|V\|_n^n)$ with n^{th} Schatten norm of V that was established in Theorem 2.1 of [9] in the case of noncompact resolvents. The form of the approximating term in (1.1) improves the previously derived one in the case of a noncompact resolvent. In Theorem 4.3, we derive an explicit integral representation for the remainder of the above approximation, which is analogous to the representation obtained in Theorem 1.1 of [9] via spectral shift functions. The result of Theorem 4.3 for H having a compact resolvent was previously known only in the case $n=2$ (see Theorem 3.10 of [10]).

The asymptotic expansion (1.1) provides a significant improvement of the dependence on f in the bound for a remainder obtained in [10]. Namely, when $\mathcal{M}$ is the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$ and τ is the canonical trace Tr on the trace class ideal, it was proved in Theorem 3.2 and Remark 3.3 of [10] that

$$\operatorname{Tr}(f(H+V)) = \sum_{k=0}^{n-1} \frac{1}{k!} \operatorname{Tr}\Big(\frac{\mathrm{d}^k}{\mathrm{d}s^k} f(H+sV)|_{s=0}\Big) + \mathcal{O}_f(\|V\|^n), \tag{1.2}$$

where, in the case $f \geqslant 0$ and $f^{2^{-1-\lfloor \log_2(n) \rfloor}} \in C_c^{n+1}(\mathbb{R})$,

$$\mathcal{O}_f(\|V\|^n) = \mathcal{O}\Big(\max_{1\leqslant m\leqslant 1+\lfloor \log_2(n)\rfloor} \|\sqrt[2^m]{f}\|_\infty \max_{\substack{0\leqslant m\leqslant 1+\lfloor \log_2(n)\rfloor \\ 1\leqslant p\leqslant n}} \{1,\, \|\sqrt[2^m]{f}\|_{G_p}^n\}\, \|V\|^n\Big)$$

and

$$\|g\|_{G_p} = \frac{\sqrt{2}}{p!}(\|g^{(p)}\|_2 + \|g^{(p+1)}\|_2). \tag{1.3}$$

We note that an asymptotic expansion of $\operatorname{Tr}(f(H+V))$ without an estimate for the remainder was derived in Theorem 18 and Corollary 19 of [12] under the additional summability assumption $\operatorname{Tr}(\mathrm{e}^{-tH^2}) < \infty$, with $t > 0$, for V satisfying $\|\delta(V)\| < \infty$, $\|\delta^2(V)\| < \infty$, where $\delta(\cdot) = [|H|, \cdot]$, and f a sufficiently nice even function.

The structure of the paper is as follows: preliminaries are collected in Section 2, our main technical estimate is established in Section 3, the asymptotic expansion is proved in Section 4.

Throughout the paper, $C_c^n(\mathbb{R})$ denotes the space of n times continuously differentiable compactly supported functions and $C_c^n((a,b))$ the subset of functions in $C_c^n(\mathbb{R})$ whose closed supports are subsets of the finite interval (a,b). We use the notation $A\eta\mathcal{M}$ for an operator A affiliated with $\mathcal{M}$, $\mathcal{M}_{\text{sa}}$ for the subset of self-adjoint elements of $\mathcal{M}$, and $H\eta\mathcal{M}_{\text{sa}}$ for a closed densely defined self-adjoint operator H affiliated with $\mathcal{M}$. The symbol E_H denotes the spectral measure of $H\eta\mathcal{M}_{\text{sa}}$.

2. PRELIMINARIES

Let $\mu_t(A)$ denote the t^{th} generalized s-number ([6], Definition 2.1) of a τ-measurable ([6], Definition 1.2) operator $A\eta\mathcal{M}$. An operator $A \in \mathcal{M}$ is said to be τ-compact if and only if $\lim\limits_{t\to\infty} \mu_t(A) = 0$. We will work with operators whose resolvents are τ-compact. Note that if the resolvent of an operator is τ-compact at one point, then it is τ-compact at all points of its domain.

PROPOSITION 2.1 ([1], Lemma 1.3). *If $H\eta\mathcal{M}_{\text{sa}}$ has τ-compact resolvent and $W \in \mathcal{M}_{\text{sa}}$, then $H + W$ also has τ-compact resolvent.*

The next result follows from combining Lemmas 1.4 and 1.7 of [1].

PROPOSITION 2.2. *Let $H\eta\mathcal{M}_{\text{sa}}$ have τ-compact resolvent and let $V \in \mathcal{M}_{\text{sa}}$. Then, for all $a, b \in \mathbb{R}$, $a < b$, the projection $E_{H+W}((a,b))$ is τ-finite and there exists a constant $\Omega_{a,b,H,V}$ such that*

$$\sup_{t\in[0,1]} \tau(E_{H+tV}((a,b))) \leqslant \Omega_{a,b,H,V} \tag{2.1}$$

and

$$\mu_{\Omega_{a,b,H,V}}((1+H^2)^{-1}) \leqslant \frac{1}{(1+\max\{a^2,b^2\})(1+\|V\|+\|V\|^2)}.$$

Let $\mathcal{L}^p$, $1 \leqslant p < \infty$, denote the noncommutative L^p-space associated with $(\mathcal{M}, \tau)$, that is,

$$\mathcal{L}^p = \{A\eta\mathcal{M} : \|A\|_p := (\tau(|A|^p))^{1/p} < \infty\}.$$

Let $\|\cdot\|_\infty$ denote the operator norm and let $\mathcal{L}^\infty$ denote the algebra $\mathcal{M}$.

PROPOSITION 2.3. *Let $H\eta\mathcal{M}_{\text{sa}}$ have τ-compact resolvent and let $f \in C_c((a,b))$. Then, $f(H) \in \mathcal{L}^p$, for every $p \in \mathbb{N}$, and*

$$\|f(H)\|_1 \leqslant \tau(E_H((a,b)))\, \|f\|_\infty. \tag{2.2}$$

Proof. It follows from the spectral theorem that $|f(H)| \leqslant \|f\|_\infty\, E_H((a,b))$. Hence, $f(H) \in \mathcal{L}^p$ for every $p \in \mathbb{N}$. Applying Proposition 2.2 completes the proof. ∎

Below we work with multilinear transformations whose symbols are divided differences of smooth functions. Recall that the divided difference of order p is an operation on functions f of one real variable defined recursively as follows:

$$f^{[0]}(\lambda_0) := f(\lambda_0),$$

$$f^{[p]}(\lambda_0,\dots,\lambda_p) := \begin{cases} \frac{f^{[p-1]}(\lambda_0,\dots,\lambda_{p-2},\lambda_{p-1}) - f^{[p-1]}(\lambda_0,\dots,\lambda_{p-2},\lambda_p)}{\lambda_{p-1}-\lambda_p} & \text{if } \lambda_{p-1} \neq \lambda_p, \\ \frac{\partial}{\partial t}(\lambda_0,\dots,\lambda_{p-2},t) f^{[p-1]}|_{t=\lambda_{p-1}} & \text{if } \lambda_{p-1} = \lambda_p. \end{cases}$$

DEFINITION 2.4. Let $H\eta\mathcal{M}_{\text{sa}}$, $n \in \mathbb{N}$, $W_k \in \mathcal{L}^{\alpha_k}$, $\alpha_k \in [1,\infty]$, $k = 1,\dots,n$. Then, for $f \in C_c^{n+1}(\mathbb{R})$,

$$T_{f^{[n]}}^{H,\dots,H}(W_1,\dots,W_n)$$

$$:= \lim_{m\to\infty} \lim_{N\to\infty} \sum_{|l_0|,\dots,|l_n|\leqslant N} f^{[n]}\Big(\frac{l_0}{m},\frac{l_1}{m},\dots,\frac{l_n}{m}\Big) E_{H,l_0,m} W_1 E_{H,l_1,m} W_2 \cdots W_n E_{H,l_n,m}, \tag{2.3}$$

where the limits are evaluated in the $\mathcal{L}^\alpha$-norm, $\frac{1}{\alpha} = \frac{1}{\alpha_1} + \dots + \frac{1}{\alpha_n}$, and $E_{H,l_k,m} = E_H\Big(\Big[\frac{l_k}{m},\frac{l_k+1}{m}\Big)\Big)$, for $k = 0,\dots,n$. Existence of the limits in (2.3) is justified in Lemma 3.5 of [9]. We call the multilinear transformation $T_{f^{[n]}}^{H,\dots,H}$ defined in (2.3) a multiple operator integral and write $T_{f^{[n]}}$ when there is no ambiguity which element H is used.

As a consequence of Theorem 2.8 in [10] adjusted to the context of a semifinite von Neumann algebra, we have the following result.

PROPOSITION 2.5. *Let $H\eta\mathcal{M}_{\text{sa}}$, $n \in \mathbb{N}$, $n \geqslant 2$, $V_k \in \mathcal{L}^{\alpha_k}$, $\alpha_k \in [1,\infty]$, $k = 1,\dots,n$. Let $\alpha \in [1,\infty]$ be such that $\frac{1}{\alpha_1} + \dots + \frac{1}{\alpha_n} = \frac{1}{\alpha}$. Then, for $f \in C_c^{n+1}((a,b))$,*

$$\|T_{f^{[n]}}(V_1,\dots,V_n)\|_\alpha \leqslant \|f\|_{G_n} \prod_{k=1}^n \|V_k\|_{\alpha_k} \leqslant \frac{\sqrt{2}}{n!}(b-a+1)^{3/2}\|f^{(n+1)}\|_\infty \prod_{k=1}^n \|V_k\|_{\alpha_k}.$$

When all entries in $(V_1,\dots,V_n)$ belong to $\mathcal{L}^\alpha$, with $n < \alpha < \infty$, the estimate in Proposition 2.5 can be substantially improved. The following estimate is a consequence of Theorem 5.3 in [9]. The case $n = 1$ is well known; it can be found in, e.g., Theorem 2.9 of [10].

PROPOSITION 2.6. *Let $H\eta\mathcal{M}_{\text{sa}}$, $n \in \mathbb{N}$, $W_k \in \mathcal{L}^{2n}$, $k = 1,\dots,n$. Then, there exists $c_n > 0$, $c_1 = 1$, such that*

$$\|T_{f^{[n]}}(W_1,\dots,W_n)\|_2 \leqslant c_n \|f^{(n)}\|_\infty \prod_{k=1}^n \|W_k\|_{2n}, \tag{2.4}$$

for $f \in C_c^{n+1}(\mathbb{R})$.

We need the following algebraic properties of a multiple operator integral derived from Theorem 2.11 of [10] and Definition 2.4.

PROPOSITION 2.7. *Let $H\eta\mathcal{M}_{\text{sa}}$, $n \in \mathbb{N}$, $W_k \in \mathcal{L}^{\alpha_k}$, with $\alpha_k \in [1,\infty]$, $k = 1,\dots,n$. The following assertions hold:*

(i) *If $f,\varphi \in C_c^{n+1}(\mathbb{R})$, then*

$$T_{(f\varphi)^{[n]}}(W_1,\dots,W_n) = \sum_{k=0}^n T_{f^{[k]}}(W_1,\dots,W_k) \cdot T_{\varphi^{[n-k]}}(W_{k+1},\dots,W_n),$$

where $T_{f^{[0]}}$ denotes $f(H)$.

(ii) *Let* $f \in C_c^{n+1}(\mathbb{R})$ *and* $\psi_1, \psi_2 : \mathbb{R} \mapsto \mathbb{C}$ *be bounded Borel functions. Then,*

$$\psi_1(H) T_{f^{[n]}}(W_1, \ldots, W_n) \psi_2(H) = T_{f^{[n]}}(\psi_1(H) W_1, W_2, \ldots, W_{n-1}, W_n \psi_2(H)).$$

PROPOSITION 2.8. *Let* $H \eta \mathcal{M}_{\mathrm{sa}}$, $n \in \mathbb{N}$, $W_k \in \mathcal{L}^{\alpha_k}$, *with* $\alpha_k \in [1, \infty]$, $k = 1, \ldots, n$, *satisfying* $\frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_n} = 1$. *Assume that* $\alpha_{j_0} = 1$ *for some* $1 \leqslant j_0 \leqslant n$. *Then, for* $f \in C_c^{n+1}(\mathbb{R})$,

$$\tau(T_{f^{[n]}}(W_1, \ldots, W_n)) = \tau(T_{f^{[n]}}(W_i, \ldots, W_n, W_1, W_2, \ldots, W_{i-1})),$$

for every $i \in \{2, \ldots, n\}$.

Proof. The result follows upon applying (2.3), continuity of the trace τ in the $\mathcal{L}^1$-norm, and cyclicity $\tau(AB) = \tau(BA)$ for $A \in \mathcal{L}^1, B \in \mathcal{M}$. ∎

3. MAIN ESTIMATE

Let $a, b \in \mathbb{R}$, $a < b$, $\varepsilon > 0$ and denote

$$a_\varepsilon = a - \varepsilon, \quad b_\varepsilon = b + \varepsilon.$$

Let φ_ε be a smoothening of the indicator function of (a, b) satisfying the properties $\sqrt[4]{\varphi_\varepsilon} \in C_c^\infty((a_\varepsilon, b_\varepsilon))$, $\varphi_\varepsilon|_{(a,b)} \equiv 1$, $0 \leqslant \varphi_\varepsilon \leqslant 1$. More precisely, let φ_ε be defined by

$$\varphi_\varepsilon(x) = (h_1(x) - h_2(x))^4, \tag{3.1}$$

where

$$h_1(x) = \frac{\int_{a_\varepsilon}^x \phi(t - a_\varepsilon)\phi(a - t)\,\mathrm{d}t}{\int_{a_\varepsilon}^a \phi(t - a_\varepsilon)\phi(a - t)\,\mathrm{d}t}, \quad h_2(x) = \frac{\int_b^x \phi(t - b)\phi(b_\varepsilon - t)\,\mathrm{d}t}{\int_b^{b_\varepsilon} \phi(t - b)\phi(b_\varepsilon - t)\,\mathrm{d}t},$$

$$\phi(x) = \begin{cases} \mathrm{e}^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0. \end{cases}$$

We utilize the function φ_ε to create summable weights and make known results for summable perturbations applicable in our unsummable setting.

THEOREM 3.1. *Let* $H \eta \mathcal{M}_{\mathrm{sa}}$ *have* τ*-compact resolvent,* $n \in \mathbb{N}$, $V_1, \ldots, V_n \in \mathcal{M}$. *Let* $a, b \in \mathbb{R}$, $a < b$, *and* $\varepsilon > 0$. *Then, there exists* $C_{n,a,b,\varepsilon,H} > 0$ *such that*

$$|\tau(T_{f^{[n]}}(V_1, \ldots, V_n))| \leqslant C_{n,a,b,\varepsilon,H} \, \|f^{(n)}\|_\infty \prod_{k=1}^n \|V_k\|, \tag{3.2}$$

for every $f \in C_c^{n+1}((a, b))$, *and*

$$\begin{aligned} C_{n,a,b,\varepsilon,H} \leqslant {} & (2^n(n+1) + c_n)\,(b - a + 1)^n\,(1 + \tau(E_H((a, b)))) \\ & \times \Big(\tau(E_H((a_\varepsilon, b_\varepsilon))) + \sqrt{2}\,(b_\varepsilon - a_\varepsilon + 1)^{3/2} \max_{1 \leqslant k \leqslant n} \frac{\|\varphi_\varepsilon^{(k+1)}\|_\infty}{k!}\Big), \end{aligned} \tag{3.3}$$

where c_n *satisfies* (2.4) *and* φ_ε *is defined in* (3.1).

Proof. Define $\gamma_{n,1}$ and $\gamma_{n,0}$ recursively by

$$\gamma_{0,1} = 1, \quad \gamma_{1,1} = 2, \tag{3.4}$$
$$\gamma_{m,1} = \sum_{k=0}^{m-1} \gamma_{k,1} + \frac{\sqrt{2}}{m!}, \quad \gamma_{m,0} = \Big\lfloor \frac{m+1}{2} \Big\rfloor \sum_{k=0}^{m-1} \gamma_{k,1} + c_m, \quad m = 2, \dots, n.$$

Note that for $n \geqslant 2$,

$$\gamma_{n,1} = 2^{n-1}\Big(\frac{3}{2} + \sqrt{2}\sum_{j=2}^{n-1} \frac{1}{2^j\, j!} + \frac{\sqrt{2}}{2^{n-1}\, n!}\Big) \leqslant 2^{n-1}\sqrt{2}\sum_{j=0}^{n-1}\frac{1}{2^j\, j!} \leqslant 2^{n-1}\sqrt{2e}.$$

Hence, for $n \in \mathbb{N}$,

$$\gamma_{n,0} \leqslant \Big\lfloor \frac{n+1}{2} \Big\rfloor \gamma_{n,1} + c_n \leqslant 2^n(n+1) + c_n. \tag{3.5}$$

Denote

$$\beta_{\varepsilon,n,H} = \max\Big\{\|\varphi_\varepsilon(H)\|_1,\ \max_{1\leqslant k\leqslant n} \|T_{\varphi_\varepsilon^{[k]}} : \mathcal{M}^{\times k} \mapsto \mathcal{M}\|\Big\},$$

where

$$\|T_{\varphi_\varepsilon^{[k]}} : \mathcal{M}^{\times k} \mapsto \mathcal{M}\| = \sup_{V_1,\dots,V_k \in \mathcal{M}} \|T_{\varphi_\varepsilon^{[k]}}(V_1,\dots,V_k)\|.$$

By Proposition 2.3,

$$\|\varphi_\varepsilon(H)\|_1 \leqslant \tau(E_H((a_\varepsilon, b_\varepsilon))) \tag{3.6}$$

and by Proposition 2.5,

$$\|T_{\varphi_\varepsilon^{[k]}} : \mathcal{M}^{\times k} \mapsto \mathcal{M}\| \leqslant \|\varphi_\varepsilon\|_{G_k}, \tag{3.7}$$

where $\|\cdot\|_{G_k}$ is defined in (1.3). It follows from (3.6) and (3.7) that

$$\beta_{\varepsilon,n,H} \leqslant \max\Big\{\tau(E_H((a_\varepsilon,b_\varepsilon))), \sqrt{2}(b_\varepsilon - a_\varepsilon)^{1/2} \max_{1\leqslant k\leqslant n} \frac{\|\varphi_\varepsilon^{(k)}\|_\infty + \|\varphi_\varepsilon^{(k+1)}\|_\infty}{k!}\Big\}. \tag{3.8}$$

Hence, to prove (3.2), it suffices to prove

$$|\tau(T_{f^{[n]}}(V_1,\dots,V_n))| \leqslant \Theta_{n,a,b,\varepsilon,H,0}\, \|f^{(n)}\|_\infty \prod_{k=1}^n \|V_k\|, \tag{3.9}$$

where

$$\Theta_{n,a,b,\varepsilon,H,0} = \gamma_{n,0}\,(b-a+1)^n\,(1+\tau(E_H((a,b))))\,\beta_{\varepsilon,n,H}.$$

Along with proving (3.9), we will also prove

$$\|T_{f^{[n]}}(V_1,\dots,V_n)\|_1 \leqslant \Theta_{n,a,b,\varepsilon,H,1}\, \|f^{(n+1)}\|_\infty \prod_{k=1}^n \|V_k\|, \tag{3.10}$$

where

$$\Theta_{n,a,b,\varepsilon,H,1} = \gamma_{n,1}\,(b-a+1)^{n+1}\,(1+\tau(E_H((a,b))))\,\beta_{\varepsilon,n,H}.$$

Note that $f = f\varphi_\varepsilon$, so $f^{[k]} = (f\varphi_\varepsilon)^{[k]}$, for every $k = 1, \dots, n$, where φ_ε is defined in (3.1). We will prove (3.9) and (3.10) for $n = 1$ and then for every $n \geqslant 2$ by induction on $n \geqslant 2$.

Case 1. $n = 1$.

By Proposition 2.7(i),

$$T_{f^{[1]}}(V_1) = f(H)T_{\varphi_\varepsilon^{[1]}}(V_1) + T_{f^{[1]}}(V_1)\varphi_\varepsilon(H). \tag{3.11}$$

By Proposition 2.7(ii),

$$T_{f^{[1]}}(V_1)\varphi_\varepsilon(H) = T_{f^{[1]}}(V_1\sqrt{\varphi_\varepsilon}(H))\sqrt{\varphi_\varepsilon}(H). \tag{3.12}$$

Applying (3.11), (3.12), Hölder's inequality and Propositions 2.3 and 2.6 implies

$$\begin{aligned}\|T_{f^{[1]}}(V_1)\|_1 \leqslant \|f\|_\infty\, \tau(E_H((a,b)))\, \|T_{\varphi_\varepsilon^{[1]}}(V_1)\| \\ + \|f'\|_\infty\, \|V_1\sqrt{\varphi_\varepsilon}(H)\|_2\, \|\sqrt{\varphi_\varepsilon}(H)\|_2.\end{aligned} \tag{3.13}$$

Since

$$\|\sqrt{\varphi_\varepsilon}(H)\|_2^2 = \|\varphi_\varepsilon(H)\|_1, \tag{3.14}$$

combination of (3.13), (3.14), and Hölder's inequality implies

$$\|T_{f^{[1]}}(V_1)\|_1 \leqslant (b-a+1)\,(1+\tau(E_H((a,b))))\,\|f'\|_\infty\,\|V_1\,\beta_{\varepsilon,1,H}\|, \tag{3.15}$$

ensuring (3.10) and (3.9) for $n = 1$.

Case 2. $n = 2$.

By Propositions 2.3 and 2.7 and Hölder's inequality,

$$\begin{aligned}\|T_{f^{[2]}}(V_1,V_2)\|_1 \leqslant \|f\|_\infty\, \tau(E_H((a,b)))\, \|T_{\varphi_\varepsilon^{[2]}}(V_1,V_2)\| + \|T_{f^{[1]}}(V_1)\|_1\, \|T_{\varphi_\varepsilon^{[1]}}(V_2)\| \\ + \|T_{f^{[2]}}(V_1,V_2)\|\, \|\varphi_\varepsilon(H)\|_1.\end{aligned} \tag{3.16}$$

By Proposition 2.5,

$$\|T_{f^{[2]}}(V_1,V_2)\| \leqslant \frac{\sqrt{2}}{2}\,(b-a+1)^{3/2}\,\|f'''\|_\infty\,\|V_1\|\,\|V_2\|. \tag{3.17}$$

Combining (3.14)–(3.17) and (3.10) for $n = 1$ gives (3.10) for $n = 2$.

By Propositions 2.3 and 2.7(i) and Hölder's inequality,

$$\begin{aligned}|\tau(T_{f^{[2]}}(V_1,V_2))| \leqslant \|f\|_\infty\, \tau(E_H((a,b)))\, \|T_{\varphi_\varepsilon^{[2]}}(V_1,V_2)\| + \|T_{f^{[1]}}(V_1)\|_1\, \|T_{\varphi_\varepsilon^{[1]}}(V_2)\| \\ + |\tau(T_{f^{[2]}}(V_1,V_2)\varphi_\varepsilon(H))|.\end{aligned} \tag{3.18}$$

By Proposition 2.7(ii) and Hölder's inequality,

$$\begin{aligned}|\tau(T_{f^{[2]}}(V_1,V_2)\varphi_\varepsilon(H))| &= |\tau(T_{f^{[2]}}(\sqrt[4]{\varphi_\varepsilon}(H)V_1, V_2\sqrt[4]{\varphi_\varepsilon}(H))\sqrt{\varphi_\varepsilon}(H))| \\ &\leqslant \|T_{f^{[2]}}(\sqrt[4]{\varphi_\varepsilon}(H)V_1, V_2\sqrt[4]{\varphi_\varepsilon}(H))\|_2\, \|\sqrt{\varphi_\varepsilon}(H)\|_2.\end{aligned} \tag{3.19}$$

By Proposition 2.6 and Hölder's inequality,

$$\|T_{f^{[2]}}(\sqrt[4]{\varphi_\varepsilon}(H)V_1, V_2\sqrt[4]{\varphi_\varepsilon}(H))\|_2 \leqslant c_2\,\|f''\|_\infty\,\|V_1\|\,\|V_2\|\,\|\sqrt[4]{\varphi_\varepsilon}(H)\|_4^2.$$

Combining the latter with (3.19) gives

$$|\tau(T_{f^{[2]}}(V_1,V_2)\varphi_\varepsilon(H))| \leqslant c_2\,\|f''\|_\infty\,\|V_1\|\,\|V_2\|\,\|\varphi_\varepsilon(H)\|_1. \tag{3.20}$$

Combining (3.18) and (3.20) with (3.15) gives (3.9) for $n=2$.

Case 3. $n \geqslant 3$.

Assume that (3.10) and (3.9) hold for every $n \leqslant p-1$. We demonstrate below that in this case (3.10) and (3.9) also hold for $n=p$. Applying Proposition 2.7 and the inductive hypothesis implies

$$\begin{aligned}\|T_{f^{[p]}}(V_1,\dots,V_p)\|_1 \leqslant \Theta_{p,a,b,\varepsilon,H,1}\,\|f^{(p)}\|_\infty \prod_{k=1}^{p-1}\|V_k\| \\ + \|T_{f^{[p]}}(V_1,\dots,V_p)\|\,\|\varphi_\varepsilon(H)\|_1.\end{aligned} \tag{3.21}$$

By Proposition 2.5, Hölder's inequality, and (3.14),

$$\|T_{f^{[p]}}(V_1,\dots,V_p)\| \leqslant \frac{\sqrt{2}}{p!}\,(b-a+1)^{3/2}\,\|f^{(p+1)}\|_\infty \prod_{k=1}^{p}\|V_k\|. \tag{3.22}$$

Combining (3.21) and (3.22) completes the proof of (3.10).

By Proposition 2.7 and the inductive hypothesis,

$$\begin{aligned}|\tau(T_{f^{[p]}}(V_1,\dots,V_p))| \leqslant \Theta_{p,a,b,\varepsilon,H,1}\,\|f^{(p)}\|_\infty \prod_{k=1}^{p}\|V_k\| \\ + |\tau(T_{f^{[p]}}(V_1,\dots,V_p)\varphi_\varepsilon(H))|.\end{aligned} \tag{3.23}$$

Denote

$$\widetilde{V}_k = V_k\sqrt{\varphi_\varepsilon}(H), \quad k=1,\dots,p.$$

By Propositions 2.7(ii) and 2.8,

$$|\tau(T_{f^{[p]}}(V_1,\dots,V_p)\varphi_\varepsilon(H))| = |\tau(T_{f^{[p]}}(V_3,\dots,V_{p-1},\widetilde{V}_p,\widetilde{V}_1^*,V_2))|. \tag{3.24}$$

Applying the reasoning like in (3.22) and (3.24) $\lfloor (p+1)/2\rfloor - 2$ times more gives

$$|\tau(T_{f^{[p]}}(V_1,\dots,V_p))| \leqslant \Theta_{p,a,b,\varepsilon,H,1}\left(\left\lfloor\frac{p+1}{2}\right\rfloor - 1\right)\|f^{(p)}\|_\infty \prod_{k=1}^{p}\|V_k\| + X_p, \tag{3.25}$$

where

$$X_p = \begin{cases} |\tau(T_{f^{[p]}}(V_{p-1},\widetilde{V}_p,\widetilde{V}_1^*,\dots,\widetilde{V}_{p-4},\widetilde{V}_{p-3}^*,V_{p-2}))| & \text{if } p \text{ is even,}\\ |\tau(T_{f^{[p]}}(\widetilde{V}_p,\widetilde{V}_1^*,\dots,\widetilde{V}_{p-3},\widetilde{V}_{p-2}^*,V_{p-1}))| & \text{if } p \text{ is odd.}\end{cases}$$

If p is even, then arguing as above ensures

$$\begin{aligned}X_p \leqslant \Theta_{p,a,b,\varepsilon,H,1}\,\|f^{(p)}\|_\infty \prod_{k=1}^{p}\|V_k\| \\ + |\tau(T_{f^{[p]}}(\sqrt[4]{\varphi_\varepsilon}(H)\,V_{p-1},\widetilde{V}_p,\widetilde{V}_1^*,\dots,\widetilde{V}_{p-4},\widetilde{V}_{p-3}^*,V_{p-2}\,\sqrt[4]{\varphi_\varepsilon}(H))\sqrt{\varphi_\varepsilon}(H))|\end{aligned} \tag{3.26}$$

and

$$|\tau(T_{f^{[p]}}(\sqrt[4]{\varphi_\varepsilon}(H)\,V_{p-1},\widetilde{V}_p,\widetilde{V}_1^*,\ldots,\widetilde{V}_{p-4},\widetilde{V}_{p-3}^*,V_{p-2}\,\sqrt[4]{\varphi_\varepsilon}(H))\sqrt{\varphi_\varepsilon}(H))|$$

$$\leqslant c_p\,\|f^{(p)}\|_\infty \prod_{k=1}^{p}\|V_k\|\,\|\sqrt[4]{\varphi_\varepsilon}(H)\|_{2p}^2\,\|\sqrt{\varphi_\varepsilon}(H)\|_{2p}^{p-2}\,\|\sqrt{\varphi_\varepsilon}(H)\|_2$$

$$\leqslant c_p\,\|f^{(p)}\|_\infty\,\|\varphi_\varepsilon(H)\|_1 \prod_{k=1}^{p}\|V_k\|. \tag{3.27}$$

If p is odd, then

$$X_p \leqslant \Theta_{p,a,b,\varepsilon,H,1}\,\|f^{(p)}\|_\infty \prod_{k=1}^{p}\|V_k\|$$

$$+\,|\tau(T_{f^{[p]}}(\widetilde{V}_p,\widetilde{V}_1^*,\ldots,\widetilde{V}_{p-3},\widetilde{V}_{p-2}^*,\widetilde{V}_{p-1})\sqrt{\varphi_\varepsilon}(H))| \tag{3.28}$$

By Proposition 2.6 and Hölder's inequality,

$$|\tau(T_{f^{[p]}}(\widetilde{V}_p,\widetilde{V}_1^*,\ldots,\widetilde{V}_{p-3},\widetilde{V}_{p-2}^*,\widetilde{V}_{p-1})\sqrt{\varphi_\varepsilon}(H))|$$

$$\leqslant c_p\,\|f^{(p)}\|_\infty \prod_{k=1}^{p}\|V_k\|\,\|\sqrt{\varphi_\varepsilon}(H)\|_{2p}^{p}\,\|\sqrt{\varphi_\varepsilon}(H)\|_2$$

$$\leqslant c_p\,\|f^{(p)}\|_\infty\,\|\varphi_\varepsilon(H)\|_1 \prod_{k=1}^{p}\|V_k\|. \tag{3.29}$$

Combining (3.25)–(3.29) completes the proof of (3.9). ∎

4. ASYMPTOTIC EXPANSION

Given $H\eta\mathcal{M}_{\mathrm{sa}}$, $V\in\mathcal{M}_{\mathrm{sa}}$, $n\in\mathbb{N}$, and $f\in C_c^{n+1}(\mathbb{R})$, denote

$$\mathcal{R}_{H,f,n}(V) = f(H+V) - \sum_{k=0}^{n-1}\frac{1}{k!}\cdot\frac{\mathrm{d}^k}{\mathrm{d}s^k}f(H+sV)|_{s=0}, \tag{4.1}$$

where the Gâteaux derivatives are evaluated in the operator norm. It follows from, e.g., Theorem 2.6 of [10] (see also references in [10]) that the above derivatives exist and can be represented in the form

$$\frac{1}{k!}\cdot\frac{\mathrm{d}^k}{\mathrm{d}s^k}f(H+sV)|_{s=t} = T_{f^{[k]}}^{H+tV,\ldots,H+tV}(\underbrace{V,\ldots,V}_{k\text{ times}}). \tag{4.2}$$

It is proved in Theorem 3.2 of [10] that these derivatives are elements of $\mathcal{L}^1$ whenever H has τ-compact resolvent.

In the next theorem, we establish the bound (1.1) for the trace of (4.1).

THEOREM 4.1. *Let $H\eta\mathcal{M}_{\mathrm{sa}}$ have τ-compact resolvent, $V \in \mathcal{M}_{\mathrm{sa}}$, $n \in \mathbb{N}$, and $\varepsilon > 0$. Then, for $f \in C_{\mathrm{c}}^{n+1}((a,b))$,*

$$|\tau(\mathcal{R}_{H,f,n}(V))| \leqslant \|f^{(n)}\|_\infty \, \|V\|^n \, (2^n(n+1) + c_n) \, (b-a+1)^n \, (1 + \Omega_{a,b,H,V}) \tag{4.3}$$
$$\times \Big(\Omega_{a_\varepsilon,b_\varepsilon,H,V} + \sqrt{2}\,(b_\varepsilon - a_\varepsilon + 1)^{3/2} \max_{1\leqslant k\leqslant n} \frac{\|\varphi_\varepsilon^{(k+1)}\|_\infty}{k!}\Big),$$

where $\mathcal{R}_{H,f,n}(V)$ is defined in (4.1), *$\Omega_{a_\varepsilon,b_\varepsilon,H,V}$ satisfies* (2.1), *c_n satisfies* (2.4), *and φ_ε is defined in* (3.1).

Proof. It follows from, e.g., Theorem 2.7 of [10] that

$$\mathcal{R}_{H,f,n}(V) = \frac{1}{(n-1)!} \int_0^1 (1-t)^{n-1} \, \frac{\mathrm{d}^n}{\mathrm{d}s^n} f(H+sV)|_{s=t} \, \mathrm{d}t, \tag{4.4}$$

where the integral is evaluated in the strong operator topology. We note that by Proposition 2.1, $H + sV$ has τ-compact resolvent for every $s \in [0,1]$. By (4.2) and (3.2) in Theorem 3.1,

$$\frac{1}{n!}\Big|\tau\Big(\frac{\mathrm{d}^n}{\mathrm{d}s^n} f(H+sV)|_{s=t}\Big)\Big| \leqslant C_{n,a,b,\varepsilon,H+tV} \, \|f^{(n)}\|_\infty \, \|V\|^n, \tag{4.5}$$

where $C_{n,a,b,\varepsilon,H+tV}$ satisfies (3.3). The estimate (4.3) follows from (4.4), (4.5), and Proposition 2.2. ∎

The spectral action functional has the following asymptotic expansion established in two steps, for $n = 1$ and $n \geqslant 2$.

PROPOSITION 4.2 ([1], Theorem 2.5). *Let $H\eta\mathcal{M}_{\mathrm{sa}}$ have τ-compact resolvent and $V \in \mathcal{M}_{\mathrm{sa}}$. Then, for $f \in C_{\mathrm{c}}^3((a,b))$,*

$$\tau(f(H+V)) = \tau(f(H)) + \int_{\mathbb{R}} f'(\lambda) \, \tau(E_H((a,\lambda]) - E_{H+V}((a,\lambda])) \, \mathrm{d}\lambda.$$

THEOREM 4.3. *Let $H\eta\mathcal{M}_{\mathrm{sa}}$ have τ-compact resolvent, $V \in \mathcal{M}_{\mathrm{sa}}$, $n \in \mathbb{N}$, $n \geqslant 2$, and $\varepsilon > 0$. Then, there exists a unique real-valued locally integrable function $\eta_{n,H,V}$ such that*

$$\tau(f(H+V)) = \sum_{k=0}^{n-1} \frac{1}{k!} \, \tau\Big(\frac{\mathrm{d}^k}{\mathrm{d}s^k} f(H+sV)|_{s=0}\Big) + \int_{\mathbb{R}} f^{(n)}(t) \, \eta_{n,H,V}(t) \, \mathrm{d}t, \tag{4.6}$$

for $f \in C_{\mathrm{c}}^{n+1}(\mathbb{R})$. The function $\eta_{n,H,V}$ satisfies the bound

$$\int_{[a,b]} |\eta_{n,H,V}(t)| \, \mathrm{d}t \leqslant \|V\|^n \, (2^n(n+1) + c_n) \, (b-a+1)^n \, (1 + \Omega_{a,b,H,V})$$
$$\times \Big(\Omega_{a_\varepsilon,b_\varepsilon,H,V} + \sqrt{2}\,(b_\varepsilon - a_\varepsilon + 1)^{3/2} \max_{1\leqslant k\leqslant n} \frac{\|\varphi_\varepsilon^{(k+1)}\|_\infty}{k!}\Big)$$

where $\Omega_{a_\varepsilon,b_\varepsilon,H,V}$ satisfies (2.1), *c_n satisfies* (2.4), *and φ_ε is defined in* (3.1).

Proof. The result follows from Theorem 4.1, the Riesz representation theorem for elements of $(C_c^{n+1}(\mathbb{R}))^*$, estimate (4.5), and integration by parts. This method is standard in derivation of trace formulas and can be found in, e.g., the proof of Theorem 3.10 in [10]. ∎

Analogs of the trace formula (4.6) have a long history in perturbation theory, and we refer the reader to [11] for details and references.

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