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MAXIMAL AMENABILITY OF THE GENERATOR SUBALGEBRA IN q-GAUSSIAN VON NEUMANN ALGEBRAS

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ABSTRACT. In this article, we develop a structural theorem for the q-Gaussian algebras, namely, we construct a Riesz basis for the q-Fock space in the spirit of Rădulescu. As an application, we show that the generator subalgebra is maximal amenable inside the q-Gaussian von Neumann algebra for any real number q with $|q| < \frac{1}{9}$.

KEYWORDS: *q-Gaussian, von Neumann algebra, Riesz basis, maximal amenability.*

MSC (2010): Primary 46Lxx; Secondary 47Lxx.

INTRODUCTION

In this paper, we investigate the structure of von Neumann algebras called the "q-Gaussian von Neumann algebras", which can be viewed as deformed free group factors. First of all, let us explain the history of the q-Gaussians. Voiculescu's free probability theory [19] identified the free group factors with von Neumann algebras acting on the full Fock spaces, leading to many deep results about the free group factors. One of the vital points in Voiculescu's analysis is that the left creation operators of the full Fock spaces satisfy the following equation:

$$\ell(\eta)^*\ell(\xi) - 0 \cdot \ell(\xi)\ell(\eta)^* = \langle \xi, \eta \rangle$$

for any $\xi, \eta \in \mathcal{H}_{\mathbb{R}}$. This equation is similar to the "Bosonic" or "Fermionic" equations

$$\ell(\eta)^*\ell(\xi) - (\pm 1) \cdot \ell(\xi)\ell(\eta)^* = \langle \xi, \eta \rangle.$$

This similarity raises a natural question: are there operators interpolating the above equations? Do there exist operators satisfying the following equation:

$$\ell(\eta)^*\ell(\xi) - q \cdot \ell(\xi)\ell(\eta)^* = \langle \xi, \eta \rangle, \quad \text{for all } -1 \leqslant q \leqslant 1?$$

As an affirmative answer to this question, Bożejko and Speicher [4] constructed the q-Fock spaces, and the left creation operators on those q-Fock spaces do the job. Similar to Voiculescu's free Gaussian functor, Bożejko–Speicher's q-Gaussian functor naturally generates a finite von Neumann algebra to any real Hilbert space, called the *q-Gaussian von Neumann algebras*. The q-Gaussian algebras are currently under intense study, and many interesting properties such as factoriality [17], non-injectivity [13], fullness [18], and strong solidity [1] are known. Furthermore, when the dimension of the real Hilbert space is finite and when $|q|$ is sufficiently small, the corresponding q-Gaussian is isomorphic to the free group factors [10].

When $q \neq 0$, the difficulty of analysing the q-Gaussian algebras lies in the wild behaviour of the inner product arising from the trace, making it hard to estimate the L^2-norms of the operators. This situation is in contrast to that of the free group factor case: when $q = 0$, given an orthonormal basis of the underlying real Hilbert space, the set of all finite words of those basis vectors naturally forms an orthonormal basis of the full Fock space, which behaves well while acted by the left and right annihilation operators. On the other hand, if $q \neq 0$, the same idea fails to give us a well-behaving basis. Therefore, it is natural to call for a new basis for the q-Fock space.

In this paper, we present a basis. More precisely, the following theorem.

THEOREM A. *Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space and let q be a real number with $|q| < \frac{1}{9}$. Let $e \in \mathcal{H}_{\mathbb{R}}$ be a unit vector and let A be the von Neumann algebra generated by $s(e) := c(e) + a(e)$. Then, there exists a Riesz basis $\{\eta^i_{r,s} : i \in I, r, s \geqslant 0\}$ for the orthogonal complement of $L^2(A)$ in the q-Fock space such that for any fixed i, the vector $a(e)^k \eta^i_{r,s}$ is a finite linear combination of the $\eta^i_{r,s}$'s.*

Conceptually, our idea comes from Rădulescu's basis for the radial masa of the free group factors. Indeed, in [8] the Rădulescu's basis is used to show that the radial masa is maximal injective inside the free group factors. One of the advantages of our basis is that it is compatible with multiplication by the right and left annihilation operators of e. In particular, the operator $s(e)$ preserves finite sums of $\eta^i_{r,s}$, which turns out to be useful in investigating the q-Gaussian algebras. As a corollary, we show the maximal amenability of the generator subalgebra.

THEOREM B. *With the same assumptions as in the above theorem, A is maximal amenable inside M. Moreover, if B is another distinct maximal amenable subalgebra of M, then $A \cap B$ is atomic.*

The above theorem can be viewed as a generalization of Popa's original result [15] on the maximal amenability of the generator masa in free group factors, which corresponds to the case $q = 0$.

In addition, one can apply the structural result to give another proof of the fullness of the q-Gaussian algebras (cf. [18]).

The paper consists of 5 sections. Section 1 is on preliminaries for the q-Fock space and we establish some notations which will be used throughout the paper. In Section 2 we develop some q-combinatorics for later use. The next two sections will take up the majority of our paper: Section 3 includes the construction of the Riesz basis in the spirit of Rădulescu and Section 4 contains the key estimates for elements in the relative commutant of A in the ultraproduct. In the last section, we establish the asymptotic orthogonality property for the inclusion of the generator subalgebra $A \subset M$ and finish the proof of the main theorem. We also include a proof for the fullness of the q-Gaussian algebras using our basis.

1. PRELIMINARIES

Throughout the paper we assume that $-1 < q < 1$ and let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space with $\dim \mathcal{H}_{\mathbb{R}} \geqslant 2$. Denote by $\mathcal{H} := \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $\mathcal{H}_{\mathbb{R}}$. Define an inner product on $\bigoplus\limits_{n\geqslant 0} \mathcal{H}^{\otimes n}$ by

$$\langle e_1 \otimes \cdots \otimes e_n, f_1 \otimes \cdots \otimes f_m \rangle_q = \delta_{n,m} \sum_{\sigma \in S_m} q^{|\sigma|} \langle e_1 \otimes \cdots \otimes e_n, f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(m)} \rangle,$$

where S_m is the group of permutations on $\{1, \ldots, m\}$, $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$ is the space spanned by the vacuum vector Ω, the inner product on the right-hand side is the usual one on $\mathcal{H}^{\otimes m}$ and by $|\sigma|$ we mean the number of inversions of $\sigma \in S_m$ given by

$$|\sigma| = \#\{(i,j) \in \{1, \ldots, m\}^2 : i < j, \sigma(i) > \sigma(j)\}.$$

The *q-deformed Fock space* $\mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$ is the completion of $(\bigoplus_{n\geqslant 0} \mathcal{H}^{\otimes n}, \langle \cdot, \cdot \rangle_q)$ and $\| \cdot \|_q$ is the norm induced by this inner product. For simplicity, sometimes we will suppress the subscript q for the inner product and the norm on the q-Fock space.

For $e \in \mathcal{H}_{\mathbb{R}}$, we define the *left creation operator* $c(e)$ and the *right creation operator* $c_{\mathrm{r}}(e)$ on $\mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$ by $c(e)(\Omega) = e = c_{\mathrm{r}}(e)(\Omega)$ and

$$\begin{aligned} c(e)(e_1 \otimes \cdots \otimes e_n) &= e \otimes e_1 \otimes \cdots \otimes e_n, \\ c_{\mathrm{r}}(e)(e_1 \otimes \cdots \otimes e_n) &= e_1 \otimes \cdots \otimes e_n \otimes e, \end{aligned} \tag{1.1}$$

for $n \geqslant 1$. Both $c(e)$ and $c_{\mathrm{r}}(e)$ are bounded operators ([4], Lemma 4) and their adjoints $a(e) = c^*(e), a_{\mathrm{r}}(e) = c_{\mathrm{r}}^*(e)$ are called the *left annihilation operator* and the *right annihilation operator*, respectively, which are given by $a(e)(\Omega) = 0 = a_{\mathrm{r}}(e)(\Omega)$ and

$$\begin{aligned} a(e)(e_1 \otimes \cdots \otimes e_n) &= \sum_{1\leqslant i\leqslant n} q^{(i-1)} \langle e, e_i \rangle e_1 \otimes \cdots \otimes \widehat{e_i} \otimes \cdots \otimes e_n, \\ a_{\mathrm{r}}(e)(e_1 \otimes \cdots \otimes e_n) &= \sum_{1\leqslant i\leqslant n} q^{(n-i)} \langle e, e_i \rangle e_1 \otimes \cdots \otimes \widehat{e_i} \otimes \cdots \otimes e_n, \end{aligned} \tag{1.2}$$

for $n \geqslant 1$, where $\widehat{e_i}$ means the letter being removed. Note that $c(e)$ and $c_{\mathrm{r}}(f)$ commute but $c(e)$ and $a_{\mathrm{r}}(f)$ do not commute in general.

The operators $c(e), c_{\mathrm{r}}(e)$ satisfy the following important *q-commutation relations* [4]:

$$a(e)c(f) - qc(f)a(e) = \langle e, f\rangle \mathrm{Id}, \quad a_{\mathrm{r}}(e)c_{\mathrm{r}}(f) - qc_{\mathrm{r}}(f)a_{\mathrm{r}}(e) = \langle e, f\rangle \mathrm{Id}. \tag{1.3}$$

For $e \in \mathcal{H}_{\mathbb{R}}$, let

$$s(e) = c(e) + a(e)$$

and let $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ be the von Neumann algebra generated by $\{s(e) : e \in \mathcal{H}_{\mathbb{R}}\}$. We call it the *q-Gaussian algebra associated with* $\mathcal{H}_{\mathbb{R}}$. It is known that $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ is a type II_1 factor ([17], Corollary 1) and Ω is a separating and cyclic vector which gives the trace for $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ ([5], Theorem 4.3, 4.4). Consequently, each element $x \in \Gamma_q(\mathcal{H}_{\mathbb{R}})$ is uniquely determined by $\xi = x \cdot \Omega \in \mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$ and we write $x = s(\xi)$. This notation is consistent with the above definition for $s(e), e \in \mathcal{H}_{\mathbb{R}}$.

One can also define $s_{\mathrm{r}}(e) = c_{\mathrm{r}}(e) + a_{\mathrm{r}}(e)$ for $e \in \mathcal{H}_{\mathbb{R}}$ and define $\Gamma_{q,\mathrm{r}}(\mathcal{H}_{\mathbb{R}}) := \{s_{\mathrm{r}}(e) : e \in \mathcal{H}_{\mathbb{R}}\}''$. Then similar to the group von Neaumann algebra case we have $\Gamma_{q,\mathrm{r}}(\mathcal{H}_{\mathbb{R}}) = \Gamma_q(\mathcal{H}_{\mathbb{R}})'$.

Here we record two facts that will be used in this paper.

(•) Let $e \in \mathcal{H}$ be a unit vector, then

$$\|e^{\otimes n}\|_q^2 = [n]_q!, \tag{1.4}$$

where $[k]_q = \frac{1-q^k}{1-q}$ and $[n]_q! = [1]_q \cdots [n]_q$. We also define $[0]_q! := 1$.

(•) [Wick formula [3], Proposition 2.7] Let $e_1 \otimes \cdots \otimes e_n \in \mathcal{H}^{\otimes n}$, then

$$\begin{aligned} s(e_1 \otimes \cdots \otimes e_n) = \sum_{i=0}^{n} \sum_{\sigma \in S_n/(S_{n-i}\times S_i)} & q^{|\sigma|} c(e_{\sigma(1)}) \cdots c(e_{\sigma(n-i)}) \\ & \times a(e_{\sigma(n-i+1)}) \cdots a(e_{\sigma(n)}), \end{aligned} \tag{1.5}$$

where σ is the representative of $S_{n-i} \times S_i$ in S_n with minimal number of inversions.

From now on we fix a unit vector $e \in \mathcal{H}_{\mathbb{R}}$ and we call the von Neumann subalgebra $\Gamma_q(\mathbb{R}e) \subset \Gamma_q(\mathcal{H}_{\mathbb{R}})$ a *generator subalgebra*. It is shown by Ricard in [17] that this gives a maximal abelian subalgebra (masa) of $\Gamma_q(\mathcal{H}_{\mathbb{R}})$.

Let $T : \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ be an $\mathbb{R}$-linear contraction. We still denote by T its complexification given by $T(\xi + \mathrm{i}\eta) = T(\xi) + \mathrm{i}T(\eta), \forall \xi, \eta \in \mathcal{H}_{\mathbb{R}}$. Then the *first quantization* $\mathcal{F}_q(T)$, is the bounded operator on $\mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$ defined by

$$\mathcal{F}_q(T) = \mathrm{Id}_{\mathbb{C}\Omega} \oplus \bigoplus_{n \geqslant 1} T^{\otimes n}.$$

The *second quantization* of T, is the unique unital completely positive map $\Gamma_q(T)$ on $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ defined as

$$\Gamma_q(T)(s(\xi)) = s(\mathcal{F}_q(T)(\xi)).$$

In particular, if $T = E_e : \mathcal{H}_{\mathbb{R}} \to \mathbb{R}e$ is the orthogonal projection, then $\mathcal{F}_q(E_e)$ is the conditional expectation of $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ onto $\Gamma_q(\mathbb{R}e)$.

To simplify notations, from now on we will write $A := \Gamma_q(\mathbb{R}e)$ for the generator subalgebra and $M := \Gamma_q(\mathcal{H}_{\mathbb{R}})$ for the q-Gaussian algebra.

2. SOME q-COMBINATORICS

In this section we will develop some formulas about combinatorics in q-Gaussians that will be needed in later sections.

For $n, m \in \mathbb{N} \cup \{0\}, n \geqslant m$, set

$$\binom{n}{m}_q = \frac{[n]_q!}{[m]_q![n-m]_q!} = \prod_{i=1}^{n-m} \frac{1-q^{m+i}}{1-q^i}.$$

We make the following convention

$$\binom{n}{m}_q = 0 \quad \text{whenever } m > n \text{ or } m < 0.$$

The following q-analogue of the Pascal's identity for these q-binomial coefficients (cf. Proposition 1.8 of [3]) can be easily checked.

LEMMA 2.1. *For all $m \in \mathbb{Z}$ and $n \geqslant 0$,*

$$\binom{n+1}{m}_q = q^m \binom{n}{m}_q + \binom{n}{m-1}_q = \binom{n}{m}_q + q^{n-m+1} \binom{n}{m-1}_q. \tag{2.1}$$

Continuing the analogy, the q-binomial coefficients $\binom{n}{m}_q$ can also be seen to count "number" of weighted paths in the "q-Pascal's triangle" from $(0,0)$ to $(n-m, m)$. The q-Pascal triangle is formed from the ordinary Pascal triangle by putting a weight of q^i on each (right) edge from (i,j) to $(i,j+1)$, as shown below. All the other (left) edges will have weight 1. The weight of a path is the product of the weights on its constituent edges.

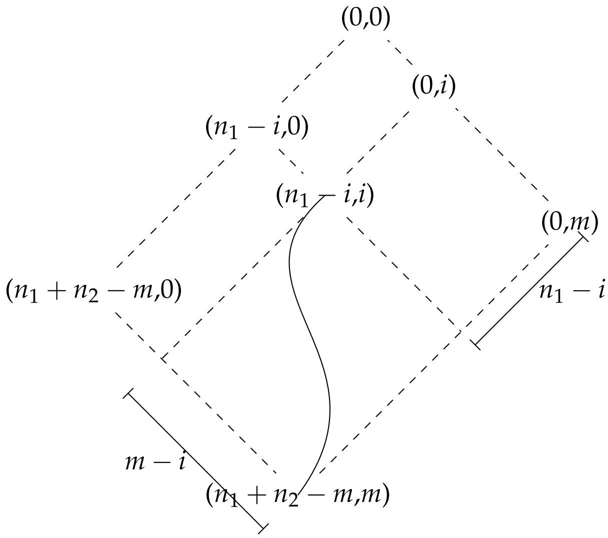
$$\begin{tikzcd}
 & & & & (0,0) \arrow[dl] \arrow[dr, "q^0"] & & & & \\
 & & & (1,0) \arrow[dl] \arrow[dr, "q^1"] & & (0,1) \arrow[dl] \arrow[dr, "q^0"] & & & \\
 & & (2,0) \arrow[dl] \arrow[dr, "q^2"] & & (1,1) \arrow[dl] \arrow[dr, "q^1"] & & (0,2) \arrow[dl] \arrow[dr, "q^0"] & & \\
 & (3,0) \arrow[dl] \arrow[dr, "q^3"] & & (2,1) \arrow[dl] \arrow[dr, "q^2"] & & (1,2) \arrow[dl] \arrow[dr, "q^1"] & & (0,3) \arrow[dl] \arrow[dr, "q^0"] & \\
(4,0) & & (3,1) & & (2,2) & & (1,3) & & (0,4)
\end{tikzcd}$$

For instance, the sum of all weighted paths from $(0,0)$ to $(1,2)$ is $1 + q + q^2 = [3]_q = \binom{3}{2}_q$. It is clear from the diagram that they satisfy the second recurrence relation mentioned above, with the other one following from the symmetry of the q-binomial coefficient.

LEMMA 2.2. *For $n_1, n_2, m \in \mathbb{N} \cup \{0\}$ with $n_1 + n_2 \geqslant m$, we have the following:*

$$\sum_{i=0}^{m} q^{(n_1-i)(m-i)} \binom{n_1}{i}_q \binom{n_2}{m-i}_q = \binom{n_1+n_2}{m}_q. \tag{2.2}$$

Proof. Any path from $(0,0)$ to (n_1+n_2-m, m) will pass through (n_1-i, i) for some $0 \vee m - n_2 \leqslant i \leqslant m \wedge n_1$. This range of index corresponds exactly to the terms with non-zero contribution in the above sum. Now $\binom{n_1}{i}_q$ counts the sum of weighted paths from $(0,0)$ to (n_1-i, i). To go from (n_1-i, i) to (n_1+n_2-m, m) involves travelling along paths counted by $\binom{n_2}{m-i}_q$.



However now every right edge in such a path has its weight multiplied by an extra factor of q^{n_1-i}. Moreover there are exactly $m-i$ right edges in such paths. The result now follows. ∎

Let X, Y, Z, W be indeterminates satisfying the following relations:

$$XY = qYX + 1, \quad XZ = ZX + W, \quad WZ = qZW, \quad XW = qWX. \tag{2.3}$$

For convenience, we make the convention that $x^0 = 1$, for all $x \in \{X, Y, Z, W\}$.

REMARK 2.3. As we will see, in the following we will sometimes use negative powers of the indeterminates, however their coefficients will all be zero.

We first discuss the commutation relations between powers of X and Y.

LEMMA 2.4. *For $n, m \in \mathbb{N}$, we have*

$$X^m Y^n = \sum_{i=0}^{m} q^{(n-i)(m-i)} [i]_q! \binom{n}{i}_q \binom{m}{i}_q Y^{n-i} X^{m-i}. \tag{2.4}$$

Proof. We prove it by induction on m and n.

An easy induction shows that

$$XY^n = q^nY^nX + [n]_qY^{n-1}, \quad X^mY = q^mYX^m + [m]_qX^{m-1},$$

which are special cases of (2.4).

Now suppose that (2.4) holds up to m, n. Then

$$\begin{aligned}
X^{m+1}Y^n &= \sum_{i=0}^{m} q^{(n-i)(m-i)}[i]_q!\binom{n}{i}_q\binom{m}{i}_q XY^{n-i}X^{m-i}\\
&= \sum_{i=0}^{m}\Big(q^{(n-i)(m-i+1)}[i]_q!\binom{n}{i}_q\binom{m}{i}_q Y^{n-i}X^{m+1-i}\\
&\qquad + q^{(n-i)(m-i)}[i]_q!\binom{n}{i}_q\binom{m}{i}_q [n-i]_qY^{n-1-i}X^{m-i}\Big)\\
&= \sum_{i=0}^{m+1}\Big[Y^{n-i}X^{m+1-i}\Big(q^{(n-i)(m+1-i)}[i]_q!\binom{n}{i}_q\binom{m}{i}_q\\
&\qquad + q^{(n-i+1)(m+1-i)}[i-1]_q!\binom{n}{i-1}_q\binom{m}{i-1}_q[n+1-i]_q\Big)\Big]\\
&= \sum_{i=0}^{m+1} Y^{n-i}X^{m+1-i}q^{(n-i)(m+1-i)}[i]_q!\binom{n}{i}_q\left(\binom{m}{i}_q + q^{m+1-i}\binom{m}{i-1}_q\right)\\
&= \sum_{i=0}^{m+1} Y^{n-i}X^{m+1-i}q^{(n-i)(m+1-i)}[i]_q!\binom{n}{i}_q\binom{m+1}{i}_q.
\end{aligned}$$

Here in the second equation we used (2.3); the third equality is due to the simple fact

$$[i-1]_q!\binom{n}{i-1}_q[n-i+1]_q = [i]_q!\binom{n}{i}_q,$$

and the last equality comes from (2.1).

The case for X^mY^{n+1} is completely similar so we omit the details. ∎

Now we turn to the relation between powers of X and Z. Naturally, W also comes into play.

LEMMA 2.5. *For $m, n \in \mathbb{N}$, we have*

$$X^mZ^n = \sum_{i=0}^{m}[i]_q!\binom{n}{i}_q\binom{m}{i}_q Z^{n-i}W^iX^{m-i}. \tag{2.5}$$

Proof. Let's proceed by induction on m and n.

One can easily show that

$$XZ^n = Z^nX + [n]_qZ^{n-1}W, \quad X^mZ = ZX^m + [m]_qWX^{m-1},$$

which are special cases for (2.5).

Suppose that (2.5) holds up to m, n. Then

$$\begin{aligned}
X^{m+1}Z^n &= \sum_{i=0}^{m}[i]_q!\binom{n}{i}_q\binom{m}{i}_q XZ^{n-i}W^iX^{m-i} \\
&= \sum_{i=0}^{m}[i]_q!\binom{n}{i}_q\binom{m}{i}_q (Z^{n-i}X+[n-i]_qZ^{n-i-1}W)W^iX^{m-i} \\
&= \sum_{i=0}^{m}[i]_q!\binom{n}{i}_q\binom{m}{i}_q (q^iZ^{n-i}W^iX^{m+1-i}+[n-i]_qZ^{n-i-1}W^{i+1}X^{m-i}) \\
&= \sum_{i=0}^{m+1}\Big[Z^{n-i}W^iX^{m+1-i} \\
&\qquad \times\Big([i]_q!\binom{n}{i}_q\binom{m}{i}_q q^i+[i-1]_q!\binom{n}{i-1}_q\binom{m}{i-1}_q[n-i+1]_q\Big)\Big] \\
&= \sum_{i=0}^{m+1}[i]_q!\binom{n}{i}_q\binom{m+1}{i}_q Z^{n-i}W^iX^{m+1-i}.
\end{aligned}$$

The other case is completely similar. ∎

PROPOSITION 2.6. *Let $X = a(e), Y = c(e), Z = c_{\mathrm{r}}(e)$ and W be the bounded operator on $\mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$ defined by*

$$W(\xi) = q^n\xi, \quad \forall \xi \in \mathcal{H}^{\otimes n}.$$

Then X, Y, Z, W satisfy the relations listed in (2.3). *Consequently,* (2.4) *and* (2.5) *hold true.*

Proof. (2.3) can be checked by direct computations hence (2.4) and (2.5) follow from the previous two lemmas. ∎

3. RĂDULESCU BASIS IN q-FOCK SPACES

In this section we construct the Rădulescu basis for $\mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$, which will be the fundamental tool to study the generator subalgebra. The construction is motivated by the original construction of Rădulescu in [16].

For each integer $k \geqslant 0$, we consider the following subspace of $\mathcal{H}^{\otimes k} \subset \mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$:

$$T^k := \{\xi \in \mathcal{H}^{\otimes k} : a(e)\xi = a_{\mathrm{r}}(e)\xi = 0\}. \tag{3.1}$$

It is clear that $T^0 = \mathbb{C}\Omega$ and each T^k is non-zero (for instance, if we choose $f \in \mathcal{H}_{\mathbb{R}}$ with $f \perp e$, then $f^{\otimes k} \in T^k$).

For each $\xi \in \mathcal{H}^{\otimes k}$ and for all $s, t \in \mathbb{N} \cup \{0\}$, define

$$\xi_{s,t} = e^{\otimes s} \otimes \xi \otimes e^{\otimes t} \in \mathcal{H}^{\otimes(k+s+t)}. \tag{3.2}$$

We also make the convention that $\xi_{s,t} = 0$ if either $s < 0$ or $t < 0$.

We start with a few important observations.

LEMMA 3.1. *For $\xi \in T^k$ and $s, t \geqslant 0$, we have the following:*

$$a(e)\xi_{s,t} = [s]_q \xi_{s-1,t} + q^{s+k}[t]_q \xi_{s,t-1}, \quad a_{\mathrm{r}}(e)\xi_{s,t} = q^{t+k}[s]_q \xi_{s-1,t} + [t]_q \xi_{s,t-1}. \tag{3.3}$$

Proof. Since the left and right annihilation operators behave similarly, we just show the first equation. This is a consequence of the following identity

$$\begin{aligned} a(e)(W_1 \otimes W_2 \otimes W_3) = (a(e)W_1) \otimes W_2 \otimes W_3 + q^{|W_1|} W_1 \otimes (a(e)W_2) \otimes W_3 \\ + q^{|W_1|+|W_2|} W_1 \otimes W_2 \otimes (a(e)W_3), \end{aligned}$$

where $W_i, i = 1, 2, 3$ are basic words and $|W_i|$ stands for the length. By linearity, the equation still holds even if W_2 is a linear combination of basic words with the same length. Thus for $\xi \in T^k$, we have

$$\begin{aligned} a(e)\xi_{s,t} &= a(e)(e^{\otimes s} \otimes \xi \otimes e^{\otimes t}) \\ &= (a(e)e^{\otimes s}) \otimes \xi \otimes e^{\otimes t} + q^s e^{\otimes s} \otimes (a(e)\xi) \otimes e^{\otimes t} + q^{s+k} e^{\otimes s} \otimes \xi \otimes (a(e)e^{\otimes t}) \\ &= [s]_q \xi_{s-1,t} + 0 + q^{s+k}[t]_q \xi_{s,t-1}. \quad \blacksquare \end{aligned}$$

LEMMA 3.2. *For $\xi \in T^k$, we have*

$$\begin{aligned} & s(e)^n s_{\mathrm{r}}(e)^m \xi \in \operatorname{span}\{\xi_{s,t} : s, t \geqslant 0\}, \quad \forall n, m \geqslant 0, \\ & \xi_{s,t} \in \operatorname{span}\{s(e)^n s_{\mathrm{r}}(e)^m \xi : n, m \geqslant 0\}, \quad \forall s, t \geqslant 0. \end{aligned} \tag{3.4}$$

Proof. For the first inclusion,

$$\begin{aligned} s(e)^n s_{\mathrm{r}}(e)^m \xi &= (c(e) + a(e))^n (c_{\mathrm{r}}(e) + a_{\mathrm{r}}(e))^m \xi \\ &= (c_{\mathrm{r}}(e) + a_{\mathrm{r}}(e))^m (c(e) + a(e))^n \xi. \end{aligned} \tag{3.5}$$

The q-commutation relations imply that we can write $(c(e) + a(e))^n$ and $(c_{\mathrm{r}}(e) + a_{\mathrm{r}}(e))^m$ as polynomials of the form

$$(c(e) + a(e))^n = \sum_{i,j \geqslant 0} a_{i,j} c(e)^i a(e)^j, \quad (c_{\mathrm{r}}(e) + a_{\mathrm{r}}(e))^m = \sum_{s,t \geqslant 0} b_{s,t} c_{\mathrm{r}}(e)^s a_{\mathrm{r}}(e)^t. \tag{3.6}$$

Thus $s(e)^n s_{\mathrm{r}}(e)^m \xi$ is a linear combination of

$$c(e)^i a(e)^j c_{\mathrm{r}}(e)^s a_{\mathrm{r}}(e)^t \xi,$$

where $i, j, s, t \in \mathbb{N} \cup \{0\}$. By the previous lemma, all such terms are in $\operatorname{span}\{\xi_{l,p} : l, p \geqslant 0\}$, which yields the first inclusion.

We prove the second inclusion by induction on $s + t$. When $s + t = 0$, the conclusion clearly holds. Suppose now that the inclusion holds for $s + t \leqslant N$. By Lemma 3.1 we have that

$$\begin{aligned} s(e)\xi_{s,t} &= \xi_{s+1,t} + [s]_q \xi_{s-1,t} + q^{s+k}[t]_q \xi_{s,t-1}, \\ s_{\mathrm{r}}(e)\xi_{s,t} &= \xi_{s,t+1} + q^{t+k}[s]_q \xi_{s-1,t} + [t]_q \xi_{s,t-1}. \end{aligned}$$

Hence the conclusion holds for $s + t = N + 1$ as well. $\blacksquare$

For $k \geqslant 0$, let

$$Q_k : \mathcal{F}_q(\mathcal{H}_{\mathbb{R}}) \to \mathcal{H}^{\otimes k},$$

be the orthogonal projections from the q-Fock space onto $\mathcal{H}^{\otimes k}$ and we define

$$S^k := \mathcal{H}^{\otimes k} \ominus T^k.$$

For notational convenience, we let $\mathcal{H}^{\otimes i} = \{0\}$, for all $i < 0$.

We first characterize S^k as in the following lemma.

LEMMA 3.3. *For all $k \geqslant 0$, $S^k = \operatorname{span}\{Q_k(s(e)\eta), Q_k(s_{\mathrm{r}}(e)\eta) : \eta \in \mathcal{H}^{\otimes l}, l < k\}$.*

Proof. It suffices to show that $\xi \in \mathcal{H}^{\otimes k}$ belongs to T^k if and only if

$$\langle \xi, Q_k(s(e)\eta)\rangle_q = \langle \xi, Q_k(s_{\mathrm{r}}(e)\eta)\rangle_q = 0$$

for any $\eta \in \mathcal{H}^{\otimes l}, l < k$.

To see this, notice that

$$\langle \xi, Q_k(s(e)\eta)\rangle_q = \langle \xi, Q_k(c(e)\eta)\rangle_q = \langle a(e)\xi, \eta\rangle_q.$$

Since $a(e)\xi \in \mathcal{H}^{\otimes(k-1)}$, we have that $\langle \xi, Q_k(s(e)\eta)\rangle_q = 0$ for any $\eta \in \mathcal{H}^{\otimes l}, l < k$ if and only if $a(e)\xi = 0$.

Similarly, $\langle \xi, Q_k(s_{\mathrm{r}}(e)\eta)\rangle_q = 0$ for any $\eta \in \mathcal{H}^{\otimes l}, l < k$ if and only if $a_{\mathrm{r}}(e)\xi = 0$. ∎

LEMMA 3.4. *For all $k \geqslant 0$, $\mathcal{H}^{\otimes k} \subset \operatorname{span}\{s(e)^n s_{\mathrm{r}}(e)^m \xi : \xi \in T^l, l \leqslant k, n, m \geqslant 0\}$.*

Proof. We prove it by induction. When $k = 0$, the statement is clearly true. Assume that the lemma holds up to $k-1$ and let $\eta \in \mathcal{H}^{\otimes k}$. We may further assume that $\eta \in S^k$.

By Lemma 3.3, η is a linear combination of $Q_k(s(e)\xi)$ and $Q_k(s_{\mathrm{r}}(e)\xi), \xi \in \mathcal{H}^{\otimes(k-1)}$. By the induction hypothesis, each $\xi \in \mathcal{H}^{\otimes(k-1)}$ is a linear combination of $s(e)^n s_{\mathrm{r}}(e)^m \xi', \xi' \in T^l, l \leqslant k-1, n, m \geqslant 0$. Thus η is a linear combination of $Q_k(s(e)^n s_{\mathrm{r}}(e)^m \xi'), \xi' \in T^l, l \leqslant k-1, n, m \geqslant 0$.

Now, by Lemma 3.2, $Q_k(s(e)^n s_{\mathrm{r}}(e)^m \xi') \in \operatorname{span}\{\xi'_{r,s} : r, s \geqslant 0, r + s + |\xi'| = k\}$ but again by Lemma 3.2, each $\xi'_{r,s} \in \operatorname{span}\{s(e)^n s_{\mathrm{r}}(e)^m \xi' : n, m \geqslant 0\}$. Therefore, we are done. ∎

LEMMA 3.5. *Suppose that $\xi \in T^t$ and $r, s, k \geqslant 0$ are non-negative integers, then we have*

$$\begin{aligned} a(e)^k \xi_{r,s} &= \sum_{i+j=k, i,j \geqslant 0} \frac{[r]_q!}{[r-j]_q!} \cdot \frac{[s]_q!}{[s-i]_q!} \cdot \binom{k}{i}_q q^{(t+r-j)i} \xi_{r-j, s-i}, \\ a_{\mathrm{r}}(e)^k \xi_{r,s} &= \sum_{i+j=k, i,j \geqslant 0} \frac{[r]_q!}{[r-i]_q!} \cdot \frac{[s]_q!}{[s-j]_q!} \cdot \binom{k}{i}_q q^{(t+s-j)i} \xi_{r-i, s-j}. \end{aligned} \tag{3.7}$$

The proof is just an induction via direct computations so we omit the details.

Now we compute the inner products between $\xi_{r,s}$.

LEMMA 3.6. *Let $\xi,\eta \in \bigcup_{k\geqslant 0} T^k$ with $\xi \perp \eta$, then for any r,s,r',s' non-negative integers, we have*

$$\langle \xi_{r,s}, \eta_{r',s'}\rangle = 0. \tag{3.8}$$

Moreover, if $r+s \neq r'+s'$, then

$$\langle \xi_{r,s}, \xi_{r',s'}\rangle = 0.$$

Proof. The second statement is trivial so we focus on the first. By Lemma 3.2 and the fact that $s(e)s_{\mathrm{r}}(e) = s_{\mathrm{r}}(e)s(e)$, it suffices to show that

$$s(e)^n s_{\mathrm{r}}(e)^m \xi \perp \eta$$

for all n,m non-negative integers. Again by Lemma 3.2, it reduces to show that

$$\xi_{r,s} \perp \eta$$

for all r,s non-negative integers. This is clear by the definition of T^k unless $r = s = 0$, but then the assumption $\xi \perp \eta$ leads to the conclusion. ∎

PROPOSITION 3.7. *Let r,s,r',s',k be non-negative integers with $r+s = r'+s'$ and $\xi \in T^k$ of norm 1, then*

$$\langle \xi_{r,s}, \xi_{r',s'}\rangle = \sum_{i=0}^{r'} q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q. \tag{3.9}$$

Proof. The proof is simply a direct but lengthy computation. However for the convenience of the readers, we include the details here. For simplicity, we write $\alpha^i_{n,m} := [i]_q!\binom{n}{i}_q\binom{m}{i}_q$ for $n,m,i \geqslant 0$. Note that by our convention, $\alpha^i_{n,m} = 0$ when either $i > n$ or $i > m$.

Now we compute

$$\begin{aligned}
\langle \xi_{r,s}, \xi_{r',s'}\rangle &= \langle c(e)^r c_{\mathrm{r}}(e)^s \xi, c(e)^{r'} c_{\mathrm{r}}(e)^{s'} \xi\rangle = \langle a(e)^{r'} c(e)^r c_{\mathrm{r}}(e)^s \xi, c_{\mathrm{r}}(e)^{s'} \xi\rangle \\
&\overset{(2.4)}{=} \Big\langle \sum_{i=0}^{r'} q^{(r-i)(r'-i)} \alpha^i_{r,r'} c(e)^{r-i} a(e)^{r'-i} c_{\mathrm{r}}(e)^s \xi, c_{\mathrm{r}}(e)^{s'} \xi \Big\rangle \\
&\overset{(2.5)}{=} \Big\langle \sum_{i=0}^{r'} q^{(r-i)(r'-i)} \alpha^i_{r,r'} c(e)^{r-i} \sum_{j=0}^{r'-i} \alpha^j_{r'-i,s} c_{\mathrm{r}}(e)^{s-j} W^j a(e)^{r'-i-j} \xi, c_{\mathrm{r}}(e)^{s'} \xi \Big\rangle \\
&\overset{(*)}{=} \Big\langle \sum_{i=0}^{r'} q^{(r-i)(r'-i)} \alpha^i_{r,r'} c(e)^{r-i} \alpha^{r'-i}_{r'-i,s} c_{\mathrm{r}}(e)^{s-(r'-i)} W^{r'-i} \xi, c_{\mathrm{r}}(e)^{s'} \xi \Big\rangle \\
&= \sum_{i=0}^{r'} q^{(r-i)(r'-i)} q^{(r'-i)k} \alpha^i_{r,r'} \alpha^{r'-i}_{r'-i,s} \langle c_{\mathrm{r}}(e)^{s-(r'-i)} \xi, a(e)^{r-i} c_{\mathrm{r}}(e)^{s'} \xi\rangle \\
&\overset{(2.5)}{=} \sum_{i=0}^{r'} \Big[q^{(r-i)(r'-i)} q^{(r'-i)k} \alpha^i_{r,r'} \alpha^{r'-i}_{r'-i,s}
\end{aligned}$$

$$\begin{aligned}
&\times \left\langle c_{\mathrm{r}}(e)^{s-(r'-i)}\xi, \sum_{j=0}^{r-i} \alpha_{r-i,s'}^{j} c_{\mathrm{r}}(e)^{s'-j} W^{j} a(e)^{r-i-j}\xi \right\rangle \Big] \\
&\overset{(*)}{=} \sum_{i=0}^{r'} [q^{(r-i)(r'-i)} q^{(r'-i)k} q^{(r-i)k} \alpha_{r,r'}^{i} \alpha_{r'-i,s}^{r'-i} \alpha_{r-i,s'}^{r-i} \\
&\quad \times \langle c_{\mathrm{r}}(e)^{s-(r'-i)}\xi, c_{\mathrm{r}}(e)^{s'-(r-i)}\xi \rangle] \\
&= \sum_{i=0}^{r'} q^{(r-i)(r'-i)} q^{(r'-i)k} q^{(r-i)k} \alpha_{r,r'}^{i} \alpha_{r'-i,s}^{r'-i} \alpha_{r-i,s'}^{r-i} [s-r'+i]_q! \\
&= \sum_{i=0}^{r'} q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q,
\end{aligned}$$

where in the equations with $(*)$ we used the fact that $a(e)\xi = 0$. ∎

For later use, we define two constants depending on q:

$$C(q) := \prod_{i=1}^{\infty} \frac{1}{1-q^i}, \quad D(q) := \prod_{i=1}^{\infty} (1+|q|^i).$$

Basic calculus shows that whenever $-1 < q < 1$, the above two limits exist unconditionally.

We record a simple but very useful estimate here for later references.

LEMMA 3.8. *For all $-1 < q < 1$ and for all $n, m \geqslant 0$, we have*

$$\left| \binom{n}{m}_q \right|^{\pm 1} \leqslant D(q) C(|q|).$$

LEMMA 3.9. *Let r, s, r', s', k be non-negative integers with $r+s = r'+s'$, $r \geqslant r'$ and $\xi \in T^k$ of norm 1. Then for each $-1 < q < 1$, there are constants $E(q), F(q)$ such that*

$$E(q)|q|^{k(r-r')}[r+s]_q! \leqslant |\langle \xi_{r,s}, \xi_{r',s'} \rangle| \leqslant F(q)|q|^{k(r-r')}[r+s]_q!. \tag{3.10}$$

Moreover, we have that

$$\lim_{q \to 0} E(q) = \lim_{q \to 0} F(q) = 1.$$

Proof. When $0 \leqslant q < 1$, we have

$$\begin{aligned}
\langle \xi_{r,s}, \xi_{r',s'} \rangle &= \sum_{i=0}^{r'} q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \\
&\leqslant q^{k(r-r')} \sum_{i=0}^{r'} q^{(r-i)(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \\
&\overset{(2.2)}{=} q^{k(r-r')} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r+s}{r'}_q \\
&= q^{k(r-r')}[r+s]_q!,
\end{aligned}$$

where the last equality comes from the assumption $r+s=r'+s'$. This proves the inequality on the right side.

For the inequality on the left side, simply note that

$$\begin{aligned}
&\sum_{i=0}^{r'} q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \\
&\quad \geqslant \sum_{i=r'}^{r'} q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \\
&\quad = q^{k(r-r')}[r']_q![s']_q! \cdot \binom{r}{r'}_q = q^{k(r-r')}[r']_q![s']_q! \cdot \frac{[r]_q!}{[r']_q![r-r']_q!} \cdot \frac{[r+s]_q!}{[r+s]_q!} \\
&\quad = q^{k(r-r')}[r+s]_q! \cdot \frac{[s']_q![r]_q!}{[r-r']_q![r+s]_q!} \\
&\quad = q^{k(r-r')}[r+s]_q! \cdot \frac{(1-q^{r-r'+1})\cdots(1-q^r)}{(1-q^{s'+1})\cdots(1-q^{r+s})} \\
&\quad \geqslant q^{k(r-r')}[r+s]_q! \cdot (1-q^{r-r'+1})\cdots(1-q^r) \geqslant \frac{1}{C(q)} q^{k(r-r')}[r+s]_q!.
\end{aligned}$$

Thus if we let $E(q)=\frac{1}{C(q)}, F(q)=1$, we are done.

Now assume $-1<q<0$. By Lemma 3.8 we have

$$[r']_q![s']_q! \leqslant [r'+s']_q! D(q)C(|q|).$$

Therefore,

$$\begin{aligned}
&\Big| \sum_{i=0}^{r'} q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \Big| \\
&\qquad \leqslant D(q)^3 C(|q|)^3 [r'+s']_q! \sum_{i=0}^{r'} |q|^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \\
&\qquad \leqslant D(q)^3 C(|q|)^3 [r'+s']_q! \cdot |q|^{k(r-r')} \cdot \frac{1}{1-|q|}.
\end{aligned}$$

Meanwhile, we have

$$\begin{aligned}
&\Big| \sum_{i=r'}^{r'} q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \Big| \\
&\quad = |q|^{k(r-r')}[r']_q! \cdot [s']_q! \cdot \binom{r}{r'}_q = |q|^{k(r-r')}[r+s]_q! \cdot \frac{(1-q^{r-r'+1})\cdots(1-q^r)}{(1-q^{s'+1})\cdots(1-q^{r+s})} \\
&\quad \geqslant |q|^{k(r-r')}[r+s]_q! \cdot \frac{(1-|q|^{r-r'+1})\cdots(1-|q|^r)}{(1+|q|^{s'+1})\cdots(1+|q|^{r+s})} \geqslant \frac{|q|^{k(r-r')}}{D(q)C(|q|)} \cdot [r+s]_q!,
\end{aligned}$$

and

$$\begin{aligned}
\Big| \sum_{i=0}^{r'-1} & q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \Big| \\
&\leqslant \sum_{i=0}^{r'-1} |q|^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \\
&\leqslant |q|^{k(r-r')} \sum_{i=0}^{r'-1} |q|^{(r-i)} [r+s]_q! D(q)^3 C(|q|)^3 \\
&\leqslant |q|^{k(r-r')} \frac{|q|}{1-|q|} D(q)^3 C(|q|)^3 [r+s]_q!.
\end{aligned}$$

Hence by the triangle inequality,

$$\begin{aligned}
\Big| \sum_{i=0}^{r'} & q^{(r-i)(r'-i)+k(r-i)+k(r'-i)} \cdot [r']_q! \cdot [s']_q! \cdot \binom{r}{i}_q \cdot \binom{s}{r'-i}_q \Big| \\
&\geqslant \frac{|q|^{k(r-r')}}{D(q)C(|q|)} \cdot [r+s]_q! - \frac{|q|^{k(r-r')+1}}{1-|q|} D(q)^3 C(|q|)^3 [r+s]_q! \\
&= |q|^{k(r-r')} \Big(\frac{1}{D(q)C(|q|)} - \frac{|q|}{1-|q|} D(q)^3 C(|q|)^3 \Big) \cdot [r+s]_q!.
\end{aligned}$$

Finally, if we let $E(q) = \frac{1}{D(q)C(|q|)} - \frac{|q|}{1-|q|} D(q)^3 C(|q|)^3$, $F(q) = \frac{D(q)^3 C(|q|)^3}{1-|q|}$, the proof is complete. ∎

The following corollary will be used multiple times later.

COROLLARY 3.10. *Let r, s, k be non-negative integers and let $\xi \in T^k$ be of norm 1. Then there is a positive number $\alpha > 0$, such that whenever $|q| \leqslant \alpha$, we have*

$$\frac{1}{2}[r+s]_q! \leqslant \|\xi_{r,s}\|_q^2 \leqslant 2[r+s]_q!. \tag{3.11}$$

REMARK 3.11. Lemma 3.9 and Corollary 3.10 hold when $-\frac{1}{7} < q < \frac{1}{4}$.

To prove the main theorem of this section, we need another lemma.

LEMMA 3.12. *Let $\alpha \in \mathbb{R}$ with $|\alpha| < 1$. For any $n \in \mathbb{N}$ we define*

$$E_\alpha = \begin{pmatrix} 0 & -\alpha & \cdots & -\alpha^{n-1} \\ -\alpha & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\alpha \\ -\alpha^{n-1} & \cdots & -\alpha & 0 \end{pmatrix},$$

then the operator norm $\|E_\alpha\| \leqslant \frac{2|\alpha|}{1-|\alpha|}$.

Proof. Clearly we have

$$\left\| \begin{pmatrix} 0 & -\alpha^k & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & -\alpha^k \\ \mathbf{0} & & & 0 \end{pmatrix} \right\| = |\alpha|^k.$$

Hence $\|E_\alpha\| \leqslant 2 \sum\limits_{i=1}^{n-1} |\alpha|^i \leqslant \frac{2|\alpha|}{1-|\alpha|}$. ∎

Take an orthonormal basis $\{\xi_j^i : j \in I_k\}$ for $T_k, k \geqslant 1$. We may re-order the set $\bigcup\limits_{i\in\mathbb{N}} \{\xi_j^i : j \in I_k\}$ as $\{\xi^i : i \in I\}$ for some index set I and we set that $\xi^0 = \Omega$.

Finally we are ready to state and prove the main result of this section.

THEOREM 3.13. *For any real number q satisfying $|q| < \frac{1}{9}$, the set*

$$\left\{ \frac{\xi_{r,s}^i}{\|\xi_{r,s}^i\|} : i \in I, r, s \geqslant 0 \right\}$$

forms a Riesz basis for $L^2(M) \ominus L^2(A)$, i.e., its linear span is dense in $L^2(M) \ominus L^2(A)$ and there exists some constants $A_q, B_q > 0$, such that for all $\lambda_{r,s}^i \in \mathbb{C}$, one has

$$A_q \sum_{r,s,i} |\lambda_{r,s}^i|^2 \|\xi_{r,s}^i\|^2 \leqslant \Big\| \sum_{r,s,i} \lambda_{r,s}^i \xi_{r,s}^i \Big\|^2 \leqslant B_q \sum_{r,s,i} |\lambda_{r,s}^i|^2 \|\xi_{r,s}^i\|^2. \tag{3.12}$$

Proof. By Lemma 3.6, it suffices to find such $A_q, B_q > 0$ which are independent of $i \in I$ and $k \geqslant 0$ such that

$$A_q \sum_{r+s=k} |\lambda_{r,s}^i|^2 \|\xi_{r,s}^i\|^2 \leqslant \Big\| \sum_{r+s=k} \lambda_{r,s}^i \xi_{r,s}^i \Big\|^2 \leqslant B_q \sum_{r+s=k} |\lambda_{r,s}^i|^2 \|\xi_{r,s}^i\|^2$$

holds for any $\lambda_{r,s}^i \in \mathbb{C}$. Fixing $\xi = \xi^i \in T^t$ for some $t \in \mathbb{N}$, for simplicity, we will omit the superscript i in the rest of the proof. Fix $0 < \varepsilon < \frac{1}{7}$. As explained in Remark 3.11, both Lemma 3.9 and Corollary 3.10 hold. Then for all q with $|q| \leqslant \varepsilon$,

$$\begin{aligned}
\Big\| \sum_{r+s=k} \lambda_{r,s} \xi_{r,s} \Big\|^2 &\geqslant \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 - \sum_{r+s=r'+s'=k,\, r\neq r'} |\lambda_{r,s}\lambda_{r',s'}| \cdot |\langle \xi_{r,s}, \xi_{r',s'} \rangle| \\
&\overset{(3.10)}{\geqslant} \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 - 2 \sum_{r+s=r'+s'=k,\, r\neq r'} |\lambda_{r,s}\lambda_{r',s'}| \cdot |q|^{t|r-r'|} [k]_q! \\
&\geqslant \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 - 2 \sum_{\substack{r+s=r'+s'=k \\ r\neq r'}} |\lambda_{r,s}\lambda_{r',s'}| \cdot |q|^{|r-r'|} [k]_q! \\
&\overset{(3.11)}{\geqslant} \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 - 4 \sum_{r+s=k,\, r'+s'=k,\, r\neq r'} |\lambda_{r,s}\lambda_{r',s'}| \cdot |q|^{|r-r'|} \|\xi_{r,s}\| \|\xi_{r',s'}\|
\end{aligned}$$

$$= \left\langle (1+4E_{|q|}) \begin{pmatrix} |\lambda_{k,0}|\|\xi_{k,0}\| \\ \vdots \\ |\lambda_{0,k}|\|\xi_{0,k}\| \end{pmatrix}, \begin{pmatrix} |\lambda_{k,0}|\|\xi_{k,0}\| \\ \vdots \\ |\lambda_{0,k}|\|\xi_{0,k}\| \end{pmatrix} \right\rangle,$$

where $E_{|q|}$ is the matrix defined in the previous lemma. When $|q| < \frac{1}{9}$, $1+4E_{|q|}$ is strictly positive since we have that $1+4E_{|q|} \geqslant 1-4\|E_{|q|}\| \geqslant 1-4\cdot\left(\frac{2|q|}{1-|q|}\right) > 0$. Also, notice that the strict positivity of $1+4E_{|q|}$ depends on neither i nor k. This shows the existence of $A_q > 0$ satisfying the first half of (3.12).

Similarly,

$$\begin{aligned}
\Big\| \sum_{r+s=k} \lambda_{r,s}\xi_{r,s} \Big\|^2 &\leqslant \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 + \sum_{r+s=r'+s'=k,\ r\neq r'} |\lambda_{r,s}\lambda_{r',s'}| \cdot |\langle \xi_{r,s}, \xi_{r',s'}\rangle| \\
&\overset{(3.10)}{\leqslant} \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 + 2 \sum_{r+s=r'+s'=k, r\neq r'} |\lambda_{r,s}\lambda_{r',s'}| \cdot |q|^{t|r-r'|}[k]_q! \\
&\leqslant \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 + 2 \sum_{r+s=r'+s'=k, r\neq r'} |\lambda_{r,s}\lambda_{r',s'}| \cdot |q|^{|r-r'|}[k]_q! \\
&\overset{(3.11)}{\leqslant} \sum_{r+s=k} |\lambda_{r,s}|^2 \|\xi_{r,s}\|^2 + 4 \sum_{r+s=k, r'+s'=k, r\neq r'} |\lambda_{r,s}\lambda_{r',s'}| \cdot |q|^{|r-r'|} \|\xi_{r,s}\| \|\xi_{r',s'}\| \\
&= \left\langle (1-4E_{|q|}) \begin{pmatrix} |\lambda_{k,0}|\|\xi_{k,0}\| \\ \vdots \\ |\lambda_{0,k}|\|\xi_{0,k}\| \end{pmatrix}, \begin{pmatrix} |\lambda_{k,0}|\|\xi_{k,0}\| \\ \vdots \\ |\lambda_{0,k}|\|\xi_{0,k}\| \end{pmatrix} \right\rangle,
\end{aligned}$$

and the existence of B_q is obvious.

The completeness is already shown in Lemma 3.4, therefore we are done. ∎

REMARK 3.14. Recall that $\xi^0 = \Omega$. If we consider

$$\left\{ \frac{\xi^0_{r,0}}{\|\xi^0_{r,0}\|} : r \geqslant 0 \right\} \cup \left\{ \frac{\xi^i_{r,s}}{\|\xi^i_{r,s}\|} : i \in I, r, s \geqslant 0 \right\},$$

then this is a Riesz basis of the entire q-Fock space $L^2(M)$.

4. LOCATING THE SUPPORTS OF ELEMENTS IN THE RELATIVE COMMUTANT

Throughout this section we will assume that q is a real number with $|q| < \frac{1}{9}$ such that the conclusions in Corollary 3.10 and Theorem 3.13 hold.

For $N \geqslant 0$, define the idempotent $L_N, R_N : L^2(M) \to L^2(M)$ by $L_N|_{\mathcal{F}_q(\mathbb{R}e)} = R_N|_{\mathcal{F}_q(\mathbb{R}e)} = 0$ and

$$\begin{aligned} L_N\Big(\sum_{i\in I, r,s\geqslant 0} c^i_{r,s}\xi^i_{r,s}\Big) &= \sum_{i\in I, s\geqslant 0, 0\leqslant r\leqslant N} c^i_{r,s}\xi^i_{r,s}, \\ R_N\Big(\sum_{i\in I, r,s\geqslant 0} c^i_{r,s}\xi^i_{r,s}\Big) &= \sum_{i\in I, r\geqslant 0, 0\leqslant s\leqslant N} c^i_{r,s}\xi^i_{r,s}. \end{aligned} \tag{4.1}$$

By Theorem 3.13, L_N and R_N are both well-defined. Moreover, with a little abuse of notation, sometimes we will also use L_N(respectively R_N) to denote the image of L_N(respectively R_N).

Let $C \subset A$ be a diffuse subalgebra and fix a free ultrafilter $\omega \in \beta(\mathbb{N})\backslash\mathbb{N}$. Let $z = (z_n)_n \in (M^\omega \ominus A^\omega) \cap C'$. Without loss of generality we assume that $\|z\| = 1$ and $\|z_n\| \leqslant 1$ (the operator norm is bounded above by 1) and $z_n \in M \ominus A, \forall n$. Just as in [20], we would like to show that the support of z eventually escapes both L_N and R_N. To this end, we need some preparations.

The first key step towards our goal is to show that L_N is asymptotically right-A modular.

Recall that Q_k is the orthogonal projection from $\mathcal{F}_q(\mathcal{H}_\mathbb{R})$ onto $\mathcal{H}^{\otimes k}$.

LEMMA 4.1. *For any $k \in \mathbb{N}$, we have*

$$\lim_{n\to\omega} Q_k(z_n) = 0.$$

Proof. Suppose this is not the case, then there exists some $k \in \mathbb{N}$ with

$$\mathcal{H}^{\otimes k} \ni z_0 = \lim_{n\to\omega} Q_k(z_n) \neq 0.$$

In particular, $z_n \to y$ weakly for some non-zero $y \in M$. Clearly $y \in C' \cap M \ominus A$.

However, it is known that the generator masa A is *mixing* in M (see [2], [21]). Thus by Proposition 5.1 in [7] we must have that $C' \cap M = A$, a contradiction. ∎

The next estimate will be essential in order to establish the right-A modularity of L_N.

LEMMA 4.2. *Let $x \in L^2(M) \ominus L^2(A)$ whose Fourier expansion along $\{\xi^i_{r,s} : i \in I, r, s \geqslant 0\}$ is of the form*

$$x = \sum_{r\geqslant N+1, s\geqslant 0} \lambda_{r,s}\xi_{r,s},$$

where $\xi = \xi^i$ for some fixed $i \in I$ with $\xi \in T^t$. Then we have

$$\|L_N(a_{\mathrm{r}}(e)^k x)\|_2^2 \leqslant \frac{4kB_q C(|q|)^3 D(q)^6}{(1-q)^k(1-q^{2t})} \sum_{r,s\geqslant 0} q^{2(t+s+r-k-N-1)} |\lambda_{r,s}|^2 \|\xi_{r,s}\|_2^2, \tag{4.2}$$

for all $k, N \geqslant 0$.

Proof. We let $\lambda_{r,s} = 0$ for all $r \leqslant N$ and $s \geqslant 0$. By (3.7), we have

$$\begin{aligned}
&L_N\Big(a_{\mathrm{r}}(e)^k \sum_{r \geqslant N+1, s\geqslant 0} \lambda_{r,s}\xi_{r,s}\Big)\\
&\quad = L_N\Big(\sum_{r\geqslant N+1, s\geqslant 0} \lambda_{r,s} \sum_{i+j=k, i,j\geqslant 0} \frac{[r]_q!}{[r-i]_q!}\cdot\frac{[s]_q!}{[s-j]_q!}\cdot\binom{k}{i}_q q^{(t+s-j)i}\xi_{r-i,s-j}\Big)\\
&\quad = \sum_{r\leqslant N, s\geqslant 0}\xi_{r,s}\sum_{i+j=k, r+i\geqslant N+1, j\geqslant 0}\lambda_{r+i,s+j}\frac{[r+i]_q!}{[r]_q!}\cdot\frac{[s+j]_q!}{[s]_q!}\cdot\binom{k}{i}_q q^{(t+s)i}.
\end{aligned}$$

Note that for all $-1 < q < 1$,

- (•) $\frac{[r+i]_q!}{[r]_q!}\cdot\frac{[s+j]_q!}{[s]_q!} \leqslant \frac{D(q)^2}{(1-q)^k}$, $\forall i+j=k$;
- (•) $\binom{k}{i}_q \leqslant D(q)C(|q|)$.

Therefore, for each $r \leqslant N$,

$$\begin{aligned}
&\Big|\sum_{i+j=k, r+i\geqslant N+1, j\geqslant 0}\lambda_{r+i,s+j}\frac{[r+i]_q!}{[r]_q!}\cdot\frac{[s+j]_q!}{[s]_q!}\cdot\binom{k}{i}_q q^{(t+s)i}\Big|^2\\
&\leqslant \Big|\sum_{i+j=k, r+i\geqslant N+1, j\geqslant 0}|\lambda_{r+i,s+j}|\|\xi_{r+i,s+j}\|_2\cdot\frac{1}{\|\xi_{r+i,s+j}\|_2}\frac{C(|q|)D(q)^3}{(1-q)^k}|q|^{(t+s)i}\Big|^2\\
&\leqslant \Big(\sum_{i+j=k, r+i\geqslant N+1, j\geqslant 0}|\lambda_{r+i,s+j}|^2\|\xi_{r+i,s+j}\|_2^2\Big)\\
&\qquad\cdot\Big(\sum_{i+j=k, r+i\geqslant N+1, j\geqslant 0}\frac{1}{\|\xi_{r+i,s+j}\|_2^2}\frac{C(|q|)^2D(q)^6}{(1-q)^{2k}}q^{2(t+s)i}\Big)\\
&\overset{(3.11)}{\leqslant}\Big(\sum_{i+j=k, r+i\geqslant N+1, j\geqslant 0}|\lambda_{r+i,s+j}|^2\|\xi_{r+i,s+j}\|_2^2\Big)\\
&\qquad\times\frac{2}{[r+s+k]_q!}\cdot\frac{C(|q|)^2D(q)^6}{(1-q)^{2k}}\sum_{i+j=k, r+i\geqslant N+1, j\geqslant 0}q^{2(t+s)i}\\
&\leqslant\Big(\sum_{i+j=k, r+i\geqslant N+1, j\geqslant 0}|\lambda_{r+i,s+j}|^2\|\xi_{r+i,s+j}\|_2^2\Big)\frac{2}{[r+s+k]_q!}\cdot\frac{C(|q|)^2D(q)^6}{(1-q)^{2k}}\\
&\qquad\cdot\frac{q^{2(t+s)(N+1-r)}}{1-q^{2(t+s)}}\\
&\leqslant\Big(\sum_{i+j=k, r+i\geqslant N+1, j\geqslant 0}|\lambda_{r+i,s+j}|^2\|\xi_{r+i,s+j}\|_2^2\Big)\frac{2C(|q|)^2D(q)^6}{(1-q)^{2k}(1-q^2)}\cdot\frac{q^{2(t+s+r-N-1)}}{[r+s+k]_q!},
\end{aligned}$$

where in the last inequality we used the fact that $ab \geqslant a-b$ for all $a, b \geqslant 1$.

Finally, we have

$$\|L_N(a_{\mathrm{r}}(e)^k x)\|_2^2$$

$$
\begin{aligned}
&= \Big\| \sum_{r\leqslant N, s\geqslant 0} \xi_{r,s} \sum_{i+j=k, r+i\geqslant N+1, j\geqslant 0} \lambda_{r+i,s+j} \frac{[r+i]_q!}{[r]_q!}\cdot\frac{[s+j]_q!}{[s]_q!}\cdot\binom{k}{i}_q q^{(t+s)i}\Big\|_2^2 \\
&\leqslant B_q \sum_{r\leqslant N, s\geqslant 0} \|\xi_{r,s}\|_2^2 \Big| \sum_{i+j=k, r+i\geqslant N+1, j\geqslant 0} \lambda_{r+i,s+j} \frac{[r+i]_q!}{[r]_q!}\cdot\frac{[s+j]_q!}{[s]_q!}\cdot\binom{k}{i}_q q^{(t+s)i}\Big|^2 \\
&\leqslant B_q \sum_{r\leqslant N, s\geqslant 0} \Big[\|\xi_{r,s}\|_2^2 \cdot \Big(\sum_{i+j=k, r+i\geqslant N+1, j\geqslant 0} |\lambda_{r+i,s+j}|^2 \|\xi_{r+i,s+j}\|_2^2\Big) \\
&\qquad \times \frac{2C(|q|)^2 D(q)^6}{(1-q)^{2k}(1-q^2)} \cdot \frac{q^{2(t+s+r-N-1)}}{[r+s+k]_q!}\Big] \\
&\overset{(3.11)}{\leqslant} 2B_q \sum_{r\leqslant N, s\geqslant 0} \Big[[r+s]_q! \Big(\sum_{i+j=k, r+i\geqslant N+1, j\geqslant 0} |\lambda_{r+i,s+j}|^2 \|\xi_{r+i,s+j}\|_2^2\Big) \\
&\qquad \times \frac{2C(|q|)^2 D(q)^6}{(1-q)^{2k}(1-q^2)} \cdot \frac{q^{2(t+s+r-N-1)}}{[r+s+k]_q!}\Big] \\
&\leqslant 4B_q \sum_{r\leqslant N, s\geqslant 0} \Big[\Big(\sum_{i+j=k, r+i\geqslant N+1, j\geqslant 0} |\lambda_{r+i,s+j}|^2 \|\xi_{r+i,s+j}\|_2^2\Big) \\
&\qquad \times \frac{C(|q|)^3 D(q)^6}{(1-q)^{k}(1-q^2)} \cdot q^{2(t+s+r-N-1)}\Big] \\
&\leqslant \frac{4kB_q C(|q|)^3 D(q)^6}{(1-q)^k(1-q^2)} \sum_{r,s\geqslant 0} q^{2(t+s+r-k-N-1)} |\lambda_{r,s}|^2 \|\xi_{r,s}\|_2^2,
\end{aligned}
$$

where the second last inequality is due to the fact that

$$\frac{[r+s]_q!}{[r+s+k]_q!} \leqslant (1-q)^k C(|q|). \quad \blacksquare$$

PROPOSITION 4.3. *For all $N, m \in \mathbb{N}$ and for any $x = (x_n) \in (L^2(M \ominus A))^\omega$ such that $\lim_{n\to\omega} Q_k(x_n) \to 0, \forall k \in \mathbb{N}$, we have*

$$\lim_{n\to\omega} \|L_N(s_{\mathrm{r}}(e^{\otimes m})x_n) - s_{\mathrm{r}}(e^{\otimes m})L_N(x_n)\|_2 = 0.$$

In particular, for all unitary u in the C^*-algebra $C^*(s(e))$ generated by $s(e)$, $N \in \mathbb{N}$ and $(z_n) \in M^\omega \ominus A^\omega \cap C'$,

$$\lim_{n\to\omega} \|L_N(z_n u) - L_N(z_n)u\|_2 = 0.$$

Proof. Each $s_{\mathrm{r}}(e^{\otimes m})$ can be written as a finite linear combination of products of the form $c_{\mathrm{r}}(e)^k a_{\mathrm{r}}(e)^l$ with $k, l \geqslant 0$ and $k + l = m$, hence it suffices to show

$$\lim_{n\to\omega} \|L_N(c_{\mathrm{r}}(e)^k a_{\mathrm{r}}(e)^l x_n) - c_{\mathrm{r}}(e)^k a_{\mathrm{r}}(e)^l L_N(x_n)\|_2 = 0,$$

for all $k, l \geqslant 0, k + l = m$.

Since $c_{\mathrm{r}}(e)L_N(\xi) = L_N(c_{\mathrm{r}}(e)\xi)$ for all $\xi \in L^2(M \ominus A)$, it then reduces to prove

$$\lim_{n\to\omega} \|L_N(a_{\mathrm{r}}(e)^k x_n) - a_{\mathrm{r}}(e)^k L_N(x_n)\|_2 = 0,$$

for all $0 \leqslant k \leqslant m$.

Suppose that $x_n = \sum_{i\in I, r,s\geqslant 0} \lambda^{i,n}_{r,s}\xi^i_{r,s}$ is the Fourier decomposition along the Riesz basis $\{\xi^i_{r,s}\}$, observe that

$$\begin{aligned}
L_N(a_{\mathrm{r}}(e)^k x_n) - a_{\mathrm{r}}(e)^k L_N(x_n) &= L_N(a_{\mathrm{r}}(e)^k x_n) - L_N(a_{\mathrm{r}}(e)^k L_N(x_n)) \\
&= L_N(a_{\mathrm{r}}(e)^k (1 - L_N)(x_n)) = L_N\Big(a_{\mathrm{r}}(e)^k \sum_{i\in I, r\geqslant N+1, s\geqslant 0} \lambda^{i,n}_{r,s}\xi^i_{r,s}\Big) \\
&= \sum_{i\in I} L_N\Big(a_{\mathrm{r}}(e)^k \sum_{r\geqslant N+1, s\geqslant 0} \lambda^{i,n}_{r,s}\xi^i_{r,s}\Big).
\end{aligned}$$

By Lemma 4.2, we have

$$\begin{aligned}
&\|L_N(a_{\mathrm{r}}(e)^k x_n) - a_{\mathrm{r}}(e)^k L_N(x_n)\|_2^2 \\
&\quad = \sum_{i\in I} \Big\| L_N\Big(a_{\mathrm{r}}(e)^k \sum_{r\geqslant N+1, s\geqslant 0} \lambda^{i,n}_{r,s}\xi^i_{r,s}\Big)\Big\|_2^2 \\
&\quad \leqslant \sum_{i\in I} \frac{4kB_q C(|q|)^3 D(q)^6}{(1-q)^k(1-q^2)} \sum_{r,s\geqslant 0} q^{2(|\xi^i|+s+r-k-N-1)} |\lambda^{i,n}_{r,s}|^2 \|\xi^i_{r,s}\|_2^2.
\end{aligned}$$

We may assume that for each n, there exists a natural number t_n such that: (1) $t_n \to \infty$ as $n \to \omega$ and (2) $\lambda^{i,n}_{r,s} = 0$ for any i, r, s with $|\xi_i| + r + s \leqslant t_n + N + 1$. Thus

$$\begin{aligned}
\|L_N(a_{\mathrm{r}}(e)^k x_n) - a_{\mathrm{r}}(e)^k L_N(x_n)\|_2^2 &\leqslant \frac{4kB_q C(|q|)^3 D(q)^6}{(1-q)^k(1-q^2)} \cdot q^{2(t_n-k)} \sum_{i,r,s} |\lambda^{i,n}_{r,s}|^2 \|\xi^i_{r,s}\|_2^2 \\
&\leqslant \frac{4kB_q C(|q|)^3 D(q)^6}{(1-q)^k(1-q^2)A_q} \cdot q^{2(t_n-k)} \|x_n\|_2^2.
\end{aligned}$$

As $n \to \omega$, t_n diverges to infinity. Therefore the right-hand side of the above inequality converges to 0 (uniformly on the unit ball of $L^2(M^\omega \ominus A^\omega)$).

The case for $u \in C^*(s(e))$ and $z = (z_n) \in M^\omega \ominus A^\omega \cap C'$ is an easy consequence of Lemma 4.1 and the fact that $\{s(e^{\otimes n}) : n \geqslant 0\}$ spans norm-densely in $C^*(s(e))$. ∎

Our next step is to show that for any $x \in M \ominus A$ and for all sequence of unitary elements $(u_k)_k$ in $C^*(s(e))$ which goes to 0 weakly, $(u_k L_N(x))_k$ is asymptotically orthogonal to the subspace L_N.

PROPOSITION 4.4. *There exists a positive number $G(q) > 0$ such that*

$$\|L_{N_1}(s(e^{\otimes n}) L_{N_2}(x))\|_2 \leqslant G(q) \cdot (n+1)^{3/2} \cdot |q|^{(n-N_1-N_2)} \|x\|_2 \tag{4.3}$$

for any $N_1, N_2 \in \mathbb{N}$ and $x \in M \ominus A$. The choice of $G(q)$ is independent of N_1, N_2, x.

Proof. By the Wick formula (1.5), one has

$$L_{N_1}(s(e^{\otimes n})L_{N_2}(x)) = \sum_{k=0}^{n} \binom{n}{k}_q L_{N_1}(c(e)^k a(e)^{n-k} L_{N_2}(x)).$$

Also, notice that in the above summation, only the terms with $0 \leqslant k \leqslant N_1$ will be able to contribute something non-zero.

Let us estimate each summand for $0 \leqslant k \leqslant N_1$. Suppose $x = \sum\limits_i \sum\limits_{r,s} \lambda^i_{r,s} \xi^i_{r,s}$ be the Fourier decomposition along $\{\xi^i_{r,s}\}$, then $L_{N_1}(c(e)^k a(e)^{n-k} L_{N_2}(x))$ can be decomposed as a sum over index i. Each summand is of the form:

$$L_{N_1}\Big(c(e)^k \sum_{r\leqslant N_2, s\geqslant 0} \lambda^i_{r,s} \sum_{j_1+j_2=n-k,\, j_2\leqslant r} \frac{[r]_q!}{[r-j_2]_q!} \frac{[s]_q!}{[s-j_1]_q!} \binom{n-k}{j_1}_q q^{(|\xi^i|+r-j_2)j_1} \xi^i_{r-j_2,s-j_1}\Big)$$

$$= L_{N_1}\Big(\sum_{r\leqslant N_2, s\geqslant 0} \lambda^i_{r,s} \sum_{j_1+j_2=n-k,\, j_2\leqslant r} \frac{[r]_q!}{[r-j_2]_q!} \frac{[s]_q!}{[s-j_1]_q!} \binom{n-k}{j_1}_q$$
$$\cdot\, q^{(|\xi^i|+r-j_2)j_1} \xi^i_{r+k-j_2,s-j_1}\Big)$$
$$= \sum_{k\leqslant r\leqslant N_1, s\geqslant 0} \Big\{ \xi^i_{r,s} \sum_{j_1+j_2=n-k, j_1\geqslant n-2k+r-N_2} \Big[\lambda^i_{r+j_2-k,s+j_1} \times \frac{[r+j_2-k]_q!}{[r-k]_q!}$$
$$\cdot\, \frac{[s+j_1]_q!}{[s]_q!} \cdot \binom{n-k}{j_1}_q q^{(|\xi^i|+r-k)j_1} \Big] \Big\}.$$

Now for $k \leqslant r \leqslant N_1$,

$$\Big| \sum_{j_1+j_2=n-k,\, j_1\geqslant n-2k+r-N_2} \lambda^i_{r+j_2-k,s+j_1} \frac{[r+j_2-k]_q!}{[r-k]_q!} \frac{[s+j_1]_q!}{[s]_q!} \cdot \binom{n-k}{j_1}_q q^{(|\xi^i|+r-k)j_1} \Big|^2$$
$$\leqslant \sum_{j_1+j_2=n-k,\, j_1\geqslant n-2k+r-N_2} |\lambda^i_{r+j_2-k,s+j_1}| \cdot \frac{D(q)^2}{(1-q)^{n-k}} \cdot C(|q|)D(q) \cdot |q|^{j_1}$$
$$\leqslant \Big(\sum_{j_1+j_2=n-k,\, j_1\geqslant n-2k+r-N_2} |\lambda^i_{r+j_2-k,s+j_1}|^2 \|\xi_{r+j_2-k,s+j_1}\|_2^2 \Big)$$
$$\times \Big(\sum_{j_1+j_2=n-k,\, j_1\geqslant n-2k+r-N_2} \frac{1}{\|\xi_{r+j_2-k,s+j_1}\|_2^2} \frac{C(|q|)^2 D(q)^6}{(1-q)^{2(n-k)}} q^{2j_1} \Big)$$
$$\leqslant \Big(\sum_{j_1+j_2=n-k,\, j_1\geqslant n-2k+r-N_2} |\lambda^i_{r+j_2-k,s+j_1}|^2 \|\xi_{r+j_2-k,s+j_1}\|_2^2 \Big)$$
$$\times \frac{2}{[r+s+n-2k]_q!} \cdot \frac{C(|q|)^2 D(q)^6}{(1-q)^{2(n-k)}} \cdot \frac{q^{2(n-N_1-N_2)}}{1-q^2}$$
$$\leqslant \Big(\sum_{j_1+j_2=n-k,\, j_1\geqslant n-2k+r-N_2} |\lambda^i_{r+j_2-k,s+j_1}|^2 \|\xi_{r+j_2-k,s+j_1}\|_2^2 \Big)$$

$$\times \frac{2C(|q|)^2 D(q)^6}{(1-q)^{2(n-k)}(1-q^2)[r+s+n-2k]_q!} \cdot q^{2(n-N_1-N_2)}.$$

Hence

$$\|L_{N_1}(c(e)^k a(e)^{n-k} L_{N_2}(x))\|_2^2$$
$$= \sum_i \Big\| \sum_{k\leqslant r\leqslant N_1,\, s\geqslant 0} \Big[\xi^i_{r,s} \sum_{j_1+j_2=n-k,\, j_1\geqslant n-2k+r-N_2} \lambda^i_{r+j_2-k,s+j_1} \frac{[r+j_2-k]_q!}{[r-k]_q!}\frac{[s+j_1]_q!}{[s]_q!}$$
$$\cdot \binom{n-k}{j_1}_q \cdot q^{(|\xi^i|+r-k)j_1}\Big]\Big\|_2^2$$
$$\leqslant B_q \sum_i \sum_{k\leqslant r\leqslant N_1, s\geqslant 0} \Big[\|\xi^i_{r,s}\|_2^2 \Big| \sum_{j_1+j_2=n-k,\, j_1\geqslant n-2k+r-N_2} \lambda^i_{r+j_2-k,s+j_1} \frac{[r+j_2-k]_q!}{[r-k]_q!}\frac{[s+j_1]_q!}{[s]_q!}$$
$$\cdot \binom{n-k}{j_1}_q \cdot q^{(|\xi^i|+r-k)j_1}\Big|^2\Big]$$
$$\leqslant B_q \sum_i \sum_{k\leqslant r\leqslant N_1,\, s\geqslant 0} \Big(\sum_{j_1+j_2=n-k,\, j_1\geqslant n-2k+r-N_2} |\lambda^i_{r+j_2-k,s+j_1}|^2 \|\xi_{r+j_2-k,s+j_1}\|_2^2\Big)$$
$$\times \frac{4C(|q|)^3 D(q)^6}{(1-q)^n(1-q^2)} \cdot q^{2(n-N_1-N_2)}$$
$$\leqslant \frac{4(n+1)B_q C(|q|)^3 D(q)^6}{(1-q)^n(1-q^2)A_q} \cdot q^{2(n-N_1-N_2)}\|x\|_2^2.$$

Finally, using the rough estimate $\binom{n}{k}_q \leqslant C(|q|)D(q)$ and the triangle inequality, we conclude that $\|L_{N_1}(s(e^{\otimes n})L_{N_2}(x))\|_2$ can be bounded above by

$$(n+1)C(|q|)D(q)\sqrt{\frac{4(n+1)B_q C(|q|)^3 D(q)^6}{(1-q)^n(1-q^2)A_q}} \cdot |q|^{(n-N_1-N_2)}\|x\|_2.$$

Setting $G(q) = \frac{2B_q^{1/2}C(|q|)^{5/2}D(q)^4}{(1-q)^{n/2}(1-q^2)^{1/2}A_q^{1/2}}$, we are done. ∎

Finally, we are ready to prove our main result in the section.

THEOREM 4.5. *For all $N \in \mathbb{N}$ and $z = (z_n)_n \in M^\omega \ominus A^\omega \cap C'$, we have*

$$\lim_{n\to\omega} \|L_N(z_n)\|_2 = \lim_{n\to\omega} \|R_N(z_n)\|_2 = 0. \tag{4.4}$$

Proof. The proof is similar to the ones in [15], [20], but for completeness we include a sketch.

Fix $N \in \mathbb{N}$. Let $(u_n)_n$ be a sequence of unitary elements in C which converges to 0 weakly. Let $(u'_n)_n$ be a sequence of unitaries in $C^*(s(e))$ such that $\|u_n - u'_n\|_2 \leqslant \frac{1}{2^n}$. Then as in the proof of Lemma 9 in [20], we further approximate u'_n with finitely supported elements in $C^*(s(e))$.

Claim. There exists a sequence (v_k) of elements in $C^*(s(e))$ which are linear combinations of $s(e^{\otimes k})$'s and increasing sequences of natural numbers (n_k) and

(M_k), a positive constant $H(q)$, such that

$$\|u'_{n_k} - v_k\|_2 \leqslant \frac{1}{2^k}, \quad \|L_N(v_k(L_{M_{k+1}} - L_{M_k})(x)) - L_N(v_k x)\|_2 < H(q)|q|^{2k/3}\|x\|_2$$

for all $x \in M \ominus A$.

Proof of the Claim. We choose (v_k), (n_k) and (M_k) inductively. Assume that we have already chosen them up to $\{M_k, n_{k-1}, v_{k-1}\}$. Since $u_n \to 0$ weakly, if we choose a large n_k, then it is possible to well-approximate u'_{n_k} in operator-norm with an element v_k in $C^*(s(e))$ which is a linear combination of $s(e^{\otimes i})$'s with $i \in [M_k + N + k + 1, N_k]$ for some $N_k \in \mathbb{N}$. By Proposition 4.4, we have $\|L_N(v_k L_{M_k}(x))\|_2 \leqslant G(q)|q|^{2(k+1)/3}\|x\|_2$ for all $x \in M \ominus A$, where $G(q)$ is a positive constant which only depends on q. If we take $M_{k+1} \geqslant N_k + N + 1$ large enough, then we have $\|L_N(v_k(L_{M_{k+1}}(x) - x))\|_2 < |q|^k$. Thus we are done. ∎

Let us continue with the proof. On one hand, using $z \in C' \cap M^\omega$ and the asymptotic right-A modularity of L_N as in Proposition 4.3, we have

$$\begin{aligned}
&\lim_{n\to\omega} \sum_{k=N_1}^{N_2} \langle L_N(u_{n_k} z_n), L_N(u_{n_k} z_n)\rangle \\
&\qquad = \lim_{n\to\omega} \sum_{k=N_1}^{N_2} \langle L_N(z_n u_{n_k}), L_N(z_n u_{n_k})\rangle \\
&\qquad \geqslant \lim_{n\to\omega} \sum_{k=N_1}^{N_2} \left(\langle L_N(z_n u'_{n_k}), L_N(z_n u'_{n_k})\rangle - \|L_N(z_n(u_{n_k} - u'_{n_k}))\|_2^2\right) \\
&\qquad \geqslant \lim_{n\to\omega} \sum_{k=N_1}^{N_2} \left(\langle L_N(z_n)u'_{n_k}, L_N(z_n)u'_{n_k}\rangle - \frac{\|L_N\|^2\|z_n\|^2}{2^{2n_k}}\right) \\
&\qquad \geqslant (N_2 - N_1) \lim_{n\to\omega} \left(\|L_N(z_n)\|_2^2 - \frac{\|L_N\|^2\|z_n\|^2}{2^{2N_1}}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\lim_{n\to\omega} \sum_{k=N_1}^{N_2} \langle L_N(u_{n_k} z_n), L_N(u_{n_k} z_n)\rangle \\
&\qquad \approx \lim_{n\to\omega} \sum_{k=N_1}^{N_2} \langle L_N(v_k(L_{M_{k+1}} - L_{M_k})(z_n)), L_N(v_k(L_{M_{k+1}} - L_{M_k})(z_n))\rangle \\
&\qquad \leqslant \lim_{n\to\omega} \|L_N\|^2\|v_k\|^2 \sum_{k=N_1}^{N_2} \langle (L_{M_{k+1}} - L_{M_k})(z_n), (L_{M_{k+1}} - L_{M_k})(z_n)\rangle \\
&\qquad \leqslant \frac{4B_q}{A_q} \lim_{n\to\omega} \|L_N\|^2\|z_n\|_2^2.
\end{aligned}$$

By combining the above two estimates and by increasing $N_2 - N_1$ and N_1, we get the conclusion for L_N. The statement about R_N follows by symmetry. ∎

5. STRONG ASYMPTOTIC ORTHOGONALITY PROPERTY

DEFINITION 5.1. Let $B \subset N$ be an inclusion of finite von Neumann algebras. We say that the inclusion has the *strong asymptotic orthogonality property* (s-AOP for short), if for all $a, b \in N \ominus B$ and $x = (x_n) \in N^\omega \ominus B^\omega \cap C'$, where $C \subset B$ is any diffuse subalgebra of B, we have

$$ax \perp xb.$$

Fix an orthonormal basis $\{e_j : j \in J\}$ of $\mathcal{H}_\mathbb{R}$ with $e = e_{j_0}$ for some $j_0 \in J$.

Recall that for any $s, t \geqslant 0$ and for any $\xi \in L^2(M)$, we set

$$\xi_{s,t} = c(e)^s c_\mathrm{r}(e)^t \xi.$$

The following lemma is a direct consequence of the definition of annihilation operators.

LEMMA 5.2. *For all $x \in L^2(M), N \in \mathbb{N}$ and $j \in J \backslash \{j_0\}$,*

$$a(e_j)x_{N,N} = q^N (a(e_j)x)_{N,N}.$$

The next estimate is the key technical result of this section.

LEMMA 5.3. *For all $x \in L^2(M \ominus A), N \in \mathbb{N}$ and $j \in J \backslash \{j_0\}$, we have*

$$\|a(e_j)x_{N,N}\|_2^2 \leqslant \frac{16B_q^2}{A_q^2} \cdot D(q)C(|q|) \cdot q^{2N} \|a(e_j)\|_\infty \|x_{N,N}\|_2^2. \tag{5.1}$$

Proof. Suppose that along the Riesz basis $\{\xi^i_{r,s} : i \in I, r, s \geqslant 0\} \cup \{\xi^0_{r,0} : r \geqslant 0\}$, we have the Fourier expansions

$$x = \sum_{i,r,s} \lambda^i_{r,s} \xi^i_{r,s}, \quad a(e_j)x = \sum_{i,r,s} \mu^i_{r,s} \xi^i_{r,s} + \sum_{r \geqslant 0} \mu^0_r \xi^0_{r,0}.$$

First note that

(•) $[r+s+2N]_q! \leqslant \frac{D(q)}{(1-q)^{2N}} \cdot [r+s]_q!$, and

(•) $[r+s+2N]_q! \geqslant \frac{1}{(1-q)^{2N} C(|q|)} \cdot [r+s]_q!$.

By the previous lemma,

$$\begin{aligned}
\|a(e_j)x_{N,N}\|_2^2 = q^{2N} \|(a(e_j)x)_{N,N}\|_2^2 &= q^{2N} \Big\| \sum_{i,r,s} \mu^i_{r,s} \xi^i_{r+N,s+N} + \sum_{r \geqslant 0} \mu^0_r \xi^0_{r+2N,0} \Big\|_2 \\
&\leqslant q^{2N} B_q \Big(\sum_{i,r,s} |\mu^i_{r,s}|^2 \|\xi^i_{r+N,s+N}\|_2^2 + \sum_{r \geqslant 0} |\mu^0_r|^2 \|\xi^0_{r+2N,0}\|_2^2 \Big) \\
&\leqslant 2q^{2N} B_q \Big(\sum_{i,r,s} |\mu^i_{r,s}|^2 [r+s+2N]_q! + \sum_{r \geqslant 0} |\mu^0_r|^2 [r+2N]_q! \Big) \\
&\leqslant \frac{2q^{2N} B_q D(q)}{(1-q)^{2N}} \Big(\sum_{i,r,s} |\mu^i_{r,s}|^2 [r+s]_q! + \sum_{r \geqslant 0} |\mu^0_r|^2 [r]_q! \Big)
\end{aligned}$$

$$\leqslant \frac{4q^{2N}B_qD(q)}{(1-q)^{2N}}\Big(\sum_{i,r,s}|\mu^i_{r,s}|^2\|\xi^i_{r,s}\|_2^2+\sum_{r\geqslant 0}|\mu^0_r|^2\|\xi^0_{r,0}\|_2^2\Big)$$
$$\leqslant \frac{4q^{2N}B_qD(q)}{(1-q)^{2N}A_q}\|a(e_j)x\|_2^2\leqslant \frac{4q^{2N}B_qD(q)}{(1-q)^{2N}A_q}\|a(e_j)\|_\infty\|x\|_2^2.$$

On the other hand,

$$\|x_{N,N}\|_2^2=\Big\|\sum_{i,r,s}\lambda^i_{r,s}\xi^i_{r+N,s+N}\Big\|_2^2\geqslant A_q\sum_{i,r,s}|\lambda^i_{r,s}|^2\|\xi^i_{r+N,s+N}\|_2^2$$
$$\geqslant \frac{A_q}{2}\sum_{i,r,s}|\lambda^i_{r,s}|^2[r+s+2N]_q!\geqslant \frac{A_q}{2(1-q)^{2N}C(|q|)}\sum_{i,r,s}|\lambda^i_{r,s}|^2[r+s]_q!$$
$$\geqslant \frac{A_q}{4(1-q)^{2N}C(|q|)}\sum_{i,r,s}|\lambda^i_{r,s}|^2\|\xi^i_{r,s}\|_2^2\geqslant \frac{A_q}{4(1-q)^{2N}B_qC(|q|)}\|x\|_2^2.$$

Combine these two inequalities, we have

$$\|a(e_j)x_{N,N}\|_2^2\leqslant \frac{16B_q^2}{A_q^2}\cdot D(q)C(|q|)\cdot q^{2N}\|a(e_j)\|_\infty\|x_{N,N}\|_2^2. \qquad \blacksquare$$

THEOREM 5.4. *The inclusion $A\subset M$ has the strong asymptotic orthogonality property, whenever $|q|$ is less than $\frac{1}{9}$.*

Proof. Suppose $C\subset A$ is a diffuse subalgebra and $z=(z_n)\in M^\omega\ominus A^\omega\cap C'$ with $\|z_n\|\leqslant 1$. By Theorem 4.5, we can assume that for any $N\in\mathbb{N}$,

$$\lim_{n\to\omega}\|L_N(z_n)\|_2=\lim_{n\to\omega}\|R_N(z_n)\|_2=0.$$

A density argument reduces the problem to showing that

$$\lim_{n\to\omega}\langle s(e_{i(1)}\otimes\cdots e_{i(k_1)})z_n, s_{\mathrm{r}}(e_{j(1)}\otimes\cdots e_{j(k_2)})z_n\rangle=0,$$

for all $k_1,k_2\geqslant 1$ and $i(1),\dots,i(k_1),j(1),\dots,j(k_2)\in J$ such that $\{i(l):1\leqslant l\leqslant k_1\}\backslash\{j_0\}\neq\emptyset$ and $\{j(l):1\leqslant l\leqslant k_2\}\backslash\{j_0\}\neq\emptyset$.

By the Wick formula (1.5), it suffices to show the inner product between the following two elements

$$c(e_{i(1)})\cdots c(e_{i(t)})a(e_{i(t+1)})\cdots a(e_{i(k_1)})z_n \quad \text{and}$$
$$c_{\mathrm{r}}(e_{j(1)})\cdots c_{\mathrm{r}}(e_{j(s)})a_{\mathrm{r}}(e_{j(s+1)})\cdots a_{\mathrm{r}}(e_{j(k_2)})z_n$$

goes to 0 as $n\to\omega$ for any $0\leqslant t\leqslant k_1$ and $0\leqslant s\leqslant k_2$. There are two cases.

Case 1. There exists some $l_1\geqslant t+1$ with $i(l_1)\neq j_0$ such that the previous lemma implies that for any $N\in\mathbb{N}$, one has

$$\|c(e_{i(1)})\cdots c(e_{i(t)})a(e_{i(t+1)})\cdots a(e_{i(k_1)})z_n\|_2\leqslant \frac{16B_q^2}{A_q^2}\cdot D(q)C(|q|)\cdot q^{2N}\|z_n\|_2,$$

once n gets sufficiently large. By letting $N\to\infty$, we clearly have that the inner product goes to 0 as $n\to\omega$.

Case 2. $i(t+1) = \cdots = i(k_1) = j(s+1) = \cdots = j(k_2) = j_0$. Let $1 \leqslant l_1 \leqslant t$ be the smallest number such that $i(l_1) \neq j_0$. Taking adjoint, we consider

$$a(e_{i(l_1)}) \cdots a(e_{i(1)}) c_{\mathrm{r}}(e_{j(1)}) \cdots c_{\mathrm{r}}(e_{j(s)}) a_{\mathrm{r}}(e_{j(s+1)}) \cdots a_{\mathrm{r}}(e_{j(k_2)}) z_n. \tag{5.2}$$

A direct computation shows that

$$a(f_1) c_{\mathrm{r}}(f_2) = c_{\mathrm{r}}(f_2) a(f_1) + \langle f_1, f_2 \rangle W,$$

where $W \in B(L^2(M))$ is defined by

$$W|_{\mathcal{H}^{\otimes n}} = q^n \mathrm{Id}, \quad \forall n \geqslant 0.$$

Observe also that $a(f)W = qWa(f)$.

Applying these relations to $a(e_{i(l_1)}) \cdots a(e_{i(1)}) c_{\mathrm{r}}(e_{j(1)}) \cdots c_{\mathrm{r}}(e_{j(s)})$, we can write (5.2) as a finite linear combination of terms of the following form:

$$D_1 \cdots D_{m_1} a(e_{i(l_1)})^t a(e)^{m_2} a_{\mathrm{r}}(e)^{k_2 - s} z_n,$$

where $m_1, m_2 \geqslant 0, t \in \{0, 1\}$ and $D_l \in \{c_{\mathrm{r}}(e_{j(1)}), \ldots, c_{\mathrm{r}}(e_{j(s)}), W\}, \forall 1 \leqslant l \leqslant m_1$. If $t = 1$, then we are back to Case 1.

If $t = 0$, then one of the $D_k, 1 \leqslant k \leqslant m_1$ must be W. But we know that W will decrease the size of the vector in an exponential rate with respect to the length of its basic words, so as $n \to \omega$, the length of z_n goes to infinity, thus

$$\| D_1 \cdots D_{m_1} a(e)^{m_2} a_{\mathrm{r}}(e)^{k_2 - s} z_n \|_2 \to 0 \quad \text{as well.} \quad ∎$$

A consequence of the theorem is a strengthening of maximal amenability.

THEOREM 5.5. *Let q be a real number with $|q| < \frac{1}{9}$; then the inclusion $A \subset M$ of the generator masa inside the q-Gaussian von Neumann algebra, has the absorbing amenability property as introduced in* [6] *(see also* [12], [20]*). That is, for any diffuse subalgebra $C \subset A$, A is the unique maximal amenable extension inside M.*

Proof. A is shown to be mixing in M by [2], [21]. Thus Theorem 8.1 of [11] applies. One can alternatively use the argument in Proposition 1 of [20]. ∎

As another application of the Rădulescu basis, we can give a very short proof of non-Gamma for the q-Gaussian algebras, whenever Theorem 3.13 is true.

COROLLARY 5.6 (See also [1]). *Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space with* $\dim \mathcal{H}_{\mathbb{R}} \geqslant 2$. *Let q be any real number with $|q| < \frac{1}{9}$. Then $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ is a full factor.*

Proof. Let $e, f \in \mathcal{H}_{\mathbb{R}}$ be two orthogonal unit vectors and let $A = \Gamma_q(\mathbb{R}e)$ (respectively $B = \Gamma_q(\mathbb{R}f)$) be the generators subalgebra associated with e (respectively f). We construct as in the previous section the Rădulescu basis $\{\xi^i_{r,s} : i \in I, r, s \geqslant 0\}$ with respect to A. Notice that $f^{\otimes n} \in T_n$, thus we may choose the basis such that for each $n \geqslant 1$, $\frac{f^{\otimes n}}{\sqrt{[n]_q!}} = \xi^{i_n}_{0,0}$ for some $i_n \in I$.

Suppose $x \in M' \cap M^\omega$ with $\tau(x) = 0$. Since $x \in A' \cap M^\omega$, by applying Theorem 4.5 for A and by the choice of the basis, we can assume that $E_{B^\omega}(x) = 0$.

Choose a Haar unitary u of A. Note that $ux=xu$ and $u\perp B$. Therefore by the s-AOP for $B\subset M$ as shown in Theorem 5.4, we have that $\tau(uxu^*x^*)=\tau(xx^*)=0$. ∎

REMARK 5.7. It is already known that for all $-1<q<1$ and for all separable real Hilbert space $\mathcal{H}_{\mathbb{R}}$ with $\dim\mathcal{H}_{\mathbb{R}}\geqslant 2$, $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ does not have property Gamma: Avsec [1] showed the strong-solidity for all $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ and it follows from an argument in [14] that solid factors do not have the property Gamma.

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