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GROUP-LIKE PROJECTIONS FOR LOCALLY COMPACT QUANTUM GROUPS

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ABSTRACT. Let $\mathbb{G}$ be a locally compact quantum group. We give a 1-1 correspondence between group-like projections in $L^\infty(\mathbb{G})$ preserved by the scaling group and idempotent states on the dual quantum group $\widehat{\mathbb{G}}$. As a byproduct we give a simple proof that normal integrable coideals in $L^\infty(\mathbb{G})$ which are preserved by the scaling group are in 1-1 correspondence with compact quantum subgroups of $\mathbb{G}$.

KEYWORDS: *Locally compact quantum group, group-like projections, idempotent states.*

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1. INTRODUCTION

Let G be a group, X a non-empty subset of G and $\mathbb{1}_X : G \to \{0,1\}$ its characteristic function. It is easy to check that X is a subgroup of G if and only if

$$\mathbb{1}_X(st)\mathbb{1}_X(t) = \mathbb{1}_X(s)\mathbb{1}_X(t) \tag{1.1}$$

for all $s, t \in G$. Let G be a locally compact group and $\Delta : L^\infty(G) \to L^\infty(G) \bar{\otimes} L^\infty(G)$ the comultiplication on $L^\infty(G)$:

$$\Delta(f)(s,t) = f(st)$$

for all $f \in L^\infty(G)$. Suppose that $P \in L^\infty(G)$ is a non-zero group-like projection, i.e. P satisfies

$$\Delta(P)(\mathbb{1}_G \otimes P) = P \otimes P. \tag{1.2}$$

Equation (1.2) implies that P is a continuous function on G (see Lemma 2.6). Denoting

$$X = \{s \in G : P(s) = 1\},$$

we have $P = \mathbb{1}_X$ and $\mathbb{1}_X$ satisfies (1.1). In particular X is a subgroup of G and the continuity of $\mathbb{1}_X$ implies that X is open. Thus we get a 1-1 correspondence between open subgroups of G and group-like projections in $L^\infty(G)$.

Let G be a locally compact group. The Banach dual $C_0(G)^*$ of $C_0(G)$ equipped with the convolution product is a Banach algebra. We say that a state $\omega \in C_0(G)^*$ is an idempotent state on $C_0(G)$ if $\omega * \omega = \omega$. In fact, as proved by Kelley ([8], Theorem 3.4) there is a 1-1 correspondence between idempotent states on $C_0(G)$ and compact subgroups of G, where given a compact subgroup $H \subset G$ the corresponding state is of the form $\omega(f) = \int_H f(h)\mathrm{d}h$ for all $f \in C_0(G)$.

Let $G \ni g \mapsto R_g \in B(L^2(G))$ be the right regular representation, $\mathrm{vN}(G) = \{R_g : g \in G\}''$ the group von Neumann algebra of G and $\widehat{\Delta} : \mathrm{vN}(G) \to \mathrm{vN}(G) \bar{\otimes} \mathrm{vN}(G)$ the comultiplication, where $\widehat{\Delta}(R_g) = R_g \otimes R_g$ for all $g \in G$. It is not difficult to see that $P = \int_H R_h \mathrm{d}h \in \mathrm{vN}(G)$ is a group-like projection in $\mathrm{vN}(G)$, i.e. it satisfies

$$\widehat{\Delta}(P)(\mathbb{1} \otimes P) = P \otimes P.$$

Theorem 4.3 and Kelley's result show that all group-like projections in $\mathrm{vN}(G)$ are of this form. In other words we have a 1-1 correspondence between idempotent states on $C_0(G)$ and group-like projections in $\mathrm{vN}(G)$. Theorem 3.1 together with Theorem 4.3 yield a generalization of this correspondence to the context of locally compact quantum groups.

A locally compact quantum group $\mathbb{G}$ is a virtual object that is assigned with a von Neumann algebra $L^\infty(\mathbb{G})$ equipped with a comultiplication $\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\mathbb{G})$. A projection $P \in L^\infty(\mathbb{G})$ is called a group-like projection if

$$\Delta(P)(\mathbb{1} \otimes P) = P \otimes P.$$

A locally compact quantum group $\mathbb{G}$ is also assigned with the C^*-algebra $C_0(\mathbb{G})$ and universal C^*-algebra $C_0^{\mathrm{u}}(\mathbb{G})$. Both Banach duals $C_0^{\mathrm{u}}(\mathbb{G})^*$ and $C_0(\mathbb{G})^*$ are in fact Banach algebras. We say that a state $\omega \in C_0^{\mathrm{u}}(\mathbb{G})^*$ is an idempotent state (on $\mathbb{G}$) if $\omega * \omega = \omega$. As already mentioned, our results establish a 1-1 correspondence between idempotent states on $\mathbb{G}$ and group-like projections on the dual $\widehat{\mathbb{G}}$ which are preserved by the scaling group of $\widehat{\mathbb{G}}$. As a byproduct of our study we get a relatively simple proof that normal integrable coideals in $L^\infty(\mathbb{G})$ which are preserved by the scaling group are in 1-1 correspondence with compact quantum subgroups of $\mathbb{G}$. Our proof, unlike the previous proof ([4], Theorem 5.15), uses only the von Neumann techniques and does not invoke the universal C^*-algebra $C_0^{\mathrm{u}}(\mathbb{G})$.

2. PRELIMINARIES

We will denote the minimal tensor product of C^*-algebras with the symbol $\otimes$. The ultraweak tensor product of von Neumann algebras will be denoted by $\bar{\otimes}$. For a C^*-subalgebra B of a C^*-algebra the multipliers $\mathrm{M}(\mathsf{A})$ of A, the norm closed linear span of the set $\{ba : b \in \mathsf{B}, a \in \mathsf{A}\}$ will be denoted by BA. A morphism between two C^*-algebras A and B is a $*$-homomorphism π from A into the multiplier algebra $\mathrm{M}(\mathsf{B})$, which is non-degenerate, i.e $\pi(\mathsf{A})\mathsf{B} = \mathsf{B}$. We will denote the set of all morphisms from A to B by $\mathrm{Mor}(\mathsf{A}, \mathsf{B})$. The non-degeneracy of a morphism π yields its natural extension to the unital $*$-homomorphism $\mathrm{M}(\mathsf{A}) \to \mathrm{M}(\mathsf{B})$ also denoted by π. Let B be a C^*-subalgebra of $\mathrm{M}(A)$. We say that B is non-degenerate if $\mathsf{BA} = \mathsf{A}$. In this case $\mathrm{M}(\mathsf{B})$ can be identified with a C^*-subalgebra of $\mathrm{M}(\mathsf{A})$. The symbol $\boldsymbol{\sigma}$ will denote the flip morphism between tensor product of operator algebras. If $X \subset \mathsf{A}$, where A is a C^*-algebra then $X^{\text{norm-cls}}$ denotes the norm closure of the linear span of X; if $X \subset \mathsf{M}$, where M is a von Neumann algebra then $X^{\sigma\text{-weak cls}}$ denotes the σ-weak closure of the linear span of X. For a C^*-algebra A, the space of all functionals on A and the state space of A will be denoted by A^* and $S(\mathsf{A})$ respectively. The predual of a von Neumann algebra N will be denoted by N_*. For a Hilbert space H the C^*-algebras of compact operators on H will be denoted by $\mathcal{K}(H)$. The algebra of bounded operators acting on H will be denoted by $B(H)$. For $\xi, \eta \in H$, the symbol $\omega_{\xi,\eta} \in B(H)_*$ is the functional $T \mapsto \langle \xi, T\eta \rangle$.

For the theory of locally compact quantum groups we refer to [9], [10], [11]. Let us recall that a von Neumann algebraic locally compact quantum group is a quadruple $\mathbb{G} = (L^\infty(\mathbb{G}), \Delta, \varphi, \psi)$, where $L^\infty(\mathbb{G})$ is a von Neumann algebra with a coassociative comultiplication $\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\mathbb{G})$, φ and ψ are, respectively, normal semifinite faithful left and right Haar weights on $L^\infty(\mathbb{G})$. The GNS Hilbert space of the right Haar weight $\psi_{\mathbb{G}}$ will be denoted by $L^2(\mathbb{G})$ and the corresponding GNS map will be denoted by $\eta_{\mathbb{G}}$. Let us recall that $\eta_{\mathbb{G}} : \mathcal{N}_\psi \to L^2(\mathbb{G})$, where $\mathcal{N}_\psi = \{x \in L^\infty(\mathbb{G}) : \psi(x^*x) < \infty\}$. The *antipode*, the *scaling group* and the *unitary antipode* will be denoted by S, $(\tau_t)_{t\in\mathbb{R}}$ and R. We have $S = R \circ \tau_{-i/2}$. Moreover, for all $a, b \in \mathcal{N}_\varphi$ the following holds (see Corollary 5.35 of [10]):

$$S((\mathrm{id} \otimes \varphi)(\Delta(a^*)(\mathbb{1} \otimes b))) = (\mathrm{id} \otimes \varphi)((\mathbb{1} \otimes a^*)\Delta(b)). \tag{2.1}$$

We will denote $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ and $(\sigma_t^\psi)_{t\in\mathbb{R}}$ the *modular automorphism groups* assigned to φ and ψ respectively.

The multiplicative unitary $\mathrm{W}^{\mathbb{G}} \in B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ is a unique unitary operator such that

$$\mathrm{W}^{\mathbb{G}}(\eta_{\mathbb{G}}(x) \otimes \eta_{\mathbb{G}}(y)) = (\eta_{\mathbb{G}} \otimes \eta_{\mathbb{G}})(\Delta_{\mathbb{G}}(x)(\mathbb{1} \otimes y))$$

for all $x, y \in D(\eta_{\mathbb{G}})$; $\mathrm{W}^{\mathbb{G}}$ satisfies the pentagonal equation $\mathrm{W}^{\mathbb{G}}_{12}\mathrm{W}^{\mathbb{G}}_{13}\mathrm{W}^{\mathbb{G}}_{23} = \mathrm{W}^{\mathbb{G}}_{23}\mathrm{W}^{\mathbb{G}}_{12}$ [1], [15]. Using $\mathrm{W}^{\mathbb{G}}$, $\mathbb{G}$ can be recovered as follows:

$$L^\infty(\mathbb{G}) = l\{(\omega \otimes \mathrm{id})\mathrm{W}^{\mathbb{G}} : \omega \in B(L^2(\mathbb{G}))_* r\}^{\sigma\text{-weak cls}}, \quad \Delta_{\mathbb{G}}(x) = \mathrm{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathrm{W}^{\mathbb{G}*}.$$

A locally compact quantum group admits a dual object $\widehat{\mathbb{G}}$. It can be described in terms of $\mathrm{W}^{\widehat{\mathbb{G}}} = \boldsymbol{\sigma}(\mathrm{W}^{\mathbb{G}})^*$:

$$L^\infty(\widehat{\mathbb{G}}) = \{(\omega \otimes \mathrm{id})\mathrm{W}^{\widehat{\mathbb{G}}} : \omega \in B(L^2(\mathbb{G}))_*\}^{\sigma\text{-weak cls}}, \quad \Delta_{\widehat{\mathbb{G}}}(x) = \mathrm{W}^{\widehat{\mathbb{G}}}(x \otimes \mathbb{1})\mathrm{W}^{\widehat{\mathbb{G}}^*}.$$

Note that $\mathrm{W}^{\mathbb{G}} \in L^\infty(\widehat{\mathbb{G}})\bar{\otimes} L^\infty(\mathbb{G})$.

DEFINITION 2.1. A von Neumann subalgebra N of $L^\infty(\mathbb{G})$ is called
(i) *left coideal* if $\Delta_{\mathbb{G}}(\mathsf{N}) \subset L^\infty(\mathbb{G})\bar{\otimes}\mathsf{N}$;
(ii) *invariant subalgebra* if $\Delta_{\mathbb{G}}(\mathsf{N}) \subset \mathsf{N}\bar{\otimes}\mathsf{N}$;
(iii) *Baaj–Vaes subalgebra* if N is an invariant subalgebra of $L^\infty(\mathbb{G})$ which is preserved by the unitary antipode R and the scaling group $(\tau_t)_{t\in\mathbb{R}}$ of $\mathbb{G}$;
(iv) *normal* if $\mathrm{W}^{\mathbb{G}}(\mathbb{1} \otimes \mathsf{N})\mathrm{W}^{\mathbb{G}*} \subset L^\infty(\widehat{\mathbb{G}})\bar{\otimes}\mathsf{N}$;
(v) *integrable* if the set of integrable elements with respect to the right Haar weight $\psi_{\mathbb{G}}$ is dense in N^+; in other words, the restriction of $\psi_{\mathbb{G}}$ to N is semifinite.

If N is a coideal of $L^\infty(\mathbb{G})$, then $\widetilde{\mathsf{N}} = \mathsf{N}' \cap L^\infty(\widehat{\mathbb{G}})$ is a coideal of $L^\infty(\widehat{\mathbb{G}})$ called the *codual* of N; it turns out that $\widetilde{\widetilde{\mathsf{N}}} = \mathsf{N}$ (see Theorem 3.9 of [6]).

The C^*-algebraic version $(C_0(\mathbb{G}), \Delta_{\mathbb{G}})$ of a given quantum group $\mathbb{G}$ is recovered from $\mathrm{W}^{\mathbb{G}}$ as follows:

$$C_0(\mathbb{G}) = \{(\omega \otimes \mathrm{id})\mathrm{W}^{\mathbb{G}} : \omega \in B(L^2(\mathbb{G}))_*\}^{\text{norm-cls}}, \quad \Delta_{\mathbb{G}}(x) = \mathrm{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathrm{W}^{\mathbb{G}*}.$$

The comultiplication can be viewed as a morphism $\Delta_{\mathbb{G}} \in \mathrm{Mor}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$ and we have $\mathrm{W}^{\mathbb{G}} \in \mathrm{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G}))$. Since $\mathrm{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G})) \subset \mathrm{M}(\mathcal{K}(L^2(\mathbb{G})) \otimes C_0(\mathbb{G}))$ we conclude that for all $x \in L^\infty(\mathbb{G})$

$$\Delta_{\mathbb{G}}(x) = \mathrm{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathrm{W}^{\mathbb{G}*} \in \mathrm{M}(\mathcal{K}(L^2(\mathbb{G})) \otimes C_0(\mathbb{G})). \tag{2.2}$$

Replacing $\Delta_{\mathbb{G}}$ with $\Delta_{\mathbb{G}^{\mathrm{op}}}$ we also get that

$$\Delta_{\mathbb{G}}(x) \in \mathrm{M}(C_0(\mathbb{G}) \otimes \mathcal{K}(L^2(\mathbb{G}))) \tag{2.3}$$

for all $x \in L^\infty(\mathbb{G})$.

Let H be a Hilbert space and $\mathrm{U} \in \mathrm{M}(C_0(\mathbb{G}) \otimes \mathcal{K}(\mathsf{H}))$ a unitary. We say that U is a representation of $\mathbb{G}$ on H if

$$(\Delta_{\mathbb{G}} \otimes \mathrm{id})(\mathrm{U}) = \mathrm{U}_{13}\mathrm{U}_{23}.$$

Let us recall the definition of an action of a quantum group $\mathbb{G}$ on a von Neumann algebra.

DEFINITION 2.2. A *(left) action* of quantum group $\mathbb{G}$ on a von Neumann algebra N is a unital injective normal $*$-homomorphism $\alpha : \mathsf{N} \to L^\infty(\mathbb{G})\bar{\otimes}\mathsf{N}$ such that $(\Delta_{\mathbb{G}} \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \alpha) \circ \alpha$. If $\mathsf{M} \subset \mathsf{N}$ is a von Neumann subalgebra then we say that M is preserved by α if $\alpha(\mathsf{M}) \subset L^\infty(\mathbb{G})\bar{\otimes}\mathsf{M}$.

Given an action $\alpha : \mathsf{N} \to L^\infty(\mathbb{G})\bar{\otimes}\mathsf{N}$ we have (see Corollary 2.6 of [6])

$$\mathsf{N} = \{(\mu \otimes \mathrm{id})\alpha(x) : x \in \mathsf{N}, \mu \in L^\infty(\mathbb{G})_*\}^{\sigma\text{-weak cls}}$$

which will be referred to as *the Podleś condition*. We can always find a unitary representation $\mathrm{U} \in \mathrm{M}(C_0(\mathbb{G}) \otimes \mathcal{K}(\mathsf{H}))$ on a Hilbert space H and a normal faithful $*$-homomorphism $\pi : \mathsf{N} \to \mathrm{B}(\mathsf{H})$ such that

$$(\mathrm{id} \otimes \pi)(\alpha(x)) = \mathrm{U}^*(\mathbb{1} \otimes \pi(x))\mathrm{U}.$$

In this case we shall say that U implements the action α. For the construction of the canonical implementation see [14].

A locally compact quantum group $\mathbb{G}$ is assigned with a universal version [9]. The universal version $C_0^{\mathrm{u}}(\mathbb{G})$ of $C_0(\mathbb{G})$ is equipped with a comultiplication $\Delta_{\mathbb{G}}^{\mathrm{u}} \in \mathrm{Mor}(C_0^{\mathrm{u}}(\mathbb{G}), C_0^{\mathrm{u}}(\mathbb{G}) \otimes C_0^{\mathrm{u}}(\mathbb{G}))$. The *counit* is a $*$-homomorphism $\varepsilon : C_0^{\mathrm{u}}(\mathbb{G}) \to \mathbb{C}$ satisfying $(\mathrm{id} \otimes \varepsilon) \circ \Delta_{\mathbb{G}}^{\mathrm{u}} = \mathrm{id} = (\varepsilon \otimes \mathrm{id}) \circ \Delta_{\mathbb{G}}^{\mathrm{u}}$. The multiplicative unitary $\mathrm{W}^{\mathbb{G}} \in \mathrm{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G}))$ admits the universal lift $\mathbb{V}\!\!\mathbb{V}^{\mathbb{G}} \in \mathrm{M}(C_0^{\mathrm{u}}(\widehat{\mathbb{G}}) \otimes C_0^{\mathrm{u}}(\mathbb{G}))$. The reducing morphisms for $\mathbb{G}$ and $\widehat{\mathbb{G}}$ will be denoted by $\Lambda_{\mathbb{G}} \in \mathrm{Mor}(C_0^{\mathrm{u}}(\mathbb{G}), C_0(\mathbb{G}))$ and $\Lambda_{\widehat{\mathbb{G}}} \in \mathrm{Mor}(C_0^{\mathrm{u}}(\widehat{\mathbb{G}}), C_0(\widehat{\mathbb{G}}))$, respectively. We have $(\Lambda_{\widehat{\mathbb{G}}} \otimes \Lambda_{\mathbb{G}})(\mathbb{V}\!\!\mathbb{V}^{\mathbb{G}}) = \mathrm{W}^{\mathbb{G}}$. We shall also use the half-lifted versions of $\mathrm{W}^{\mathbb{G}}$, $\mathbb{W}^{\mathbb{G}} = (\mathrm{id} \otimes \Lambda_{\mathbb{G}})(\mathbb{V}\!\!\mathbb{V}^{\mathbb{G}}) \in \mathrm{M}(C_0^{\mathrm{u}}(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G}))$ and $\mathbb{W}\!\!\!\!\mathbb{W}^{\mathbb{G}} = (\Lambda_{\widehat{\mathbb{G}}} \otimes \mathrm{id})(\mathbb{V}\!\!\mathbb{V}^{\mathbb{G}}) \in \mathrm{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0^{\mathrm{u}}(\mathbb{G}))$. They satisfy the appropriate versions of pentagonal equation:

$$\mathbb{W}_{12}^{\mathbb{G}}\mathbb{W}_{13}^{\mathbb{G}}\mathrm{W}_{23}^{\mathbb{G}} = \mathrm{W}_{23}^{\mathbb{G}}\mathbb{W}_{12}^{\mathbb{G}}, \quad \mathrm{W}_{12}^{\mathbb{G}}\mathbb{W}\!\!\!\!\mathbb{W}_{13}^{\mathbb{G}}\mathbb{W}\!\!\!\!\mathbb{W}_{23}^{\mathbb{G}} = \mathbb{W}\!\!\!\!\mathbb{W}_{23}^{\mathbb{G}}\mathrm{W}_{12}^{\mathbb{G}}.$$

The half-lifted versions of comultiplication are denoted by $\Delta_r^{r,\mathrm{u}} \in \mathrm{Mor}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes C_0^{\mathrm{u}}(\mathbb{G}))$ and $\widehat{\Delta}_r^{r,\mathrm{u}} \in \mathrm{Mor}(C_0(\widehat{\mathbb{G}}), C_0(\widehat{\mathbb{G}}) \otimes C_0^{\mathrm{u}}(\widehat{\mathbb{G}}))$, e.g.

$$\Delta_r^{r,\mathrm{u}}(x) = \mathbb{W}\!\!\!\!\mathbb{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathbb{W}\!\!\!\!\mathbb{W}^{\mathbb{G}*}, \quad x \in C_0(\mathbb{G}).$$

We have

$$(\Lambda_{\mathbb{G}} \otimes \mathrm{id}) \circ \Delta_{\mathbb{G}}^{\mathrm{u}} = \Delta_r^{r,\mathrm{u}} \circ \Lambda_{\mathbb{G}}, \quad (\Lambda_{\widehat{\mathbb{G}}} \otimes \mathrm{id}) \circ \Delta_{\widehat{\mathbb{G}}}^{\mathrm{u}} = \widehat{\Delta}_r^{r,\mathrm{u}} \circ \Lambda_{\widehat{\mathbb{G}}}. \tag{2.4}$$

If $\mathrm{U} \in \mathrm{M}(C_0(\mathbb{G}) \otimes \mathcal{K}(\mathsf{H}))$ is a unitary representation of $\mathbb{G}$ on a Hilbert space then there exists a unique unitary $\mathbb{U} \in \mathrm{M}(C_0^{\mathrm{u}}(\mathbb{G}) \otimes \mathcal{K}(\mathsf{H}))$ such that $\mathrm{U} = (\Lambda_{\mathbb{G}} \otimes \mathrm{id})(\mathbb{U})$ and

$$(\Delta^u \otimes \mathrm{id})(\mathbb{U}) = \mathbb{U}_{13}\mathbb{U}_{23}.$$

Actually $\mathbb{U}_{23} = \mathrm{U}_{13}^*(\Delta_r^{r,\mathrm{u}} \otimes \mathrm{id})(\mathrm{U})$.

Given a locally compact quantum group $\mathbb{G}$, the comultiplications $\Delta_{\mathbb{G}}$ and $\Delta_{\mathbb{G}}^{\mathrm{u}}$ induce Banach algebra structures on $L^\infty(\mathbb{G})_*$ and $C_0^{\mathrm{u}}(\mathbb{G})^*$, respectively. The corresponding multiplications will be denoted by $\underline{*}$ and $\overline{*}$. We shall identify $L^\infty(\mathbb{G})_*$ with a subspace of $C_0^{\mathrm{u}}(\mathbb{G})^*$ when convenient. Under this identification $L^\infty(\mathbb{G})_*$ forms a two sided ideal in $C_0^{\mathrm{u}}(\mathbb{G})^*$. Following [9], for any $\mu \in C_0^{\mathrm{u}}(\mathbb{G})^*$ we define a normal map $L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ such that $x \mapsto (\mathrm{id} \otimes \mu)(\mathbb{W}\!\!\!\!\mathbb{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathbb{W}\!\!\!\!\mathbb{W}^{\mathbb{G}*})$ for all $x \in L^\infty(\mathbb{G})$. We shall use the notation $\mu \,\overline{*}\, x = (\mathrm{id} \otimes \mu)(\mathbb{W}\!\!\!\!\mathbb{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathbb{W}\!\!\!\!\mathbb{W}^{\mathbb{G}*})$.

THEOREM 2.3. *Let* N *be a von Neumann algebra and* $\alpha : \mathsf{N} \to L^\infty(\mathbb{G})\bar{\otimes}\mathsf{N}$ *an action of* $\mathbb{G}$ *on* N*. Let* $x \in \mathsf{N}$, $x^* = x$ *and*

$$\mathsf{N}_x = \{(\mu \otimes \mathrm{id})(\alpha(x)) : \mu \in L^\infty(\mathbb{G})_*\}''.$$

Then N_x *is the smallest unital von Neumann subalgebra of* N *preserved by* $\mathbb{G}$ *and containing* x*.*

Proof. Let us consider

$$\mathsf{S} = \{(\mu \otimes \mathrm{id})(\alpha(x)) : \mu \in L^\infty(\mathbb{G})_*\}.$$

Then S forms a selfadjoint subset of S. In particular N_x is the (unital) von Neumann algebra generated by S. Noting that

$$(\omega_1 \otimes \mathrm{id})(\alpha((\omega_2 \otimes \mathrm{id})\alpha(x))) = (\omega_2 \,\underline{*}\, \omega_1 \otimes \mathrm{id})(\alpha(x)) \in \mathsf{N}_x$$

we conclude that N_x is preserved by $\mathbb{G}$.

Every $\mathsf{M} \subset \mathsf{N}$ preserved by $\mathbb{G}$ and containing x must contain N_x, so it remains to prove that $x \in \mathsf{N}_x$. For this we may assume that $\mathsf{N} \subset \mathsf{B}(\mathsf{H})$ and α is implemented by a unitary representation $\mathsf{U} \in \mathrm{M}(\mathrm{C}_0(\mathbb{G}) \otimes \mathcal{K}(\mathsf{H}))$

$$\alpha(x) = \mathsf{U}^*(\mathbb{1} \otimes x)\mathsf{U}.$$

Unitary implementation enables us to define a morphism $\alpha_0 \in \mathrm{Mor}(\mathcal{K}(\mathsf{H}), C_0^{\mathrm{u}}(\mathbb{G}) \otimes \mathcal{K}(\mathsf{H}))$, where $\alpha_0(x) = \mathsf{U}^*(\mathbb{1} \otimes x)\mathsf{U}$. Thus, using natural extension of the morphism α_0 to $\mathsf{B}(\mathsf{H}) = \mathrm{M}(\mathcal{K}(\mathsf{H}))$ we can further extend α to an action on $\mathsf{B}(\mathsf{H})$ and we shall assume in what follows that $\mathsf{N} = \mathsf{B}(\mathsf{H})$. As the conclusion of the above observation we see that, given a C^*-algebra B, an element $X \in \mathrm{M}(\mathsf{B} \otimes \mathcal{K}(\mathsf{H}))$ and a functional $\mu \in \mathsf{B}^*$ we have

$$\alpha((\mu \otimes \mathrm{id})(X)) = (\mu \otimes \mathrm{id} \otimes \mathrm{id})((\mathrm{id} \otimes \alpha)(X)). \tag{2.5}$$

Let $\mathbb{U} \in \mathrm{M}(C_0^{\mathrm{u}}(\mathbb{G}) \otimes \mathcal{K}(H))$ be the universal lift of U. Let us note that

$$\mathsf{M} := \{(\mu \otimes \mathrm{id})(\mathbb{U}^*(\mathbb{1} \otimes x)\mathbb{U}) : \mu \in C_0^{\mathrm{u}}(\mathbb{G})^*\}''$$

is a von Neumann subalgebra of $\mathsf{B}(\mathsf{H})$ containing x (for the latter take $\mu = \varepsilon$) and $\mathsf{N}_x \subset \mathsf{M}$. Furthermore, for every $\omega \in L^\infty(\mathbb{G})_*$ we have

$$(\omega \otimes \mathrm{id})(\alpha((\mu \otimes \mathrm{id})(\mathbb{U}^*(\mathbb{1} \otimes x)\mathbb{U})) = (\mu \,\overline{*}\, \omega \otimes \mathrm{id})(\alpha(x)) \in \mathsf{N}_x \subset \mathsf{M} \tag{2.6}$$

where we used (2.5). This shows that M is preserved by $\mathbb{G}$ (note that for the proof of the containment "$\in$" in equation (2.6) we use $\mu \,\overline{*}\, \omega \in L^\infty(\mathbb{G})_*$). Since the action of $\mathbb{G}$ on M satisfies the Podleś condition, M is generated by elements of the form $(\mu \,\overline{*}\, \omega \otimes \mathrm{id})(\alpha(x))$, $\mu \in C_0^{\mathrm{u}}(\mathbb{G})^*, \omega \in L^\infty(\mathbb{G})_*$. Since $\mu \,\overline{*}\, \omega \in L^\infty(\mathbb{G})_*$, we conclude that $\mathsf{M} \subset \mathsf{N}_x$ and in particular $x \in \mathsf{N}_x$. ∎

REMARK 2.4. If in the context of Theorem 2.3 we start with a not necessary self-adjoint $x \in \mathsf{N}$, then the smallest von Neumann subalgebra of N containing x is given by

$$\mathsf{N}_x = \{(\mu \otimes \mathrm{id})(\alpha(x)), (\mu \otimes \mathrm{id})(\alpha(x^*)) : \mu \in L^\infty(\mathbb{G})_*\}''.$$

DEFINITION 2.5. Let N be a von Neumann algebra with an action $\alpha : \mathsf{N} \to L^\infty(\mathbb{G})\bar{\otimes}\mathsf{N}$ of a locally compact quantum group $\mathbb{G}$ and let $x \in \mathsf{N}$. We say that N is $\mathbb{G}$-generated by x if $\mathsf{N}_x = \mathsf{N}$.

A state $\omega \in S(C_0^{\mathrm{u}}(\mathbb{G}))$ is said to be an *idempotent state* if $\omega \,\overline{*}\, \omega = \omega$. For a nice survey describing the history and motivation behind the study of idempotent states see [12]. For the theory of idempotent states we refer to [13]. We shall use Proposition 4 of [13] which in particular states that an idempotent state

$\omega \in S(C_0^{\mathrm{u}}(\mathbb{G}))$ is preserved by the universal scaling group τ_t^{u} and the universal unitary antipode $R^{\mathrm{u}} : C_0^{\mathrm{u}}(\mathbb{G}) \to C_0^{\mathrm{u}}(\mathbb{G})$, i.e.

$$\omega \circ \tau_t^{\mathrm{u}} = \omega = \omega \circ R^{\mathrm{u}} \tag{2.7}$$

for all $t \in \mathbb{R}$. An idempotent state $\omega \in S(C_0^{\mathrm{u}}(\mathbb{G}))$ yields a conditional expectation $E_\omega : C_0(\mathbb{G}) \to C_0(\mathbb{G})$ (see [13]),

$$E_\omega(x) = \omega \,\overline{*}\, x$$

for all $x \in C_0(\mathbb{G})$. Using (2.7), we easily get

$$\tau_t(E_\omega(x)) = E_\omega(\tau_t(x)). \tag{2.8}$$

The conditional expectation extends to $E_\omega : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ and clearly (2.8) holds for all $x \in L^\infty(\mathbb{G})$. The image $\mathsf{N} = E_\omega(L^\infty(\mathbb{G}))$ of E_ω forms a coideal in $L^\infty(\mathbb{G})$.

Let $\mathbb{H}$ and $\mathbb{G}$ be locally compact quantum groups. A morphism $\pi \in \mathrm{Mor}(C_0^{\mathrm{u}}(\mathbb{G}), C_0^{\mathrm{u}}(\mathbb{H}))$ such that

$$(\pi \otimes \pi) \circ \Delta_{\mathbb{G}}^{\mathrm{u}} = \Delta_{\mathbb{H}}^{\mathrm{u}} \circ \pi$$

is said to define a homomorphism from $\mathbb{H}$ to $\mathbb{G}$. If $\pi(C_0^{\mathrm{u}}(\mathbb{G})) = C_0^{\mathrm{u}}(\mathbb{H})$, then $\mathbb{H}$ is called a *Woronowicz-closed quantum subgroup* of $\mathbb{G}$ [2]. A homomorphism from $\mathbb{H}$ to $\mathbb{G}$ admits the dual homomorphism $\widehat{\pi} \in \mathrm{Mor}(C_0^{\mathrm{u}}(\widehat{\mathbb{H}}), C_0^{\mathrm{u}}(\widehat{\mathbb{G}}))$ such that

$$(\mathrm{id} \otimes \pi)(\mathbb{V}\!\!\mathbb{V}^{\mathbb{G}}) = (\widehat{\pi} \otimes \mathrm{id})(\mathbb{V}\!\!\mathbb{V}^{\mathbb{H}}).$$

A homomorphism from $\mathbb{H}$ to $\mathbb{G}$ identifies $\mathbb{H}$ as a *closed quantum subgroup* of $\mathbb{G}$ if there exists an injective normal unital $*$-homomorphism $\gamma : L^\infty(\widehat{\mathbb{H}}) \to L^\infty(\widehat{\mathbb{G}})$ such that

$$\Lambda_{\widehat{\mathbb{G}}} \circ \widehat{\pi}(x) = \gamma \circ \Lambda_{\widehat{\mathbb{H}}}(x)$$

for all $x \in C_0^{\mathrm{u}}(\widehat{\mathbb{H}})$. Let $\mathbb{H}$ be a closed quantum subgroup of $\mathbb{G}$, then $\mathbb{H}$ acts on $L^\infty(\mathbb{G})$ (in the von Neumann algebraic sense) by the following formula

$$\alpha : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\mathbb{H}), \quad x \mapsto V(x \otimes \mathbb{1})V^*,$$

where

$$V = (\gamma \otimes \mathrm{id})(\mathrm{W}^{\mathbb{H}}). \tag{2.9}$$

The fixed point space of α is denoted by

$$L^\infty(\mathbb{G}/\mathbb{H}) = \{x \in L^\infty(\mathbb{G}) : \alpha(x) = x \otimes \mathbb{1}\}$$

and referred to as the algebra of bounded functions on the quantum homogeneous space $\mathbb{G}/\mathbb{H}$. If $\mathbb{H}$ is a compact quantum subgroup of $\mathbb{G}$, then there is a conditional expectation $E : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ onto $L^\infty(\mathbb{G}/\mathbb{H})$ which is defined by

$$E = (\mathrm{id} \otimes \psi_{\mathbb{H}}) \circ \alpha, \tag{2.10}$$

where $\psi_{\mathbb{H}}$ is the Haar state of $\mathbb{H}$.

According to Definition 2.2 of [9] we say that $\mathbb{H}$ is an *open quantum subgroup* of $\mathbb{G}$ if there is a surjective normal $*$-homomorphism $\rho : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{H})$ such that

$$\Delta_{\mathbb{H}} \circ \rho = (\rho \otimes \rho) \circ \Delta_{\mathbb{G}}.$$

Every open quantum subgroup is closed ([3], Theorem 3.6). We recall that a projection $P \in L^\infty(\mathbb{G})$ is a *group-like projection* if $\Delta_{\mathbb{G}}(P)(\mathbb{1} \otimes P) = P \otimes P$. Note that (2.3) implies that $\Delta_{\mathbb{G}}(P)(\mathbb{1} \otimes P) \in \mathrm{M}(C_0(\mathbb{G}) \otimes \mathcal{K}(L^2(\mathbb{G})).$ In particular we have the following lemma.

LEMMA 2.6. *Let $P \in L^\infty(\mathbb{G})$ be a group-like projection. Then $P \in \mathrm{M}(C_0(\mathbb{G}))$.*

There is a 1-1 correspondence between (isomorphism classes of) open quantum subgroups of $\mathbb{G}$ and central group-like projections in $\mathbb{G}$ ([3], Theorem 4.3). The group-like projection assigned to $\mathbb{H}$, i.e. the central support of ρ, will be denoted by $\mathbb{1}_{\mathbb{H}}$.

3. FROM IDEMPOTENT STATES TO GROUP-LIKE PROJECTIONS

Let $\mathbb{G}$ be a locally compact quantum group and $\omega \in C_0^{\mathrm{u}}(\mathbb{G})^*$ an idempotent state on $\mathbb{G}$ and let $E_\omega : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ be the conditional expectation assigned to ω:

$$E_\omega(x) = \omega \,\overline{*}\, x.$$

We note that

$$\eta_{\mathbb{G}}(E_\omega(x)) = \eta_{\mathbb{G}}(\omega \,\overline{*}\, x) = (\mathrm{id} \otimes \omega)(\mathbb{W})\eta_{\mathbb{G}}(x),$$

where in the last equality we use Proposition 7.4 of [5]. The element $(\mathrm{id} \otimes \omega)(\mathbb{W}) \in L^\infty(\widehat{\mathbb{G}})$ is a hermitian projection which we denote by P_ω. In particular

$$\eta_{\mathbb{G}}(E_\omega(x)) = P_\omega \eta_{\mathbb{G}}(x). \tag{3.1}$$

Let $\mathsf{N} = E_\omega(L^\infty(\mathbb{G}))$ be the coideal assigned to ω. The set

$$E_\omega(\{(\mu \otimes \mathrm{id})(\mathrm{W}) : \mu \in L^\infty(\widehat{\mathbb{G}})_*\}) = \{(P_\omega \cdot \mu \otimes \mathrm{id})(\mathrm{W}) : \mu \in L^\infty(\widehat{\mathbb{G}})_*\} \tag{3.2}$$

is weakly dense in N.

Let us recall that $\widetilde{\mathsf{N}} \subset L^\infty(\widehat{\mathbb{G}})$ denotes the codual coideal of N. Since N is preserved by $\tau^{\mathbb{G}}$, $\widetilde{\mathsf{N}}$ is preserved by $\tau^{\widehat{\mathbb{G}}}$.

THEOREM 3.1. *Adopting the above notation we have*

$$\mathsf{N} = \{x \in L^\infty(\mathbb{G}) : P_\omega x = x P_\omega\} \quad \textit{and} \quad \widetilde{\mathsf{N}} = \{y \in L^\infty(\widehat{\mathbb{G}}) : \Delta_{\widehat{\mathbb{G}}}(y)(\mathbb{1} \otimes P_\omega) = y \otimes P_\omega\}.$$

Moreover, $P_\omega \in \widetilde{\mathsf{N}}$ is a minimal central projection of $\widetilde{\mathsf{N}}$ and it satisfies:

(i) $\tau_t^{\widehat{\mathbb{G}}}(P_\omega) = P_\omega$ *for all* $t \in \mathbb{R}$;

(ii) $R^{\widehat{\mathbb{G}}}(P_\omega) = P_\omega$;

(iii) $\sigma_t^{\widehat{\psi}}(P_\omega) = P_\omega$ *for all* $t \in \mathbb{R}$;

(iv) $\sigma_t^{\widehat{\varphi}}(P_\omega) = P_\omega$ *for all* $t \in \mathbb{R}$;
(v) $\Delta_{\widehat{\mathbb{G}}}(P_\omega)(\mathbb{1} \otimes P_\omega) = P_\omega \otimes P_\omega = \Delta_{\widehat{\mathbb{G}}}(P_\omega)(P_\omega \otimes \mathbb{1})$.

Proof. The equalities $\tau_t^{\widehat{\mathbb{G}}}(P_\omega) = P_\omega$ and $R^{\widehat{\mathbb{G}}}(P_\omega) = P_\omega$ follow easily from (2.7). Let $x \in L^\infty(\mathbb{G})$. Using (3.1) we see that the condition

$$P_\omega x = x P_\omega \tag{3.3}$$

holds if and only if

$$\eta_{\mathbb{G}}(E_\omega(xz)) = \eta_{\mathbb{G}}(xE_\omega(z))$$

for all $z \in \mathcal{N}_\psi$. The latter is equivalent to the identity $E_\omega(xz) = xE_\omega(z)$ holding for all $z \in \mathcal{N}_\psi$. Since $\mathcal{N}_\psi \subset L^\infty(\mathbb{G})$ forms a dense subset of $L^\infty(\mathbb{G})$, we see that (3.3) is equivalent with $E_\omega(x) = x$.

Using (3.2), we can see that $y \in \widetilde{\mathsf{N}}$ if and only if

$$(\mu \otimes \mathrm{id})((\mathbb{1} \otimes y)\mathrm{W}(P_\omega \otimes \mathbb{1})) = (\mu \otimes \mathrm{id})(\mathrm{W}(P_\omega \otimes y))$$

for all $\mu \in L^\infty(\widehat{\mathbb{G}})_*$. Equivalently $y \in \widetilde{\mathsf{N}}$ if and only if

$$\mathrm{W}^*(\mathbb{1} \otimes y)\mathrm{W}(P_\omega \otimes \mathbb{1}) = P_\omega \otimes y$$

which is in turn equivalent with

$$\Delta_{\widehat{\mathbb{G}}}(y)(\mathbb{1} \otimes P_\omega) = y \otimes P_\omega.$$

Since $P_\omega \in \widetilde{\mathsf{N}}$ we get $\Delta_{\widehat{\mathbb{G}}}(P_\omega)(\mathbb{1} \otimes P_\omega) = P_\omega \otimes P_\omega$.

Using Podleś condition $\widetilde{\mathsf{N}} = \{(\mu \otimes \mathrm{id})(\Delta_{\widehat{\mathbb{G}}}(y)) : y \in \widetilde{\mathsf{N}}, \mu \in L^\infty(\widehat{\mathbb{G}})_*\}^{\sigma\text{-weak cls}}$ we conclude that P_ω is a minimal central projection in $\widetilde{\mathsf{N}}$. Indeed, for all $y \in \widetilde{\mathsf{N}}$ and $\mu \in L^\infty(\widehat{\mathbb{G}})_*$ we have

$$(\mu \otimes \mathrm{id})(\Delta_{\widehat{\mathbb{G}}}(y))P_\omega = \mu(y)P_\omega = P_\omega(\mu \otimes \mathrm{id})(\Delta_{\widehat{\mathbb{G}}}(y)).$$

Thus $\widetilde{\mathsf{N}}P_\omega = \mathbb{C}P_\omega$ (i.e. P_ω is minimal in $\widetilde{\mathsf{N}}$) and $P_\omega \in Z(\widetilde{\mathsf{N}})$. Minimality and centrality of $P_\omega \in \widetilde{\mathsf{N}}$ yield a unique normal character $\varepsilon_\omega : \widetilde{\mathsf{N}} \to \mathbb{C}$ such that $yP_\omega = \varepsilon_\omega(y)P_\omega$ for all $y \in \widetilde{\mathsf{N}}$.

Using $\Delta_{\widehat{\mathbb{G}}} \circ \sigma_t^{\widehat{\psi}} = (\sigma_t^{\widehat{\psi}} \otimes \tau_{-t}^{\widehat{\mathbb{G}}}) \circ \Delta_{\widehat{\mathbb{G}}}$ (see Proposition 6.8 of [10]) we get

$$\begin{aligned}\Delta_{\widehat{\mathbb{G}}}(\sigma_t^{\widehat{\psi}}(P_\omega))(\mathbb{1} \otimes P_\omega) &= (\sigma_t^{\widehat{\psi}} \otimes \tau_{-t}^{\widehat{\mathbb{G}}})(\Delta_{\widehat{\mathbb{G}}}(P_\omega))(\mathbb{1} \otimes P_\omega)\\ &= (\sigma_t^{\widehat{\psi}} \otimes \tau_{-t}^{\widehat{\mathbb{G}}})(\Delta_{\widehat{\mathbb{G}}}(P_\omega)(\mathbb{1} \otimes P_\omega)) = \sigma_t^{\widehat{\psi}}(P_\omega) \otimes P_\omega\end{aligned}$$

and $\sigma_t^{\widehat{\psi}}(P_\omega) \in \widetilde{\mathsf{N}}$. In particular $P_\omega\sigma_t^{\widehat{\psi}}(P_\omega) = \varepsilon_\omega(\sigma_t^{\widehat{\psi}}(P_\omega))P_\omega$, where $\varepsilon_\omega(\sigma_t^{\widehat{\psi}}(P_\omega)) \in \{0, 1\}$ for all $t \in \mathbb{R}$. Since the map $\mathbb{R} \ni t \mapsto \varepsilon_\omega(\sigma_t^{\widehat{\psi}}(P_\omega)) \in \mathbb{R}$ is continuous and $\varepsilon_\omega(\sigma_t^{\widehat{\psi}}(P_\omega))|_{t=0} = 1$, we conclude that $P_\omega\sigma_t^{\widehat{\psi}}(P_\omega) = P_\omega$, i.e. $\sigma_t^{\widehat{\psi}}(P_\omega) \geqslant P_\omega$ for all $t \in \mathbb{R}$. Thus also $\sigma_{-t}^{\widehat{\psi}}(P_\omega) \leqslant P_\omega$ for all $t \in \mathbb{R}$ and $\sigma_t^{\widehat{\psi}}(P_\omega) = P_\omega$.

Since P_ω is preserved by $R^{\widehat{\mathbb{G}}}$, the identity $\Delta_{\widehat{\mathbb{G}}}(P_\omega)(\mathbb{1} \otimes P_\omega) = P_\omega \otimes P_\omega$ implies that

$$\Delta_{\widehat{\mathbb{G}}}(P_\omega)(P_\omega \otimes \mathbb{1}) = P_\omega \otimes P_\omega.$$

Finally using $\sigma_t^{\widehat{\varphi}} = R^{\widehat{\mathbb{G}}} \circ \sigma_{-t}^{\widehat{\psi}} \circ R^{\widehat{\mathbb{G}}}$ we get $\sigma_t^{\widehat{\varphi}}(P_\omega) = P_\omega$ for all $t \in \mathbb{R}$. ∎

For the concept of $\widehat{\mathbb{G}}$-generation used in the next lemma, see Definition 2.5.

LEMMA 3.2. *Let* $\omega \in C_0^{\mathrm{u}}(\mathbb{G})^*$ *be an idempotent state,* $\mathsf{N} = E_\omega(L^\infty(\mathbb{G}))$ *the corresponding coideal and* $\widetilde{\mathsf{N}} \subset L^\infty(\widehat{\mathbb{G}})$ *the codual of* N. *Then* $\widetilde{\mathsf{N}}$ *is* $\widehat{\mathbb{G}}$*-generated by* $P_\omega \in \widetilde{\mathsf{N}}$.

Proof. Let us recall that $x \in \mathsf{N}$ if and only if $x \in L^\infty(\mathbb{G})$ and $xP_\omega = P_\omega x$. Let $\widehat{V} = (J \otimes J)\mathrm{W}^*(J \otimes J) \in L^\infty(\widehat{\mathbb{G}}) \bar{\otimes} L^\infty(\mathbb{G})'$ where $J : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ is the Tomita–Takesaki antiunitary conjugation assigned to ψ. Then for all $y \in L^\infty(\widehat{\mathbb{G}})$ we have

$$\Delta_{\widehat{\mathbb{G}}}(y) = \widehat{V}^*(\mathbb{1} \otimes y)\widehat{V}.$$

In particular if $x \in L^\infty(\mathbb{G})$ and $P_\omega x = xP_\omega$ then

$$\Delta_{\widehat{\mathbb{G}}}(P_\omega)(\mathbb{1} \otimes x) = \widehat{V}^*(\mathbb{1} \otimes P_\omega)\widehat{V}(\mathbb{1} \otimes x) = (\mathbb{1} \otimes x)\Delta_{\widehat{\mathbb{G}}}(P_\omega). \tag{3.4}$$

Conversely, if (3.4) holds then

$$P_\omega \otimes P_\omega x = \Delta_{\widehat{\mathbb{G}}}(P_\omega)(\mathbb{1} \otimes x)(P_\omega \otimes \mathbb{1}) = (\mathbb{1} \otimes x)\Delta_{\widehat{\mathbb{G}}}(P_\omega)(P_\omega \otimes \mathbb{1}) = P_\omega \otimes xP_\omega$$

and we get $P_\omega x = xP_\omega$. In particular $\mathsf{N} = \mathsf{S}' \cap L^\infty(\mathbb{G})$, where

$$\mathsf{S} = \{(\mu \otimes \mathrm{id})(\Delta_{\widehat{\mathbb{G}}}(P_\omega)) : \mu \in L^\infty(\widehat{\mathbb{G}})_*\}.$$

Let us note that S'' is the smallest coideal of $L^\infty(\widehat{\mathbb{G}})$ containing P_ω (see Theorem 2.3). Since $\mathsf{N} = \mathsf{S}' \cap L^\infty(\mathbb{G}) = (\mathsf{S}'')' \cap L^\infty(\mathbb{G})$ we get $\mathsf{S}'' = \widetilde{\mathsf{N}}$. ∎

LEMMA 3.3. *Adopting the above notation we have* $\tau_t^{\widehat{\mathbb{G}}}(x) = \sigma_t^{\widehat{\varphi}}(x)$ *for all* $x \in \widetilde{\mathsf{N}}$ *and* $t \in \mathbb{R}$.

Proof. Using the formula $\Delta^{\widehat{\mathbb{G}}} \circ \sigma_t^{\widehat{\varphi}} = (\tau_t^{\widehat{\mathbb{G}}} \otimes \sigma_t^{\widehat{\varphi}}) \circ \Delta^{\widehat{\mathbb{G}}}$ (see Proposition 5.38 of [10]) and $\Delta^{\widehat{\mathbb{G}}} \circ \tau_t^{\widehat{\mathbb{G}}} = (\tau_t^{\widehat{\mathbb{G}}} \otimes \tau_t^{\widehat{\mathbb{G}}}) \circ \Delta^{\widehat{\mathbb{G}}}$ (see Result 5.12 of [10]), we conclude that for all $\mu \in L^\infty(\widehat{\mathbb{G}})_*$

$$\tau_t^{\widehat{\mathbb{G}}}((\mu \otimes \mathrm{id})(\Delta_{\widehat{\mathbb{G}}}(P_\omega))) = \sigma_t^{\widehat{\varphi}}((\mu \otimes \mathrm{id})(\Delta_{\widehat{\mathbb{G}}}(P_\omega)))$$

(note that for the latter we also use $\tau^{\widehat{\mathbb{G}}}$-invariance and $\sigma^{\widehat{\varphi}}$-invariance of P_ω). Since $\widetilde{\mathsf{N}}$ is $\widehat{\mathbb{G}}$-generated by P_ω, we are done. ∎

Next result is a strengthening of Lemma 3.2.

THEOREM 3.4. *Adopting the assumptions and notation of Lemma* 3.2 *we have*

$$\widetilde{\mathsf{N}} = \overline{\{(\mu \otimes \mathrm{id})(\Delta_{\widehat{\mathbb{G}}}(P_\omega)) : \mu \in L^\infty(\widehat{\mathbb{G}})_*\}}^{\,\mathrm{weak}}. \tag{3.5}$$

Proof. From $\tau^{\widehat{\mathbb{G}}}$-invariance of $\widetilde{\mathsf{N}}$ it follows that $\widetilde{\mathsf{N}} \cap D(S_{\widehat{\mathbb{G}}}^{-1})$ is a dense subset of $\widetilde{\mathsf{N}}$. Suppose that $x \in \widetilde{\mathsf{N}} \cap D(S_{\widehat{\mathbb{G}}}^{-1})$. We shall prove that

$$\Delta_{\widehat{\mathbb{G}}}(P_\omega)(\mathbb{1} \otimes x) = \Delta_{\widehat{\mathbb{G}}}(P_\omega)(S_{\widehat{\mathbb{G}}}^{-1}(x) \otimes \mathbb{1}). \tag{3.6}$$

From this, it follows that $\overline{\{(\mu\otimes\mathrm{id})(\Delta_{\widehat{\mathbb{G}}}(P_\omega)) : \mu\in L^\infty(\widehat{\mathbb{G}})_*\}}^{\,\mathrm{weak}}$ is an ideal in $\widetilde{\mathsf{N}}$ (in particular a von Neumann subalgebra of $\widetilde{\mathsf{N}}$). It is also easy to check that the right hand side of equation (3.5) is $\widehat{\mathbb{G}}$-invariant. By ergodicity of the action of $\widehat{\mathbb{G}}$ on $\widetilde{\mathsf{N}}$, we conclude that equation (3.5) holds (here we use the same argument as in the final part of the proof of Theorem 3.3 in [3]). It remains to prove equation (3.6). To this end, we continue assuming that x is $\tau^{\widehat{\mathbb{G}}}$-analytic. Note that by Corollary 3.3, it is also $\sigma^{\widehat{\varphi}}$-analytic. Let $a,b\in\mathcal{N}_{\widehat{\varphi}}$. We compute

$$
\begin{aligned}
&(\mathrm{id}\otimes\widehat{\varphi})((\mathbb{1}\otimes a^*)\Delta_{\widehat{\mathbb{G}}}(bP_\omega)(S_{\widehat{\mathbb{G}}}(x)\otimes\mathbb{1}))\\
&=S_{\widehat{\mathbb{G}}}((\mathrm{id}\otimes\widehat{\varphi})((x\otimes\mathbb{1})\Delta_{\widehat{\mathbb{G}}}(a^*)(\mathbb{1}\otimes bP_\omega)))=S_{\widehat{\mathbb{G}}}((\mathrm{id}\otimes\widehat{\varphi})((x\otimes P_\omega)\Delta_{\widehat{\mathbb{G}}}(a^*)(\mathbb{1}\otimes b)))\\
&=S_{\widehat{\mathbb{G}}}((\mathrm{id}\otimes\widehat{\varphi})((\mathbb{1}\otimes P_\omega)\Delta_{\widehat{\mathbb{G}}}(xa^*)(\mathbb{1}\otimes b)))=S_{\widehat{\mathbb{G}}}((\mathrm{id}\otimes\widehat{\varphi})(\Delta_{\widehat{\mathbb{G}}}(xa^*)(\mathbb{1}\otimes bP_\omega)))\\
&=(\mathrm{id}\otimes\widehat{\varphi})((\mathbb{1}\otimes xa^*)\Delta_{\widehat{\mathbb{G}}}(bP_\omega))=(\mathrm{id}\otimes\widehat{\varphi})((\mathbb{1}\otimes a^*)\Delta_{\widehat{\mathbb{G}}}(bP_\omega)(\mathbb{1}\otimes\sigma^{\widehat{\varphi}}_{-\mathrm{i}}(x)))
\end{aligned}
$$

where in the first and the fifth equality, we use equation (2.1) and in the second and the fourth equality, we use $\sigma^{\widehat{\varphi}}$-invariance of P_ω. Thus we get

$$\Delta_{\widehat{\mathbb{G}}}(P_\omega)(S_{\widehat{\mathbb{G}}}(x)\otimes\mathbb{1})=\Delta_{\widehat{\mathbb{G}}}(P_\omega)(\mathbb{1}\otimes\sigma^{\widehat{\varphi}}_{-\mathrm{i}}(x)).$$

Replacing x with $\sigma^{\widehat{\varphi}}_{\mathrm{i}}(x)$ and using Corollary 3.3, we get (3.6) for $\tau^{\widehat{\mathbb{G}}}$-analytic x. Since the space of $\tau^{\widehat{\mathbb{G}}}$-analytic elements forms a core of $S^{-1}_{\widehat{\mathbb{G}}}$ we get (3.6). ∎

Theorem 3.4 is a generalization of Theorem 3.3 in [3]. Note that in the proof of Theorem 3.3 in [3], which treats the case of central P_ω, a small mistake was done where instead of equation (3.6) the following formula was derived:

$$\Delta_{\widehat{\mathbb{G}}}(P_\omega)(\mathbb{1}\otimes x)=\Delta_{\widehat{\mathbb{G}}}(P_\omega)(R_{\widehat{\mathbb{G}}}(x)\otimes\mathbb{1}).$$

The next theorem was first proved in Theorem 5.15 of [4]. The previous proof strongly uses the universal C^*-version of $\mathbb{G}$. In what follows we give a simpler proof which is based on the von Neumann version of $\mathbb{G}$.

THEOREM 3.5. *Let $\mathsf{N}\subset L^\infty(\mathbb{G})$ be an integrable normal coideal preserved by $\tau^{\mathbb{G}}$. Then there exists a unique compact quantum subgroup $\mathbb{H}\subset\mathbb{G}$ such that $\mathsf{N}=L^\infty(\mathbb{G}/\mathbb{H})$.*

Proof. Using Theorem 4.2 of [4] we conclude the existence of an idempotent state $\omega\in C^{\mathrm{u}}_0(\mathbb{G})^*$ such that $\mathsf{N}=E_\omega(L^\infty(\mathbb{G}))$. Let $\widetilde{\mathsf{N}}$ be the codual coideal. Then, since N is preserved by $\tau^{\mathbb{G}}$, $\widetilde{\mathsf{N}}$ is preserved by $\tau^{\widehat{\mathbb{G}}}$ (see Proposition 3.2 of [7]). Normality of N is equivalent with $\Delta_{\widehat{\mathbb{G}}}(\widetilde{\mathsf{N}})\subset\widetilde{\mathsf{N}}\bar{\otimes}\widetilde{\mathsf{N}}$ (see Proposition 4.3 of [7]). Moreover, using Theorem 3.4 we see that

$$\mathsf{S}=\{(\mu\otimes\mathrm{id})(\Delta_{\widehat{\mathbb{G}}}(P_\omega)) : \mu\in L^\infty(\widehat{\mathbb{G}})_*\}$$

is weakly dense in $\widetilde{\mathsf{N}}$. Let us note that $R^{\widehat{\mathbb{G}}}((\mu\otimes\mathrm{id})(\Delta_{\widehat{\mathbb{G}}}(P_\omega))=(\mathrm{id}\otimes\mu\circ R^{\widehat{\mathbb{G}}})(\Delta_{\widehat{\mathbb{G}}}(P_\omega))$. Since $P_\omega\in\widetilde{\mathsf{N}}$ we have $\Delta_{\widehat{\mathbb{G}}}(P_\omega)\in\widetilde{\mathsf{N}}\bar{\otimes}\widetilde{\mathsf{N}}$ and we see that $R^{\widehat{\mathbb{G}}}(\mathsf{S})\subset\widetilde{\mathsf{N}}$. Thus we conclude that $R^{\widehat{\mathbb{G}}}(\widetilde{\mathsf{N}})\subset\widetilde{\mathsf{N}}$. Summarizing $\widetilde{\mathsf{N}}$ forms a Baaj–Vaes subalgebra of $L^\infty(\widehat{\mathbb{G}})$

and there exists $\mathbb{H} \subset \mathbb{G}$ such that $\widetilde{\mathsf{N}} = L^\infty(\widehat{\mathbb{H}})$. Since $\mathsf{N} = L^\infty(\mathbb{G}/\mathbb{H})$ is integrable, we use Theorem A.3 of [5] for concluding that $\mathbb{H}$ is a compact quantum group. ∎

4. FROM GROUPS-LIKE PROJECTIONS TO IDEMPOTETNT STATES

Let ψ be an n.s.f. weight on a von Neumann algebra N and $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathsf{N})$ the KMS-group of automorphisms assigned to ψ. We denote

$$\mathcal{T}_\psi = \{x \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^* : x \text{ is } \sigma\text{-analytic and } \sigma_z(x) \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^* \text{ for all } z \in \mathbb{C}\}.$$

Note that if $x \in \mathcal{T}_\psi$ then $\sigma_z(x) \in \mathcal{T}_\psi$ for all $z \in \mathbb{C}$. Let us recall that the KMS-condition for σ yields that if $x \in \mathcal{N}_\psi \cap \mathrm{Dom}(\sigma_{\mathrm{i}/2})$ then $\sigma_{\mathrm{i}/2}(x)^* \in \mathcal{N}_\psi$ and

$$\psi(x^*x) = \psi(\sigma_{\mathrm{i}/2}(x)\sigma_{\mathrm{i}/2}(x)^*). \tag{4.1}$$

LEMMA 4.1. *Let $x \in \mathcal{T}_\psi$ and suppose that y is σ-analytic. Then $yx \in \mathcal{T}_\psi$.*

Proof. Let $x \in \mathcal{T}_\psi$. Clearly yx is σ-analytic. Since $\mathcal{N}_\psi$ forms a left ideal in N we have $yx \in \mathcal{N}_\psi$. Moreover $(yx)^*$ is also σ-analytic and

$$\begin{aligned}\psi((yx)^{**}(yx)^*) &= \psi(\sigma_{\mathrm{i}/2}((yx)^*)\sigma_{\mathrm{i}/2}((yx)^*)^*) = \psi(\sigma_{-i/2}(yx)^*\sigma_{-i/2}(yx)) \\ &= \psi(\sigma_{-i/2}(x)^*\sigma_{-i/2}(y)^*\sigma_{-i/2}(y)\sigma_{-i/2}(x)) \\ &\leqslant \|\sigma_{-i/2}(y)\|^2\psi(\sigma_{-i/2}(x)^*\sigma_{-i/2}(x)) < \infty.\end{aligned}$$

Thus we get $yx \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^*$. Replacing x with $\sigma_z(x)$ and y with $\sigma_z(y)$ in the above reasoning, we conclude that $\sigma_z(yx) \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^*$. Thus $yx \in \mathcal{T}_\psi$ and we are done. ∎

REMARK 4.2. Let $\mathbb{G}$ be a locally compact quantum group. In the course of the proof of the next theorem, the symbol $\widehat{\eta}$ denotes the GNS map for the Haar weight $\widehat{\psi}$ on $\widehat{\mathbb{G}}$. We will use the fact that if $a, b \in \mathcal{T}_{\widehat{\psi}}$ then the slice $(\mu_{\widehat{\eta}(a),\widehat{\eta}(b)} \otimes \mathrm{id})(\mathrm{W})$ is an element of $\mathcal{N}_\psi$ (see Lemma 8.4 and Proposition 8.13 of [10] with the roles of $\mathbb{G}$ and $\widehat{\mathbb{G}}$ reversed).

THEOREM 4.3. *Let $\mathbb{G}$ be a locally compact quantum group and let $P \in L^\infty(\widehat{\mathbb{G}})$ be a non-zero group-like projection such that $\tau_t^{\widehat{\mathbb{G}}}(P) = P$ for all $t \in \mathbb{R}$. Then there exists an idempotent state $\omega \in \mathrm{C}_0^{\mathrm{u}}(\mathbb{G})$ such that $P = (\mathrm{id} \otimes \omega)(\mathbb{W})$.*

Proof. Let us consider $\widetilde{\mathsf{N}} \subset L^\infty(\widehat{\mathbb{G}})$, where

$$\widetilde{\mathsf{N}} = \{y \in L^\infty(\widehat{\mathbb{G}}) : \Delta_{\widehat{\mathbb{G}}}(y)(\mathbb{1} \otimes P) = y \otimes P \text{ and } \Delta_{\widehat{\mathbb{G}}}(y^*)(\mathbb{1} \otimes P) = y^* \otimes P\}.$$

We will show that $\widetilde{\mathsf{N}}$ forms a coideal in $L^\infty(\widehat{\mathbb{G}})$ and we will focus on its codual $\mathsf{N} \subset L^\infty(\mathbb{G})$. Let us first note that $P \in \widetilde{\mathsf{N}}$ and $\widetilde{\mathsf{N}}$ is $\tau^{\widehat{\mathbb{G}}}$-invariant. Moreover it is easy to see that $\widetilde{\mathsf{N}}$ is a von Neumann subalgebra of $L^\infty(\widehat{\mathbb{G}})$. Let us check that $\widetilde{\mathsf{N}}$

forms a coideal of $L^\infty(\widehat{\mathbb{G}})$. For $y \in \widetilde{\mathsf{N}}$ we have

$$\begin{aligned}(\mathrm{id}\otimes\Delta_{\widehat{\mathbb{G}}})&(\Delta_{\widehat{\mathbb{G}}}(y))(\mathbb{1}\otimes\mathbb{1}\otimes P)\\ &= (\Delta_{\widehat{\mathbb{G}}}\otimes\mathrm{id})(\Delta_{\widehat{\mathbb{G}}}(y))(\mathbb{1}\otimes\mathbb{1}\otimes P) = (\Delta_{\widehat{\mathbb{G}}}\otimes\mathrm{id})(\Delta_{\widehat{\mathbb{G}}}(y)(\mathbb{1}\otimes P))\\ &= (\Delta_{\widehat{\mathbb{G}}}\otimes\mathrm{id})(y\otimes P) = \Delta_{\widehat{\mathbb{G}}}(y)\otimes P.\end{aligned}$$

Similarly we show that $(\mathrm{id}\otimes\Delta_{\widehat{\mathbb{G}}})(\Delta_{\widehat{\mathbb{G}}}(y)^*)(\mathbb{1}\otimes\mathbb{1}\otimes P) = \Delta_{\widehat{\mathbb{G}}}(y)^*\otimes P$ and we get $\Delta_{\widehat{\mathbb{G}}}(y) \in L^\infty(\widehat{\mathbb{G}})\bar{\otimes}\widetilde{\mathsf{N}}$. Repeating the reasoning presented in the fourth paragraph of the proof of Theorem 3.1 we conclude that P is a minimal central projection of $\widetilde{\mathsf{N}}$. Using $\tau_t^{\widehat{\mathbb{G}}}$ invariance of P and repeating the reasoning presented in the fifth paragraph of the proof of Theorem 3.1, we see that $\sigma_t^{\widehat{\psi}}(P) = P$. In particular P is $\sigma^{\widehat{\psi}}$-analytic.

Let $\mathsf{N} \subset L^\infty(\mathbb{G})$ denote the codual of $\widetilde{\mathsf{N}}$. Since $\widetilde{\mathsf{N}}$ is preserved by $\tau^{\widehat{\mathbb{G}}}$, N is preserved by $\tau^{\mathbb{G}}$. Moreover following backward the reasoning presented in the third paragraph of the proof of Theorem 3.1 we show that $(P\cdot\mu\otimes\mathrm{id})(\mathrm{W}) \in \mathsf{N}$ for all $\mu \in L^\infty(\widehat{\mathbb{G}})_*$.

Let $a, b \in \mathcal{T}_{\widehat{\psi}}$ and let us consider $\mu = \mu_{\widehat{\eta}(a),\widehat{\eta}(b)} \in L^\infty(\widehat{\mathbb{G}})_*$ and $x = (P\cdot\mu\otimes\mathrm{id})(\mathrm{W})$ (note that $P\cdot\mu = \mu_{\widehat{\eta}(a),\widehat{\eta}(Pb)}$). Using Lemma 4.1, we see that $Pb \in \mathcal{T}_{\widehat{\psi}}$. In particular, as explained in Remark 4.2, $x \in \mathcal{N}_\psi$. Clearly there exists $a, b \in \mathcal{T}_{\widehat{\psi}}$ such that the corresponding x is non-zero. Indeed, suppose the converse holds: $(P\cdot\mu_{\widehat{\eta}(a),\widehat{\eta}(b)}\otimes\mathrm{id})(\mathrm{W}) = 0$ for all $a, b \in \mathcal{T}_{\widehat{\psi}}$. Then $P\cdot\mu_{\widehat{\eta}(a),\widehat{\eta}(b)}(y) = 0$ for all $y \in L^\infty(\widehat{\mathbb{G}})$. Thus, taking $y = \mathbb{1}$ we get $(\widehat{\eta}(a)|P\widehat{\eta}(b)) = 0$ for all $a, b \in \mathcal{T}_{\widehat{\psi}}$. Since $\widehat{\eta}(\mathcal{T}_{\widehat{\psi}})$ is dense in $L^2(\mathbb{G})$, we conclude that $P = 0$, contradiction. In particular N contains a nonzero element $x \in \mathsf{N}\cap\mathcal{N}_\psi$. Since $(\psi\otimes\mathrm{id})\Delta_{\mathbb{G}}(x^*x) = \psi(x^*x)$ we see that N contains a non-zero integrable element with respect to the action $\Delta_{\mathbb{G}}|_{\mathsf{N}}$ and using Proposition 3.2 of [5] we conclude that N is integrable.

Summarizing, N is an integrable coideal of $L^\infty(\mathbb{G})$ preserved by $\tau^{\mathbb{G}}$. Using Theorem 4.2 of [4] we see that there exists an idempotent state $\omega \in C_0^{\mathrm{u}}(\mathbb{G})^*$ such that $\mathsf{N} = E_\omega(L^\infty(\mathbb{G}))$, where E_ω is the conditional expectation assigned to ω.

Let $P_\omega = (\mathrm{id}\otimes\omega)(\mathbb{W})$. Then $P_\omega \in \widetilde{\mathsf{N}}$ is a minimal central projection. Moreover,

$$(P\cdot\mu\otimes\mathrm{id})(\mathrm{W}) = E_\omega((P\cdot\mu\otimes\mathrm{id})(\mathrm{W})) = (P_\omega P\cdot\mu\otimes\mathrm{id})(\mathrm{W})$$

for all $\mu \in L^\infty(\widehat{\mathbb{G}})_*$. Thus $P = P_\omega P$ and we see that $P_\omega \geqslant P$. Using the minimality of P_ω we get $P_\omega = P$. ∎

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