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EXISTENCE OF COMMON HYPERCYCLIC VECTORS FOR TRANSLATION OPERATORS

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ABSTRACT. Let $\mathcal{H}(\mathbb{C})$ be the set of entire functions endowed with the topology $\mathcal{T}_{\mathrm{u}}$ of local uniform convergence. Fix a sequence of non-zero complex numbers (λ_n) with $|\lambda_n|\to+\infty$ and $|\lambda_{n+1}|/|\lambda_n|\to 1$. We prove that there exists a residual set $G\subset\mathcal{H}(\mathbb{C})$ so that for every $f\in G$ and every non-zero complex number a the set $\{f(z+\lambda_n a) : n=1,2,\dots\}$ is dense in $(\mathcal{H}(\mathbb{C}),\mathcal{T}_{\mathrm{u}})$. This provides a very strong extension of a theorem by G. Costakis and M. Sambarino in *Adv. Math.* **182**(2004), 278–306. Actually, in that article, the above result is proved only for the case $\lambda_n=n$. Our result is in a sense best possible, since there exist sequences (λ_n), with $|\lambda_{n+1}|/|\lambda_n|\to l$ for certain $l>1$, for which the above result fails to hold, cf. F. Bayart, *Int. Math. Res. Notices* **21**(2016), 6512–6552.

KEYWORDS: *Hypercyclic operator, common hypercyclic functions, translation operator.*

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1. INTRODUCTION

Let us first fix some standard notation and terminology. Throughout this paper, we denote $\mathbb{N}=\{1,2,\dots\}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ for the sets of natural, rational, real and complex numbers, respectively. By $\mathcal{H}(\mathbb{C})$ we denote the set of entire functions endowed with the topology $\mathcal{T}_{\mathrm{u}}$ of local uniform convergence. For a subset A of $\mathcal{H}(\mathbb{C})$ the symbol $\overline{A}$ denotes the closure of A with respect to the topology $\mathcal{T}_{\mathrm{u}}$. Let X be a topological vector space. A subset G of X is called G_δ if it can be written as a countable intersection of open sets in X and a subset Y of X is called residual if it contains a G_δ and dense subset of X.

A classical result of Birkhoff [14], which goes back to 1929, says that there exist entire functions of which the integer translates are dense in the space of all entire functions endowed with the topology $\mathcal{T}_{\mathrm{u}}$ of local uniform convergence (see also Luh [33] for a more general statement). Birkhoff's proof was constructive.

Much later, during the 80's, Gethner and Shapiro [28] and independently Grosse-Erdmann [29] showed that Birkhoff's result can be recovered as a particular case of a much more general theorem, through the use of Baire's category theorem. This approach simplified Birkhoff's argument substantially and in addition gave us precise information on the topological size of these functions. In particular, Grosse-Erdmann proved that for every fixed sequence of complex numbers (w_n) with $w_n \to \infty$, the set

$$\{f \in \mathcal{H}(\mathbb{C}) \mid \overline{\{f(z+w_n) : n \in \mathbb{N}\}} = \mathcal{H}(\mathbb{C})\}$$

is G_δ and dense in $\mathcal{H}(\mathbb{C})$, and hence "large" in the topological sense.

Let us now rephrase the above results using the modern language of hypercyclicity. Let $(T_n : X \to X)$ be a sequence of continuous linear operators on a topological vector space X. For $x \in X$, the set $\mathrm{Orb}(\{T_n\}, x) := \{T_n(x) : n = 1, 2, \ldots\}$ is called the *orbit* of x under (T_n). If $(T_n(x))_{n\geqslant 1}$ is dense in X for some $x \in X$, then x is called *hypercyclic* for (T_n) and we say that (T_n) is *hypercyclic* [12], [31]. The symbol $HC(\{T_n\})$ stands for the collection of all hypercyclic vectors for (T_n). In the case where the sequence (T_n) comes from the iterates of a single operator $T : X \to X$, i.e. $T_n := T^n$, then we simply say that T is *hypercyclic* and x is *hypercyclic* for T. If $T : X \to X$ is hypercyclic, then the symbol $HC(T)$ stands for the collection of all hypercyclic vectors for T. Following the standard terminology, for an operator T on X, the set $\mathrm{Orb}(T, x) := \{x, T(x), T^2(x), \ldots\}$ is called the *orbit* of x under T. A simple consequence of Baire's category theorem is that for every continuous linear operator T on a separable topological vector space X, if $HC(T)$ is non-empty, then it is necessarily (G_δ and) dense. For an account of results on the subject of hypercyclicity, we refer to the recent books [12], [31]; also see the very influential survey article [30].

For every fixed $a \in \mathbb{C} \setminus \{0\}$ consider the translation operator $T_a : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ defined by

$$T_a(f)(z) = f(z+a), \quad f \in \mathcal{H}(\mathbb{C}), z \in \mathbb{C}.$$

Thus, for $a = 1$ Birkhoff's result says that T_1 is hypercyclic. We note that the choice $a = 1$ is not significant. The same proof works nicely for every $a \in \mathbb{C} \setminus \{0\}$, that is, for every a, T_a is hypercyclic and hence $HC(T_a)$ is G_δ and dense in $\mathcal{H}(\mathbb{C})$.

Recently, Costakis and Sambarino [24] established a notable strengthening of Birkhoff's result. Namely, they showed that, for almost all entire functions f, in the sense of the Baire category, the set of the translates of f with respect to na, $n \in \mathbb{N}$, is dense in the space of all entire functions for every non-zero complex number a. The significant new element here is the uncountable range of a. In the language of hypercyclicity, their result takes the following form: the family $\{T_a : a \in \mathbb{C} \setminus \{0\}\}$ has a residual set of common hypercyclic vectors i.e.,

$$\text{the set} \bigcap_{a\in\mathbb{C}\setminus\{0\}} HC(T_a) \text{ is residual in } \mathcal{H}(\mathbb{C}),$$

or equivalently, the set

$$\bigcap_{a\in\mathbb{C}\setminus\{0\}} \{f \in \mathcal{H}(\mathbb{C}) \mid \overline{\{f(z+na) : n \in \mathbb{N}\}} = \mathcal{H}(\mathbb{C})\}$$

is residual in $\mathcal{H}(\mathbb{C})$. In particular, it is non-empty.

Subsequently, Costakis [21] asked whether, in this result, n can be replaced by more general sequences (λ_n) of non-zero complex numbers.

QUESTION 1.1 ([21]). Fix a sequence (λ_n) of non-zero complex numbers so that $|\lambda_n|\to\infty$. Are there entire functions f so that, *for all* $a \in \mathbb{C}\setminus\{0\}$, the set $\{f(z+\lambda_n a) : n \in \mathbb{N}\}$ is dense in the space of all entire functions?

In this direction Costakis [21] showed that, if the sequence (λ_n) satisfies a certain condition, then the desired conclusion holds if we restrict attention to $a \in C(0,1) := \{z \in \mathbb{C}/|z| = 1\}$. The precise condition is that for every $M > 0$, there exists a subsequence (λ_{n_k}) of (λ_n) so that

(i) $|\lambda_{n_{k+1}}| - |\lambda_{n_k}| > M$ for every $k = 1, 2, \ldots$ and

(ii) $\sum\limits_{k=1}^{\infty} (1/|\lambda_{n_k}|) = +\infty$.

Obviously, sequences of the form $\lambda_n = bn + c$, where $b, c \in \mathbb{C}$, $b \neq 0$, $\lambda_n = n(\log n)^p$, where $0 < p \leqslant 1$ or $\lambda_n = n \log n \log\log n$, etc., satisfy the above condition. Costakis asked [21] whether the same result holds for the set $\mathbb{C}\setminus\{0\}$ instead of $C(0,1)$. We proved in [40] that this question has a positive reply. On the other hand, the case where the sequence λ_n is sparse, say n^2, is left open, since in this case condition (ii) is not satisfied. And in reality this is not accidental; it reflects the limitation of the method developed in [21]. This drawback is due to a specific "one-dimensional partition" that the author chooses. Here we overcome this difficulty by constructing a "two dimensional" partition, which turns out to be the right one in order to handle sequences (λ_n) where the corresponding series in condition (ii) converges. The purpose of this paper is to give a positive answer in general for $a \in \mathbb{C}\setminus\{0\}$ that applies to a wide family of sequences (λ_n). In particular, our main result, Theorem 1.2, covers the case where (λ_n) is of the form $(p(n))$, and p is any non-constant complex polynomial, as well as the case where $\lambda_n = \mathrm{e}^{n^b}$ for $0 < b < 1$; hence for every $0 < b < 1$ we have

$$\bigcap_{a\in\mathbb{C}\setminus\{0\}} \{f \in \mathcal{H}(\mathbb{C}) \mid \overline{\{f(z+\mathrm{e}^{n^b}a) : n \in \mathbb{N}\}} = \mathcal{H}(\mathbb{C})\} \neq \emptyset.$$

We would like to stress that the allowed growth e^{n^b}, $0 < b < 1$ in the previously mentioned example is in a sense optimal, since the answer to the above question is negative if (λ_n) grows exponentially, [7], [25], that is,

$$\bigcap_{a\in C(0,1)} \{f \in \mathcal{H}(\mathbb{C}) \mid \overline{\{f(z+\mathrm{e}^{n}a) : n \in \mathbb{N}\}} = \mathcal{H}(\mathbb{C})\} = \emptyset.$$

So some restriction on the nature of (λ_n) is clearly necessary.

Let (λ_n) be a sequence of non-zero complex numbers. We attach a non-negative real number to the sequence $\Lambda := (\lambda_n)$ as the following:

$$i(\Lambda) := \inf\Big\{a \in \mathbb{R} \cup \{+\infty\} : \text{there exists a subsequence } (\mu_n) \text{ of } (\lambda_n) \text{ so that } a = \limsup_{n\to+\infty} |\mu_{n+1}/\mu_n|\Big\}.$$

Of course $i(\Lambda) \in [0, +\infty]$. If $\lambda_n \to \infty$ as $n \to +\infty$ then $i(\Lambda) \in [1, +\infty]$. Our main result is the following theorem.

THEOREM 1.2. *Let $\Lambda := (\lambda_n)$ be a fixed sequence of non-zero complex numbers such that $\lambda_n \to \infty$ as $n \to +\infty$ and $i(\Lambda) = 1$. Then, the set*

$$\bigcap_{a \in \mathbb{C}\setminus\{0\}} HC(\{T_{\lambda_n a}\})$$

is a G_δ, dense subset of $(\mathcal{H}(\mathbb{C}), \mathcal{T}_{\mathrm{u}})$. In particular, there exists $f \in \mathcal{H}(\mathbb{C})$ such that for every $a \in \mathbb{C} \setminus \{0\}$

$$\overline{\{f(z + \lambda_n a) : n = 1, 2, \ldots\}} = \mathcal{H}(\mathbb{C}).$$

All the work in this article has been done in order to prove Theorem 1.2. F. Bayart examined similar problems recently in his important paper [7]. Here seems to be an appropriate place to comment on the ideas developed in [21], [24] and to compare them with our approach. Costakis and Sambarino's result mentioned above consists of two steps. The first one is to show that the set $\bigcap_{a \in C(0,1)} HC(T_a)$ is residual in $\mathcal{H}(\mathbb{C})$. This is accomplished by choosing a suitable partition of the unit circle $C(0, 1)$ and then an application of Runge's theorem on specific compact sets depending on the partition, which concludes the argument. We stress, however that what we just mentioned is a very rough idea of their proof. In the second step they show that for any fixed $\theta \in \mathbb{R}$, $HC(T_{e^{i\theta}}) = HC(T_{re^{i\theta}})$, for every $r > 0$. The proof of the latter is based on two important results: the minimality of the irrational rotation, see for instance [26], and Ansari's theorem [2], which says that if T is hypercyclic, then for every $n \in \mathbb{N}$, T^n is hypercyclic, as is $HC(T) = HC(T^n)$. One key element to prove Ansari's theorem, is that the orbit $\mathrm{Orb}(T, x)$ has a semigroup structure, that is, if $T^n(x), T^m(x) \in \mathrm{Orb}(T, x)$ then $T^n \circ T^m(x) \in \mathrm{Orb}(T, x)$. Some nice extensions of Ansari's theorem even in a non-linear setting, can be found in [34], [37], where the semigroup structure property still plays important role in the proofs. Observe now that in our case, say $\Lambda := (\lambda_n)$, $\lambda_n \to \infty$ and assume for simplicity $\lambda_n \in \mathbb{N}$, the semigroup structure of the orbit breaks down. The very simple reason for this "unpleasant" situation is that we now need consider parts of the full orbit $\{f(z + an) : n = 1, 2, \ldots\}$, which may be very sparse. For instance, consider the sequence $\lambda_n = n^2$ (for which Theorem 1.2 holds). Clearly for $a \in \mathbb{C} \setminus \{0\}$, $f \in \mathcal{H}(\mathbb{C})$, we have $T_{m^2 a} \circ T_{l^2 a}(f) \notin \mathrm{Orb}(\{T_{n^2 a}\}, f)$ in general. In view of this obstacle, we need to follow a different approach and therefore we tried to concentrate on the first step in Costakis and Sambarino's approach. Now the problem is how to find a suitable partition, not only for the set $C(0,1)$, which is quite "thin", but for any given bounded sector S. So our main task is: for a given sequence (λ_n) satisfying the hypothesis of Theorem 1.2, and a given bounded sector $S \subset \mathbb{C}$ to find a suitable partition of S in order to show that the set $\bigcap_{a \in S} HC(\{T_{\lambda_n a}\})$ is G_δ and dense in $\mathcal{H}(\mathbb{C})$. Then, covering the complex plane by many countable such sectors and applying Baire's category theorem, we are done. We mention that the second step of Costakis and Sambarino's result can be also obtained as a particular case of a general result due to Conejero, Müller and Peris [20] concerning hypercyclic C_0 semigroups, see also [12].

There is a fast growing literature on the subject of common hypercyclic vectors for certain uncountable families of sequences of operators. For instance, Bayart and Matheron [11], answering a question from [22], show, among other things, the existence of entire functions f so that for every non-negative real number $s \geqslant 0$ and for every $a \in \mathbb{C} \setminus \{0\}$, $\overline{\{n^s f(z+na) : n = 1, 2, \ldots\}} = \mathcal{H}(\mathbb{C})$. Shkarin in [38], extending Costakis and Sambarino's result above, proves the following: the set $\bigcap_{a,b \in \mathbb{C} \setminus \{0\}} HC(bT_a)$ is residual in $\mathcal{H}(\mathbb{C})$. There are also several results concerning the existence of common hypercyclic vectors for other type of operators such as weighted shifts, adjoints of multiplication operators, differentiation and composition operators; see for instance, [1], [3]–[13], [15]–[27], [31], [32], [34], [36]–[38], [41], [39]. There are also results going in the opposite direction, namely the non-existence of common hypercyclic vectors for certain families of operators, see [6], [7], [9], [25], [38]. A most worthy and very general result, due to Shkarin [38], is the following: for any given linear and continuous operator T acting on a complex topological vector space with non-trivial dual, the family $\{rT + aI : r > 0, a \in \mathbb{C}\}$ does not have a common hypercyclic vector.

The paper is organized as follows. Sections 2–7 occupy the proof of Theorem 1.2. In the last section, Section 8, we give some illustrating examples of sequences (λ_n) satisfying the hypothesis of Theorem 1.2, which fall into four distinct classes.

2. A SPECIAL CASE OF THEOREM 1.2: AN OUTLINE OF THE PROOF AND NOTATION

In this section we provide a general framework for attacking our problem, by considering a particular case of the sequence (λ_n). It turns out that handling this case is actually all that we need in order to establish our main result, namely Theorem 1.2. This reduction is explained and presented in full detail in Section 7. Let us now describe the extra properties we impose on the sequence (λ_n).

Let (λ_n) be a sequence of non-zero complex numbers satisfying the following:

(1) $|\lambda_{n+1}| - |\lambda_n| \to +\infty$ as $n \to +\infty$;
(2) $\lambda_{n+1}/\lambda_n \to 1$ as $n \to +\infty$;
(3) $\liminf\limits_{n\to+\infty}(n(|\lambda_{n+1}/\lambda_n| - 1)) > 0$.

A sample of sequences satisfying the above three properties is: $\lambda_n = n^c$, $c > 1$, $\lambda_n = n^\beta \log n$, $\beta \geqslant 1$, $\lambda_n = n^\gamma / \log(n+1)$, $\gamma > 2$, etc. Our main task is to prove the following special case of Theorem 1.2.

THEOREM 2.1. *Fix a sequence* (λ_n) *of non-zero complex numbers which satisfies the above properties* (1), (2), (3). *Then* $\bigcap\limits_{a\in\mathbb{C}\setminus\{0\}} HC(\{T_{\lambda_n a}\})$ *is a* G_δ *and dense subset of* $(\mathcal{H}(\mathbb{C}), \mathcal{T}_{\mathrm{u}})$.

Let us now describe the steps for the proof of Theorem 2.1. Consider the sectors

$$S_n^k := \{a \in \mathbb{C} : \exists\, r \in [\tfrac{1}{n}, n] \text{ and } t \in [\tfrac{k}{4}, \tfrac{k+1}{4}] \text{ such that } a = r e^{2\pi i t}\}$$

for $k = 0, 1, 2, 3$ and $n = 2, 3, \dots$. Since

$$\bigcap_{a\in\mathbb{C}\setminus\{0\}} HC(\{T_{\lambda_n a}\}) = \bigcap_{k=0}^{3} \bigcap_{n=2}^{\infty} \bigcap_{a\in S_n^k} HC(\{T_{\lambda_n a}\}),$$

an appeal to Baire's category theorem reduces the proof of Theorem 2.1 to the following proposition.

PROPOSITION 2.2. *Fix a sequence* (λ_n) *of non-zero complex numbers which satisfies the above properties* (1), (2), (3). *Fix four real numbers* $r_0, R_0, \theta_0, \theta_T$ *so that* $0 < r_0 < 1 < R_0 < +\infty$, $0 \leqslant \theta_0 < \theta_T \leqslant 1$, $\theta_T - \theta_0 = 1/4$ *and consider the sector* S *defined by*

$$S := \{a \in \mathbb{C} : \text{there exist } r \in [r_0, R_0] \text{ and } t \in [\theta_0, \theta_T] \text{ such that } a = r e^{2\pi i t}\}.$$

Then $\bigcap\limits_{a\in S} HC(\{T_{\lambda_n a}\})$ *is a* G_δ *and dense subset of* $(\mathcal{H}(\mathbb{C}), \mathcal{T}_{\mathrm{u}})$.

For the proof of Proposition 2.2 we introduce a notation which will be carried out throughout this paper. Let (p_j), $j = 1, 2, \dots$ be a dense sequence of $(\mathcal{H}(\mathbb{C}), \mathcal{T}_{\mathrm{u}})$, (for instance, all the polynomials in one complex variable with coefficients in $\mathbb{Q} + i\mathbb{Q}$). For every $m, j, s, k \in \mathbb{N}$ we consider the set

$$E(m,j,s,k) := \Big\{ f \in \mathcal{H}(\mathbb{C}) \mid \forall a \in S\ \exists\, n \in \mathbb{N}, n \leqslant m : \sup_{|z|\leqslant k} |f(z+\lambda_n a) - p_j(z)| < 1/s \Big\}.$$

By Baire's category theorem and the three lemmas stated below, Proposition 2.2 readily follows.

LEMMA 2.3. *We have:*

$$\bigcap_{a\in S} HC(\{T_{\lambda_n a}\}) = \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} E(m,j,s,k).$$

LEMMA 2.4. *For every $m,j,s,k \in \mathbb{N}$ the set $E(m,j,s,k)$ is open in $(\mathcal{H}(\mathbb{C}), \mathcal{T}_{\mathrm{u}})$.*

LEMMA 2.5. *For every $j,s,k \in \mathbb{N}$ the set $\bigcup_{m=1}^{\infty} E(m,j,s,k)$ is dense in $(\mathcal{H}(\mathbb{C}), \mathcal{T}_{\mathrm{u}})$.*

The proof of Lemma 2.4 is similar to that in Lemma 9 of [24] and it is omitted. Referring now to Lemma 2.3, the inclusion

$$\bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} E(m,j,s,k) \subset \bigcap_{a \in S} HC(\{T_{\lambda_n a}\})$$

is easy to establish, therefore it is left as an exercise to the interested reader. At this point, we would like to stress that Lemmas 2.4, 2.5, along with the above inclusion, immediately imply that the set $\bigcap_{a \in S} HC(T_{\lambda_n a})$ is residual, hence non-empty. However, one can get more precise information concerning the topological structure of the set $\bigcap_{a \in S} HC(T_{\lambda_n a})$ which is actually G_δ. The proof of the last fact, which is not so obvious, is postponed till the last section, i.e. Section 6. We now move on to Lemma 2.5. This lemma is the heart of our argument and its proof is long and challenging. In order to present it in a more digestive form, we give a very rough sketch of the main ideas involved in the proof below. As the reader may notice, our strategy shares certain similarities with the proof of Lemma 10 from [24]. Therefore, we will indicate the points at which our argument differentiates from that in [24].

We start by fixing $j_1, s_1, k_1 \in \mathbb{N}$. We also need to prove that $\bigcup_{m=1}^{\infty} E(m, j_1, s_1, k_1)$ is dense in $(\mathcal{H}(\mathbb{C}), \mathcal{T}_{\mathrm{u}})$. For simplicity we write $p_{j_1} = p$. Consider $g \in \mathcal{H}(\mathbb{C})$, a compact set $C \subset \mathbb{C}$ and $\epsilon_0 > 0$. We seek $f \in \mathcal{H}(\mathbb{C})$ and a positive integer m_1 so that

$$f \in E(m_1, j_1, s_1, k_1) \quad \text{and} \tag{2.1}$$

$$\sup_{z \in C} |f(z) - g(z)| < \epsilon_0. \tag{2.2}$$

WHAT IS DONE IN LEMMA 10 FROM [24]. The authors in [21], [24] deal with the unit circle instead of sector S. Then they define a suitable one dimensional partition of the unit circle $\{a_1, a_2, \ldots, a_n\}$ and choose appropriate terms $\lambda_{\mu_1}, \ldots, \lambda_{\mu_n}$ of the sequence (λ_n) so that the discs

$$B, \quad B_i := B + a_i \lambda_{\mu_i}, \quad i = 1, \ldots, n$$

are pairwise disjoint, where B is a closed disc centered at zero with sufficiently large radius R and R only depends on fixed initial conditions of the problem. Then by setting

$$L := B \cup \left(\bigcup_{i=1}^{n} B_i \right),$$

defining a suitable holomorphic function on L and using Runge's theorem they conclude the existence of a polynomial which satisfies a finite number of the desired inequalities. Taking advantage of the fact that the partition $\{a_1, a_2, \ldots, a_n\}$ is very thin, i.e. a_i is close enough to a_{i+1} for $i = 1, \ldots, n-1$, they are able to check the validity of the remaining inequalities for all the points of the unit circle.

WHAT WE DO. Our argument boils down to finding a desired two dimensional partition $\{a_1, \ldots, a_n\}$ of the above sector S. The construction of this partition consists of five steps and is presented in Section 3. Then we adjust a specific term $\lambda(a_j)$, $j = 1, \ldots, n$ of the sequence (λ_n) to each one of the above numbers a_j of the partition and we define the discs

$$B, \quad B_j := B + a_j\lambda(a_j), \quad j = 1, \ldots, n$$

so that they are pairwise disjoint. Once this is established, we more or less follow the procedure mentioned above in order to prove (2.1), (2.2).

2.1. GOOD PROPERTIES OF THE SEQUENCE (λ_n). Let (λ_n) be a sequence of non-zero complex numbers satisfying the following:

(1) $|\lambda_{n+1}| - |\lambda_n| \to +\infty$ as $n \to +\infty$;
(2) $\lambda_{n+1}/\lambda_n \to 1$ as $n \to +\infty$;
(3) $\liminf_{n\to+\infty}(n(|\lambda_{n+1}/\lambda_n| - 1)) > 0$.

Let $r_0, R_0, \theta_0, \theta_T$ be positive numbers so that $0 < r_0 < 1 < R_0 < +\infty$, $0 \leqslant \theta_0 < \theta_T \leqslant 1$. Let also c_0, c_1, c_2, c_3, c_4 be positive numbers such that $c_0 > 2$, $c_1 > 2$, $0 < c_2 < 1$, $c_3 > 0$, $2c_3 < \liminf_n(n(|\lambda_{n+1}|/|\lambda_n| - 1))$ and $c_4 := r_0c_3/2$. Finally, define

$$m_0 := \left[\frac{R_0c_1}{r_0}\right] + 1, \quad k_0 := \left[\frac{2c_0}{c_2}\right] + 1,$$

where the symbol $[x]$ stands for the integer part of a real number $x \in \mathbb{R}$. Using elementary calculus and the above properties of (λ_n) it is easy to see that there exists a fixed natural number n_0 so that for every $n \in \mathbb{N}$, $n \geqslant n_0$ all the following 8 inequalities hold:

$$|\lambda_n| \cdot \sum_{k=0}^{m_0-1} \frac{1}{|\lambda_{n+k}|} > \frac{R_0}{r_0}c_1; \tag{2.3}$$

$$|\lambda_{n+1}| - |\lambda_n| > 4\frac{c_0}{r_0}; \tag{2.4}$$

$$|\lambda_n| > \frac{4c_0}{r_0}; \tag{2.5}$$

$$|\lambda_n| \cdot \sum_{i=1}^{k_0} \frac{1}{|\lambda_{n+im_0-1}|} > \frac{2c_0}{c_2}; \tag{2.6}$$

$$n\left(\left|\frac{\lambda_{n+1}}{\lambda_n}\right| - 1\right) > 2c_3; \tag{2.7}$$

$$\frac{n}{n+m_0k_0} > \frac{1}{2}; \tag{2.8}$$

$$\frac{n}{|\lambda_n|}\cdot 2c_0 < c_4; \tag{2.9}$$

$$\frac{n}{|\lambda_n|} < \frac{c_4}{2c_2k_0}. \tag{2.10}$$

Of course, inequality (2.8) has nothing to do with the sequence (λ_n); however, we chose to isolate it here since it will be needed later in the main construction of the partition and in the construction of the disks. At first glance, it may look strange why the above properties play an important role. It turns out that these properties fully characterize the sequences (λ_n) that appear in Theorem 1.2; see Lemma 7.3 in Section 7.

3. CONSTRUCTION OF THE PARTITION OF THE SECTOR S

For the rest of this section we fix a sequence (λ_n) of non-zero complex numbers satisfying the following:

(1) $|\lambda_{n+1}| - |\lambda_n| \to +\infty$ as $n \to +\infty$;
(2) $\lambda_{n+1}/\lambda_n \to 1$ as $n \to +\infty$;
(3) $\liminf\limits_{n\to+\infty}(n(|\lambda_{n+1}/\lambda_n| - 1)) > 0$.

We also fix the numbers $r_0, R_0, \theta_0, \theta_T, c_0, c_1, c_2, c_3, c_4, m_0, k_0$, which are defined in Subsection 2.1.

3.1. STEP 1. PARTITIONS OF THE INTERVAL $[\theta_0, \theta_T]$. In this step we achieve the elementary structure of our construction. All the following steps are based on this first one. For every positive integer m we shall construct a corresponding partition Δ_m. So, let $m \in \mathbb{N}$ be fixed. We have (see subsection 2.1)

$$m_0 := \left[\frac{R_0c_1}{r_0}\right] + 1.$$

Recall that the symbol $[x]$ stands for the integer part of the real number x. For every $j = 0, 1, \ldots, m_0 - 1$ choose real numbers $\theta_j^{(m)}$, $\theta_{j+1}^{(m)}$ so that

$$\theta_j^{(m)}, \theta_{j+1}^{(m)} \in [\theta_0, \theta_T)$$

and

$$\frac{c_0}{2R_0c_1|\lambda_{m+j}|} < \theta_{j+1}^{(m)} - \theta_j^{(m)} < \frac{c_0}{R_0c_1|\lambda_{m+j}|} \tag{3.1}$$

where $\theta_0^{(m)} = \theta_0$. We consider three cases.

Case 1. Assume that

$$\frac{c_0}{2R_0c_1|\lambda_m|} \geqslant \theta_T - \theta_0.$$

Then we define

$$\Delta_m = \{\theta_0^{(m)}\}.$$

Case 2. Assume that

$$\frac{c_0}{2R_0 c_1} \sum_{j=0}^{j'} \frac{1}{|\lambda_{m+j}|} \geqslant \theta_T - \theta_0$$

for a certain $j' \in \{1, \ldots, m_0\}$. Consider the lowest number $j_0 \in \{1, \ldots, m_0\}$ so that the previous inequality holds. Then we define our partition to be

$$\Delta_m = \{\theta_j^{(m)} : j = 0, \ldots, j_0 - 1\}.$$

Case 3. Assume that none of the Cases 1, 2 hold. Then by inequality (3.1) we can assume that

$$\theta_0 = \theta_0^{(m)} < \theta_1^{(m)} < \cdots < \theta_{m_0}^{(m)} < \theta_T.$$

By setting $\sigma_m := \theta_{m_0}^{(m)} - \theta_0$, we have $0 < \sigma_m < \theta_T - \theta_0$. For every positive integer k with $k \geqslant m_0 + 1$ there exist a unique $\nu \in \mathbb{N}$ and a unique $j \in \{0, 1, \ldots, m_0 - 1\}$ so that $k = \nu m_0 + j$. For every k as before, set

$$\theta_k^{(m)} = \theta_{\nu m_0 + j}^{(m)} := \theta_j^{(m)} + \nu \sigma_m.$$

It is obvious that the sequence $(\theta_k^{(m)})_{k=1}^{\infty}$ is strictly increasing and tends to $+\infty$. Without loss of generality we may assume that

$$\theta_k^{(m)} \neq \theta_T \quad \text{for every } k \geqslant m_0 + 1.$$

Otherwise, if $\theta_{k'}^{(m)} = \theta_T$ for some $k' \geqslant m_0 + 1$, and since $(\theta_k^{(m)})_{k=1}^{\infty}$ is strictly increasing, k' is the only integer having this property. Then we subtract a sufficiently small positive number $\varepsilon > 0$ from $\theta_{k'}^{(m)}$ so that replacing $\theta_{k'}^{(m)}$ by $\theta_{k'}^{(m)} - \varepsilon$ in the sequence $(\theta_k^{(m)})_{k=1}^{\infty}$, inequality (3.1) still holds.

Finally we define ν_m to be the biggest integer ν with the properties $\nu \geqslant m_0 + 1$ and $\theta_\nu^{(m)} < \theta_T$. We are ready to describe the desired partition Δ_m:

$$\Delta_m := \{\theta_0^{(m)}, \theta_1^{(m)}, \ldots, \theta_{\nu_m}^{(m)}\}.$$

The partitions $\Delta_1, \Delta_2, \ldots$ constructed above can be chosen so that the following important property holds:

$$\text{“almost disjoint property”:} \quad \text{if } m_1 \neq m_2 \text{ then } \Delta_{m_1} \cap \Delta_{m_2} = \{\theta_0\}.$$

The “almost disjoint property” turns out to be very important in the rest of the construction.

3.2. STEP 2. PARTITIONS OF THE ARC $\phi_r([\theta_0, \theta_T])$. Consider the function $\phi : [\theta_0, \theta_T] \times (0, +\infty) \to \mathbb{C}$ given by

$$\phi(t, r) := re^{2\pi i t}, \quad (t, r) \in [\theta_0, \theta_T] \times (0, +\infty)$$

and where for every $r > 0$ we define the corresponding curve $\phi_r : [\theta_0, \theta_T] \to \mathbb{C}$ by

$$\phi_r(t) := \phi(t, r), \quad t \in [\theta_0, \theta_T].$$

For any given positive integer m, $\phi_r(\Delta_m)$ is a partition of the arc $\phi_r([\theta_0, \theta_T])$, where Δ_m is the partition of the interval $[\theta_0, \theta_T]$ constructed in Step 1. For every $r > 0$, $m \in \mathbb{N}$ define

$$P_0^{r,m} := \phi_r(\Delta_m)$$

which we call partition of the arc $\phi_r([\theta_0, \theta_T])$ with height r, density m and order 0.

3.3. STEP 3. PARTITIONS OF ORDER 1 FOR A SECTOR OF OPENING $\pi/2$. In this step we elaborate on the construction of Step 2 and we aim to define a suitable partition for a sector of opening $\pi/2$. For reasons that will become apparent later on, this partition is called a partition of order 1. To explain why we deal with such a sector, notice that $\theta_T - \theta_0 = 1/4$. Therefore the set $\phi([\theta_0, \theta_T] \times (0, +\infty))$ is nothing else but a sector of opening $\pi/2$, where ϕ is defined in Section 2.

We continue with the construction of the desired partition. Recall that

$$k_0 := \left[\frac{2c_0}{c_2}\right] + 1.$$

The fixed positive constant c_2 appears in Section 2. For every $r > 0$, $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, k_0 - 1\}$ define the positive numbers

$$\mu(r, m, k) := r + \sum_{j=1}^{k} \frac{c_2}{|\lambda_{m+jm_0-1}|}, \quad k \geqslant 1$$
$$\mu(r, m, 0) := r.$$

Roughly speaking, our new partition will be obtained as a suitable finite union of partitions of order 0 with different heights and densities. More precisely for every $m \in \mathbb{N}$, $r > 0$, define the set

$$P_1^{r,m} := \bigcup_{k=0}^{k_0-1} P_0^{\mu(r,m,k), m+km_0},$$

where $P_0^{\mu(r,m,k), m+km_0}$ is the partition of the arc $\phi_{\mu(r,m,k)}([\theta_0, \theta_T])$ with height $\mu(r, m, k)$, density $m + km_0$ and order 0. We call the set $P_1^{r,m}$ a partition with basis r, density m and order 1. Observe that in this way we obtain the first partition in two dimensions, that is a partition of a sector. We will built our next two dimensional partition by stacking several partitions of order 1.

3.4. STEP 4. STACKING SEVERAL PARTITIONS OF ORDER 1: PARTITIONS OF ORDER 2. The positive number

$$c_4 := \frac{r_0 c_3}{2}$$

is fixed from the beginning of this section. For every positive integer m and every $r > 0$ we define the positive number

$$\mu_1(m) := \sum_{j=1}^{k_0} \frac{c_2}{|\lambda_{m+jm_0-1}|}$$

and observe that by Step 3 we have

$$\mu_1(m) = \mu(r, m, k_0 - 1) + \frac{c_2}{|\lambda_{m+k_0m_0-1}|} - r,$$

for every $r > 0$. Let $r > 0$ and $m \in \mathbb{N}$. We shall describe the new partition corresponding to r, m.

Case 1. Assume that

$$\mu_1(m) \geqslant \frac{c_4}{m}.$$

Then we stop the process and define

$$P_2^{r,m} := P_1^{r,m},$$

where $P_1^{r,m}$ is the partition defined in Step 3.

Case 2. Assume that

$$\mu_1(m) < \frac{c_4}{m}.$$

It trivially follows that $|w| < r + (c_4/m)$ for every $w \in P_1^{r,m}$. Consider now the following partitions of order 1

$$P_1^{r+\nu\mu_1(m),m}, \quad \text{for every } \nu = 0, 1, 2, \dots.$$

Then for every $w \in P_1^{r+\nu\mu_1(m),m}$, $\nu = 0, 1, 2, \dots$ we get

$$|w| \geqslant r + \nu\mu_1(m). \tag{3.2}$$

Let us consider the following subset of the positive integers

$$\mathcal{A} := \Big\{ N \in \mathbb{N} : |w| < r + \frac{c_4}{m}, \ \forall w \in \bigcup_{\nu=0}^{N} P_1^{r+\nu\mu_1(m),m} \Big\}.$$

Since $r + \nu\mu_1(m) \to +\infty$ as $\nu \to +\infty$, $|w| < r + \mu_1(m)$ for every $w \in P_1^{r,m}$ and in view of (3.2) we conclude that the set $\mathcal{A}$ is non-empty and finite. Take the biggest integer in this set, i.e.,

$$\nu_0^{r,m} := \max \mathcal{A}.$$

This integer describes the stopping time of the process. Then define the set

$$P_2^{r,m} := \bigcup_{\nu=0}^{\nu_0^{r,m}} P_1^{r+\nu\mu_1(m),m}.$$

Throughout the rest of the paper we call the set $P_2^{r,m}$ a partition with basis r, density m and order 2.

3.5. STEP 5. THE FINAL PARTITION. In this step, we complete the construction of the desired partition of S. For every positive integer m with $m \geqslant n_0$ and every $r > 0$ define the positive numbers

$$M^{r,m} := \max\{|w| : w \in P_2^{r,m}\},$$

where $P_2^{r,m}$ is the partition with basis r, density m and order 2, and is defined in Step 4. We call the number $M^{r,m}$ the height of the partition $P_2^{r,m}$ and define the number

$$l(P_2^{r,m}) := M^{r,m} - r,$$

which we call length of partition $P_2^{r,m}$. The proof that the length of partition $P_2^{r,m}$ is positive will be postponed till the next subsection. Let us now consider the sequence $(r_\nu^{(m)})$ of the positive numbers, defined recursively as follows:

$$\begin{aligned} r_0^{(m)} &:= r_0, \\ r_1^{(m)} - r_0^{(m)} &:= l(P_2^{r,m}), \\ r_{\nu+1}^{(m)} - r_\nu^{(m)} &:= l(P_2^{r_\nu^{(m)}, m+\nu k_0 m_0}), \end{aligned}$$

for every $\nu = 1, 2, \ldots$. In the next subsection, it will be proven that $r_\nu^{(m)} \to +\infty$ as $\nu \to +\infty$ for every $m \geqslant n_0$. Therefore, for every $m \geqslant n_0$ there exists a positive integer $\nu(m)$ so that $r_{\nu(m)}^{(m)} \geqslant R_0$. Let $\nu_1^{(m)}$ be the smallest positive integer with the previous property. Now we define

$$P_m := S \cap \Big(\bigcup_{\nu=0}^{\nu_1^{(m)}} P_2^{r_\nu^{(m)}, m+\nu k_0 m_0} \Big),$$

for every positive integer m with $m \geqslant n_0$. For every m, as before the set P_m defines a partition of the sector S, and throughout the rest of this paper P_m will be called the partition of S with order m.

3.6. PROPERTIES OF THE PARTITIONS. In the next lemma, we transfer the "almost disjoint property" of the partitions of interval $[\theta_0, \theta_T)$ to an arc.

LEMMA 3.1. *Consider the partitions P_0^{r,m_1}, P_0^{r,m_2} for given $r > 0$ and $m_1, m_2 \in \mathbb{N}$. The following property holds:*

"almost disjoint property": if $m_1 \neq m_2$ then $P_0^{r,m_1} \cap P_0^{r,m_2} = \{re^{2\pi i\theta_0}\}$.

Proof. The result is an immediate result of the corresponding property of the partitions of interval $[\theta_0, \theta_T)$ and the definition of partition $P_0^{r,m}$; see Steps 1,2. ∎

LEMMA 3.2. *Consider the partition* $P_1^{r,m} := \bigcup_{k=0}^{k_0-1} P_0^{\mu(r,m,k),m+km_0}$ *defined in Step* 3, *for fixed* $r > 0$ *and* $m \in \mathbb{N}$. *Take* $k_1, k_2 \in \{0, \dots, k_0 - 1\}$ *with* $k_1 < k_2$. *Then we have*

$$\mu(r, m, k_1) < \mu(r, m, k_2),$$

where $\mu(r, m, k_1), \mu(r, m, k_2)$ *are the heights of the partitions* $P_0^{\mu(r,m,k_1),m+k_1m_0}$, $P_0^{\mu(r,m,k_2),m+k_2m_0}$, *respectively. In particular*

$$P_0^{\mu(r,m,k_1),m+k_1m_0} \cap P_0^{\mu(r,m,k_2),m+k_2m_0} = \emptyset.$$

Proof. By the definition of $\mu(r, m, k)$ (see Step 3) it follows that $\mu(r, m, k_1) < \mu(r, m, k_2)$. ∎

LEMMA 3.3. *Consider the partition* $P_2^{r,m} := \bigcup_{\nu=0}^{\nu_0^{r,m}} P_1^{r+\nu\mu_1(m),m}$ *defined in Step* 4, *for fixed* $r > 0$ *and* $m \in \mathbb{N}$. *Take* $\nu_1, \nu_2 \in \{0, \dots, \nu_0^{r,m}\}$ *with* $\nu_1 < \nu_2$. *Then we have*

$$\max\{|w| : w \in P_1^{r+\nu_1\mu_1(m),m}\} < \min\{|w| : w \in P_1^{r+\nu_2\mu_1(m),m}\}.$$

In particular

$$P_1^{r+\nu_1\mu_1(m),m} \cap P_1^{r+\nu_2\mu_1(m),m} = \emptyset.$$

Proof. Take any $k_1, k_2 \in \{0, \dots, k_0 - 1\}$. We have

$$\begin{aligned}\mu(r + \nu_1\mu_1(m), m, k_1) &= r + \nu_1\mu_1(m) + \sum_{j=1}^{k_1} \frac{c_2}{|\lambda_{m+jm_0-1}|} < r + (\nu_1 + 1)\mu_1(m) \\ &\leqslant r + \nu_2\mu_1(m) \leqslant r + \nu_2\mu_1(m) + \sum_{j=1}^{k_2} \frac{c_2}{|\lambda_{m+jm_0-1}|} \\ &= \mu(r + \nu_2\mu_1(m), m, k_2),\end{aligned}$$

where in the case $k_1 = 0$ or $k_2 = 0$ the corresponding sum above disappears. The last implies that the height of any partition of order 0 used to build the partition $P_1^{r+\nu_1\mu_1(m),m}$ is strictly lower than the height of every partition of order 0 used to build the partition $P_1^{r+\nu_2\mu_1(m),m}$. The conclusion follows. ∎

LEMMA 3.4. *Fix any positive integer* m *with* $m \geqslant n_0$. *Then for every positive number* r, *the length of the partition* $P_2^{r,m}$, *i.e. the number* $l(P_2^{r,m})$ *defined in Step* 5, *is positive and hence the sequence* $(r_\nu^{(m)})_{\nu=0}^{\infty}$, *defined in Step* 5, *is strictly increasing.*

Proof. Recall that $r_0^{(m)} := r_0 > 0$ and $r_{\nu+1}^{(m)} - r_\nu^{(m)} := l(P_2^{r_\nu^{(m)},m+\nu k_0m_0})$; see Step 5. Hence, it suffices to show that $l(P_2^{r_\nu^{(m)},m+\nu k_0m_0}) > 0$. On the other hand, by the definition of the length of partition $P_2^{r,m}$, we have

$$l(P_2^{r,m}) := M^{r,m} - r,$$

where

$$M^{r,m} := \max\{|w| : w \in P_2^{r,m}\}.$$

Partition $P_2^{r,m}$ is defined as a union of partitions $P_1^{r',m'}$ for certain r', m'. Pick such a $P_1^{r',m'}$ which in turn is defined as a union of partitions $P_0^{r'',m''}$ for certain r'', m''. By the choice of k_0 we conclude that $P_1^{r',m'}$ contains at least five partitions $P_0^{r'',m''}$ with pairwise different heights, hence by Lemma 3.2 we get

$$\min\{|w| : w \in P_1^{r',m'}\} < \max\{|w| : w \in P_1^{r',m'}\}.$$

Observe now that

$$r \leqslant \min\{|w| : w \in P_1^{r',m'}\} \quad \text{and} \quad \max\{|w| : w \in P_1^{r',m'}\} \leqslant M^{r,m}.$$

The above inequalities imply that $l(P_2^{r,m}) > 0$, and this completes the proof of the lemma. ∎

LEMMA 3.5. *Consider the partition $P_m := S \cap \left(\bigcup_{\nu=0}^{\nu_1^{(m)}} P_2^{r_\nu^{(m)}, m+\nu k_0 m_0} \right)$ defined in Step 5, for fixed $m \in \mathbb{N}$ with $m \geqslant n_0$. Take $\nu_1, \nu_2 \in \{0, \dots, \nu_1^{(m)}\}$, with $\nu_1 < \nu_2$ and $\nu_2 - \nu_1 \geqslant 2$. Then we have*

$$\max\{|w| : w \in P_2^{r_{\nu_1}^{(m)}, m+\nu_1 k_0 m_0}\} < \min\{|w| : w \in P_2^{r_{\nu_2}^{(m)}, m+\nu_2 k_0 m_0}\}.$$

In particular,

$$P_2^{r_{\nu_1}^{(m)}, m+\nu_1 k_0 m_0} \cap P_2^{r_{\nu_2}^{(m)}, m+\nu_2 k_0 m_0} = \emptyset.$$

Proof. We proceed by induction on $\nu \in \{0, \dots, \nu_1^{(m)}\}$. Recall the following quantities from Step 5:

$$M^{r,m} := \max\{|w| : w \in P_2^{r,m}\}, \tag{3.3}$$

$$l(P_2^{r,m}) := M^{r,m} - r, \tag{3.4}$$

$$r_0^{(m)} := r_0, \; r_{\nu+1}^{(m)} - r_\nu^{(m)} := l(P_2^{r_\nu^{(m)}, m+\nu k_0 m_0}). \tag{3.5}$$

Applying (3.3), (3.4), (3.5) we get

$$M^{r_\nu^{(m)}, m+\nu k_0 m_0} = \max\{|w| : w \in P_2^{r_\nu^{(m)}, m+\nu k_0 m_0}\} = r_\nu^{(m)} + l(P_2^{r_\nu^{(m)}, m+\nu k_0 m_0}) = r_{\nu+1}^{(m)} \tag{3.6}$$

for $\nu \in \{0, \dots, \nu_1^{(m)}\}$. Using (3.6) and the fact that the sequence $(r_\nu^{(m)})_{\nu=0}^\infty$ is strictly increasing, see Lemma 3.4, we have

$$\max\{|z| : z \in P_2^{r_{\nu_1}^{(m)}, m+\nu_1 k_0 m_0}\} = M^{r_{\nu_1}^{(m)}, m+\nu_1 k_0 m_0} = r_{\nu_1+1}^{(m)} < r_{\nu_2}^{(m)}. \tag{3.7}$$

Combining the last with the (trivial) equality

$$r_{\nu_2}^{(m)} = \min\{|w| : w \in P_2^{r_{\nu_2}^{(m)}, m+\nu_2 k_0 m_0}\} \tag{3.8}$$

the result follows. ∎

LEMMA 3.6. *Consider the partition* $P_m := S \bigcap \left(\bigcup_{\nu=0}^{\nu_1^{(m)}} P_2^{r_\nu^{(m)}, m+\nu k_0 m_0} \right)$ *defined in Step* 5, *for fixed* $m \in \mathbb{N}$ *with* $m \geqslant n_0$ *and* $\nu_1^{(m)} \geqslant 1$. *Take* $\nu \in \{0, \dots, \nu_1^{(m)} - 1\}$. *Then we have*

$$\max\{|w| : w \in P_2^{r_\nu^{(m)}, m+\nu k_0 m_0}\} = \min\{|w| : w \in P_2^{r_{\nu+1}^{(m)}, m+(\nu+1)k_0 m_0}\} \quad \textit{and}$$

$$P_2^{r_\nu^{(m)}, m+\nu k_0 m_0} \cap P_2^{r_{\nu+1}^{(m)}, m+(\nu+1)k_0 m_0} = \{r_{\nu+1}^{(m)} e^{2\pi i \theta_0}\}.$$

Proof. By (3.7), (3.8) we get

$$\max\{|z| : z \in P_2^{r_\nu^{(m)}, m+\nu k_0 m_0}\} = r_{\nu+1}^{(m)} = \min\{|w| : w \in P_2^{r_{\nu+1}^{(m)}, m+(\nu+1)k_0 m_0}\}.$$

Observe that $P_2^{r_\nu^{(m)}, m+\nu k_0 m_0} = \bigcup\limits_{r', m'} P_0^{r', m'}$ and $P_2^{r_{\nu+1}^{(m)}, m+(\nu+1)k_0 m_0} = \bigcup\limits_{r'', m''} P_0^{r'', m''}$. Therefore, the partitions $P_2^{r_\nu^{(m)}, m+\nu k_0 m_0}$, $P_2^{r_{\nu+1}^{(m)}, m+(\nu+1)k_0 m_0}$ have a non-empty intersection if and only if

$$P_0^{r', m'} \cap P_0^{r'', m''} \neq \emptyset \quad \text{for some } r', r'', m', m''.$$

Clearly, two partitions $P_0^{r', m'}$ $P_0^{r'', m''}$ of zero order have a non-empty intersection if and only if the heights r', r'' are the same. In our case the last happens if and only if $r' = r'' = r_{\nu+1}^{(m)}$. On the other hand, it is not difficult to see that in every partition $P_2^{r,m} = \bigcup P_0^{R,M}$ there do not exist P_0^{R,M_1}, P_0^{R,M_2} "members" of $P_2^{r,m}$ with $M_1 \neq M_2$. Hence, by the definition of the partition of order 2, we have that the partition of order 0 and height $r_{\nu+1}^{(m)}$, which is a member of $P_2^{r_\nu^{(m)}, m+\nu k_0 m_0}$, is the one with density $m + (\nu+1)k_0 m_0 - m_0$. In a similar manner, we have that the partition of order 0 and height $r_{\nu+1}^{(m)}$ which is a member of $P_2^{r_{\nu+1}^{(m)}, m+(\nu+1)k_0 m_0}$ is the one with density $m + (\nu+1)k_0 m_0$. Since $m + (\nu+1)k_0 m_0 - m_0 < m + (\nu+1)k_0 m_0$, by Lemma 3.1, it follows that

$$P_0^{r_{\nu+1}^{(m)}, m+(\nu+1)k_0 m_0 - m_0} \cap P_0^{r_{\nu+1}^{(m)}, m+(\nu+1)k_0 m_0} = \{r_{\nu+1}^{(m)} e^{2\pi i \theta_0}\}$$

and this finishes the proof of the lemma. ∎

LEMMA 3.7. *Fix a positive integer* m *with* $m \geqslant n_0$. *Then the sequence* $(r_\nu^{(m)})_{\nu=1}^{\infty}$, *defined in Step* 4, *is strictly increasing and*

$$\lim_{\nu \to +\infty} r_\nu^{(m)} = +\infty.$$

Proof. We shall prove that

$$r_{\nu+1}^{(m)} - r_\nu^{(m)} > \frac{c_4}{2(m + \nu k_0 m_0)}, \quad \text{for every } \nu = 0, 1, \dots. \tag{3.9}$$

Fix $\nu \in \{0,1,\ldots\}$ and in order to simplify notation set

$$r := r_\nu^{(m)}, \quad r' := r_{\nu+1}^{(m)}, \quad m_1 := m + \nu k_0 m_0.$$

By definition (see Step 4) we have

$$r' - r := l(P_2^{r,m_1})$$

and again by definition (see Step 4) and since $(|\lambda_n|)$ is strictly increasing we get

$$\mu_1(m_1) := \sum_{j=1}^{k_0} \frac{c_2}{|\lambda_{m_1+jm_0-1}|} < \frac{k_0 c_2}{|\lambda_{m_1}|}. \tag{3.10}$$

By the definition of the partition P_2^{r,m_1} we obtain the inequality

$$r + l(P_2^{r,m_1}) + \mu_1(m_1) \geqslant r + \frac{c_4}{m_1},$$

which, in view of (3.10), gives the following lower bound on the length of P_2^{r,m_1}:

$$l(P_2^{r,m_1}) > \frac{c_4}{m_1} - \frac{k_0 c_2}{|\lambda_{m_1}|}. \tag{3.11}$$

By (2.8) we have

$$\frac{m_1}{|\lambda_{m_1}|} < \frac{c_4}{2c_2k_0}.$$

Combining the last inequality with (3.11) we get

$$r' - r := l(P_2^{r,m_1}) > \frac{c_4}{2m_1},$$

which proves (3.9). Clearly (3.9) implies that $\lim_{\nu\to+\infty} r_\nu^{(m)} = +\infty$. ∎

4. CONSTRUCTION AND PROPERTIES OF THE DISKS

For the rest of this section, we fix a sequence (λ_n) of non-zero complex numbers satisfying the following:

(1) $|\lambda_{n+1}| - |\lambda_n| \to +\infty$ as $n \to +\infty$;
(2) $\lambda_{n+1}/\lambda_n \to 1$ as $n \to +\infty$;
(3) $\liminf_{n\to+\infty} (n(|\lambda_{n+1}/\lambda_n| - 1)) > 0$.

We also fix the numbers $r_0, R_0, \theta_0, \theta_T, c_0$, c_1, c_2, c_3, c_4, m_0, k_0, which are defined in Subsection 2.1. Finally, on the basis of the above, for every positive integer m we consider the partition P_m constructed in the previous section.

4.1. CONSTRUCTION OF THE DISKS. The strategy in this subsection is to construct a certain family of pairwise disjoint disks, which will allow us later to apply Runge's theorem in order to prove Proposition 2.1 successfully. What we will do is assign to each point w of partition P_m a suitable closed disk with center $w\lambda(w)$ and radius c_0 (the radius will be the same for every member of the family of the disks), where $\lambda(w)$ will be chosen from the sequence (λ_n). We shall see that the construction of the partition P_m ensures on the one hand that the points of the partition are close enough to each other on the sector S, and on the other hand that the disks centered at these points with fixed radius c_0 are pairwise disjoint. This is the hard part of our argument, and also shows that the required construction is very delicate. So, let us begin with the construction of the disks.

We set

$$B := \{z \in \mathbb{C} : |z| \leqslant c_0\}.$$

Fix a positive integer $m \geqslant n_0$ and let w be any point of the partition P_m of the sector S. We distinguish two cases.

Case 1. Assume that

$$w \in \{r_\nu^{(m)} \mathrm{e}^{2\pi \mathrm{i}\theta_0} : \nu = 1, \ldots, \nu_1^{(m)}\}.$$

Then $w = r_\nu^{(m)} \mathrm{e}^{2\pi \mathrm{i}\theta_0}$ for some $\nu \in \{1, \ldots, \nu_1^{(m)}\}$. We define

$$\lambda(w) := \lambda_{m+\nu k_0 m_0 - m_0} \quad \text{and} \quad B_w := B + w\lambda(w).$$

Case 2. Assume that

$$w \in P_m \setminus \{r_\nu^{(m)} \mathrm{e}^{2\pi \mathrm{i}\theta_0} : \nu = 1, \ldots, \nu_1^{(m)}\}.$$

By Lemmas 3.5, 3.6, there exists a unique $\nu \in \{0, 1, \ldots, \nu_1^{(m)}\}$ so that

$$w \in P_2^{r_\nu^{(m)}, m+\nu k_0 m_0} = \bigcup_{r', m'} P_0^{r', m'}.$$

Applying Lemmas 3.2, 3.3, 3.5, 3.6, we conclude that there is a unique pair (r', m') so that

$$w = r' \mathrm{e}^{2\pi \mathrm{i}\theta_k^{(m')}} \quad \text{for some } k \in \{0, 1, \ldots, \nu_{m'}\}.$$

Observe that k can be uniquely written in the form

$$k = \rho m_0 + j, \quad \text{for some } \rho \in \mathbb{N},\ j \in \{0, \ldots, m_0 - 1\}.$$

From the above and the definition of the partition $\Delta_{m'}$, see Step 1, we have

$$\theta_k^{(m')} = \theta_{\rho m_0 + j}^{(m')} = \theta_j^{(m')} + \rho\sigma_{m'}.$$

Finally we define

$$\lambda(w) := \lambda_{m'+j} \quad \text{and} \quad B_w := B + w\lambda(w).$$

Therefore for every $w \in P_m$ we assigned a disk B_w according to the above rules. This completes the desired construction of the disks assigned to the partition P_m.

4.2. PROPERTIES OF THE DISKS. Our aim in this subsection is to prove that for a fixed positive integer m, the disks B_w for $w \in P_m$ (corresponding to the partition P_m), that have been constructed in the previous subsection, are pairwise disjoint.

LEMMA 4.1. *Fix a positive integer m with $m \geqslant n_0$. Then we have*

$$B \cap B_w = \emptyset \quad \text{for every } w \in P_m.$$

Proof. Take $w \in P_m$. The closed disks B, B_w are centered at $0, w\lambda(w)$, respectively, and they have the same radius c_0. Hence, we have to show that $|w\lambda(w)| > 2c_0$. Since $|w| \geqslant r_0$, it suffices to prove that

$$|\lambda(w)| > \frac{2c_0}{r_0}.$$

Observe now that, by the definition of $\lambda(w)$ in the previous subsection, $\lambda(w) = \lambda_n$ for some positive integer n with $n \geqslant m \geqslant n_0$. By property (2.3), we conclude that $|\lambda_n| > 2c_0/r_0$, and this finishes the proof of the lemma. ∎

LEMMA 4.2. *Fix a positive integer m with $m \geqslant n_0$. If $w_1, w_2 \in P_m$ with $w_1 \neq w_2, |w_1| \leqslant |w_2|, |\lambda(w_1)| < |\lambda(w_2)|$ then $B_{w_1} \cap B_{w_2} = \emptyset$.*

Proof. Take w_1, w_2 satisfying the hypothesis of the lemma. We need to show that $|w_1\lambda(w_1) - w_2\lambda(w_2)| > 2c_0$. Observe that $\lambda(w_j) = \lambda_{n_j}$ for some positive integer $n_j \geqslant m \geqslant n_0$, $j = 1, 2$. Since $|\lambda(w_1)| < |\lambda(w_2)|$ and the sequence $(|\lambda_n|)$ is strictly increasing we conclude that $n_1 < n_2$. We have

$$\begin{aligned}|w_1\lambda(w_1) - w_2\lambda(w_2)| &\geqslant \big|\,|w_1\lambda(w_1)| - |w_2\lambda(w_2)|\,\big| = |w_2\lambda(w_2)| - |w_1\lambda(w_1)| \\ &\geqslant r_0(|\lambda(w_2)| - |\lambda(w_1)|) = r_0(|\lambda_{n_2}| - |\lambda_{n_1}|) \\ &\geqslant r_0(|\lambda_{n_1+1}| - |\lambda_{n_1}|) > 2c_0,\end{aligned}$$

where the last inequality above follows by property (2.2). ∎

LEMMA 4.3. *Fix a positive integer m with $m \geqslant n_0$. If $w_1, w_2 \in P_m$ with $w_1 \neq w_2$ and $|w_1| = |w_2|$ then $B_{w_1} \cap B_{w_2} = \emptyset$.*

Proof. Fix w_1, w_2 satisfying the hypothesis of the lemma. Then we have $w_1 = re^{2\pi i\theta_1}$, $w_2 = re^{2\pi i\theta_2}$, for some $r \in [r_0, R_0]$ and some $\theta_1, \theta_2 \in [\theta_0, \theta_T)$. Since $w_1, w_2 \in P_m = \bigcup_{(r',m') \in J} P_0^{r',m'}$, where J is a suitable set of indices, then

$$\begin{aligned}&\text{either } w_1 \in P_0^{r,m_1} \text{ and } w_2 \in P_0^{r,m_2} \quad \text{for } (r, m_1), (r, m_2) \in J,\ m_1 \neq m_2 \\ &\text{or } w_1, w_2 \in P_0^{r,m'} \quad \text{for some } (r, m') \in J.\end{aligned}$$

Let us first consider the case where w_1, w_2 belong to different partitions of zero order. Then necessarily we have $|\lambda(w_1)| \neq |\lambda(w_2)|$ and since $|w_1| = |w_2|$, Lemma 4.2 implies that the disks B_{w_1}, B_{w_2} are disjoint.

We turn now to the case where both w_1, w_2 belong to the same partition of zero order $P_0^{r,m'}$. By the definition of the partition $P_0^{r,m'}$ there exist $k_1, k_2 \in$

$\{0,\dots,\nu_{m'}\}$ such that

$$\theta_1 = \theta_{k_1}^{(m')} \quad \text{and} \quad \theta_2 = \theta_{k_2}^{(m')}.$$

We also have that

$$k_1 = \rho_1 m_0 + j_1, \quad k_2 = \rho_2 m_0 + j_2$$

for $\rho_1, \rho_2 \in \mathbb{N}$ and $j_1, j_2 \in \{0,\dots,m_0-1\}$ and by the definition of the partition $\Delta_{m'}$, see Step 1, it follows that

$$\theta_{k_1}^{(m')} = \theta_{j_1}^{(m')} + \rho_1\sigma_{m'}, \quad \theta_{k_2}^{(m')} = \theta_{j_2}^{(m')} + \rho_2\sigma_{m'},$$

where (recall from Step 1),

$$\sigma_{m'} := \theta_{m_0}^{(m')} - \theta_0.$$

We shall consider two cases.

Case 1. Assume that $j_1 \neq j_2$. Since $\lambda(w_1) = \lambda_{m'+j_1}, \lambda(w_2) = \lambda_{m'+j_2}$ it readily follows that $|\lambda(w_1)| \neq |\lambda(w_2)|$. In view of Lemma 4.2 we conclude that the disks B_{w_1}, B_{w_2} are disjoint.

Case 2. It remains to handle the case $j_1 = j_2$. Observe that in this situation we have

$$\theta_2 - \theta_1 = (\rho_2 - \rho_1)\sigma_{m'}. \tag{4.1}$$

Since $w_1 \neq w_2$ and $|w_1| = |w_2|$ we may assume with no loss of generality that $\theta_1 < \theta_2$. We establish below a "sufficiently large" lower bound on $\sigma_{m'}$. Inequality (3.1) in Step 1 implies the following:

$$\begin{aligned}
\theta_1^{(m')} - \theta_0^{(m')} &> \frac{c_0}{2R_0c_1}\frac{1}{|\lambda_{m'}|}\\
\theta_2^{(m')} - \theta_1^{(m')} &> \frac{c_0}{2R_0c_1}\cdot\frac{1}{|\lambda_{m'+1}|}\\
&\vdots\\
\theta_{m_0}^{(m')} - \theta_{m_0-1}^{(m')} &> \frac{c_0}{2R_0c_1}\cdot\frac{1}{|\lambda_{m'+m_0-1}|}.
\end{aligned}$$

Adding the previous inequalities by pairs we get

$$\sigma_{m'} := \theta_{m_0}^{(m')} - \theta_0 > \frac{c_0}{2R_0c_1}\cdot\sum_{j=0}^{m_0-1}\frac{1}{|\lambda_{m'+j}|}. \tag{4.2}$$

We also need the following inequality, so called Jordan's inequality:

$$\sin x > \frac{2}{\pi}x, \quad x \in (0, \tfrac{\pi}{2}). \tag{4.3}$$

Since $r \geqslant r_0$, $(|\lambda_n|)_n$ is strictly increasing, $\theta_1 < \theta_2$ and with (4.1), (4.2), (4.3) and property (2.1) we get

$$\begin{aligned} |w_2\lambda(w_2) - w_1\lambda(w_1)| &= r|\lambda_{m'+j_0}||\mathrm{e}^{2\pi \mathrm{i}\theta_2} - \mathrm{e}^{2\pi \mathrm{i}\theta_1}| \geqslant r_0|\lambda_{m'}||\mathrm{e}^{2\pi \mathrm{i}\theta_2} - \mathrm{e}^{2\pi \mathrm{i}\theta_1}| \\ &= r_0|\lambda_{m'}|2\sin(\pi(\theta_2-\theta_1)) > 2r_0|\lambda_{m'}|\frac{2}{\pi}(\pi(\theta_2-\theta_1)) \\ &= 4r_0|\lambda_{m'}|(\rho_2-\rho_1)\sigma_{m'} \geqslant 4r_0|\lambda_{m'}|\sigma_{m'} \\ &> \frac{2r_0c_0}{R_0c_1}|\lambda_{m'}|\sum_{j=0}^{m_0-1}\frac{1}{|\lambda_{m'+j}|} > 2c_0, \end{aligned}$$

where the last inequality is a result of property (2.1). This finishes the proof for the Case 2, and hence that of the lemma. ∎

LEMMA 4.4. *Fix a positive integer m with $m \geqslant n_0$. If $w_1, w_2 \in P_m$ with $w_1 \neq w_2$ and $\lambda(w_1) = \lambda(w_2)$ then $B_{w_1} \cap B_{w_2} = \varnothing$.*

Proof. Fix w_1, w_2, satisfying the hypothesis of the lemma. If $|w_1| = |w_2|$, then by Lemma 4.3 the conclusion follows. So assume that $|w_1| \neq |w_2|$. By the definition of the partition P_m, there exist $\nu_1, \nu_2 \in \{0, \dots, \nu_1^{(m)}\}$ so that

$$w_1 \in P_2^{r_{\nu_1}^{(m)}, m+\nu_1 k_0 m_0}, \quad w_2 \in P_2^{r_{\nu_2}^{(m)}, m+\nu_2 k_0 m_0}.$$

Claim 1. $\nu_1 = \nu_2$.

Proof of Claim 1. We argue by contradiction, so assume that $\nu_1 \neq \nu_2$. Without loss of generality, suppose that $\nu_1 < \nu_2$. By definition of the partition of order 2 we have

$$P_2^{r_{\nu_1}^{(m)}, m+\nu_1 k_0 m_0} = \bigcup_{\nu=0}^{\nu_0^{r_{\nu_1}^{(m)}, m+\nu_1 k_0 m_0}} P_1^{r_{\nu_1}^{(m)} + \nu\mu_1(m+\nu_1 k_0 m_0), m+\nu_1 k_0 m_0} \quad \text{and}$$

$$P_2^{r_{\nu_2}^{(m)}, m+\nu_2 k_0 m_0} = \bigcup_{\nu=0}^{\nu_0^{r_{\nu_2}^{(m)}, m+\nu_2 k_0 m_0}} P_1^{r_{\nu_2}^{(m)} + \nu\mu_1(m+\nu_2 k_0 m_0), m+\nu_2 k_0 m_0}.$$

Hence there exist $\nu' \in \{0, \dots, \nu_0^{r_{\nu_1}^{(m)}, m+\nu_1 k_0 m_0}\}$, $\nu'' \in \{0, \dots, \nu_0^{r_{\nu_2}^{(m)}, m+\nu_2 k_0 m_0}\}$ such that

$$w_1 \in P_1^{r_{\nu_1}^{(m)} + \nu'\mu_1(m+\nu_1 k_0 m_0), m+\nu_1 k_0 m_0} \quad \text{and} \tag{4.4}$$

$$w_2 \in P_1^{r_{\nu_2}^{(m)} + \nu''\mu_1(m+\nu_2 k_0 m_0), m+\nu_2 k_0 m_0}. \tag{4.5}$$

Recall that, see Step 1, for every $r > 0$ and every positive integer m the partition $P_1^{r,m}$ is defined as a union of partitions of order 0 as follows:

$$P_1^{r,m} = \bigcup_{k=0}^{k_0-1} P_0^{\mu(r,m,k), m+km_0}.$$

Thus, by (4.4), (4.5), there exist $k_1, k_2 \in \{0, \ldots, k_0 - 1\}$ and r_1, r_2 positive numbers so that

$$w_1 \in P_0^{r_1, m+\nu_1 k_0 m_0 + k_1 m_0}, \quad w_2 \in P_0^{r_2, m+\nu_2 k_0 m_0 + k_2 m_0}.$$

From the previous and the definition of $\lambda(w)$ for $w \in P_m$ we have

$$\lambda(w_1) = \lambda_{n'+j'}, \ \ \lambda(w_2) = \lambda_{n''+j''},$$

where $n' = m + \nu_1 k_0 m_0 + k_1 m_0$, $n'' = m + \nu_2 k_0 m_0 + k_2 m_0$ and $j', j'' \in \{0, \ldots, m_0 - 1\}$. Observe now that

$$n' + l' < n'' + l'' \quad \text{for every } l', l'' \in \{0, \ldots, m_0 - 1\}. \tag{4.6}$$

By (4.6) and the fact that $(|\lambda_n|)$ is strictly increasing we arrive at

$$|\lambda(w_1)| = |\lambda_{n'+j'}| < |\lambda_{n''+j''}| = |\lambda(w_2)|,$$

which is a contradiction. This finishes the proof of the Claim 1. ∎

For simplicity reasons let us define

$$\nu := \nu_1 = \nu_2.$$

By the proof of Claim 1, we have that

$$w_1 \in P_0^{r_1, m' + k_1 m_0}, \ \ w_2 \in P_0^{r_2, m' + k_2 m_0}$$

and

$$\lambda_{m'+k_1 m_0 + j'} = \lambda(w_1) = \lambda(w_2) = \lambda_{m'+k_2 m_0 + j''}, \tag{4.7}$$

where $m' := m + \nu k_0 m_0$, $r_1, r_2 > 0$ and $j', j'' \in \{0, \ldots, m_0 - 1\}$.

Claim 2. $k_1 = k_2$.

Proof of Claim 2. We argue by contradiction, so assume that $k_1 \neq k_2$. Without loss of generality assume that $k_1 < k_2$. Then we have

$$m' + k_1 m_0 + j' \leqslant m' + (k_1 + 1) m_0 - 1 < m' + k_2 m_0 + j''.$$

The last implies that $|\lambda_{m'+k_1 m_0 + j'}| < |\lambda_{m'+k_2 m_0 + j''}|$, which contradicts (4.7). ∎

Observe now that we also have $j' = j''$. Set $r' := r_\nu^{(m)}$, $j := j' = j''$ and $k := k_1 = k_2$. Recall that

$$w_1, w_2 \in P_2^{r', m'}.$$

By the proof of Claim 1 and the previous notations, we immediately get the following

$$w_1 \in P_0^{\mu(r' + \nu' \mu_1(m'), m', k), m' + k m_0}, \quad w_2 \in P_0^{\mu(r' + \nu'' \mu_1(m'), m', k), m' + k m_0},$$

for a certain $\nu', \nu'' \in \{0, \ldots, \nu_0^{r', m'}\}$. It is now clear that

$$|w_1| = \mu(r' + \nu' \mu_1(m'), m', k) = r' + \nu' \mu_1(m') + \sum_{N=1}^{k} \frac{c_2}{|\lambda_{m'+N m_0 - 1}|},$$

$$|w_2| = \mu(r' + \nu''\mu_1(m'), |m'|, k) = r' + \nu''\mu_1(m') + \sum_{N=1}^{k} \frac{c_2}{|\lambda_{m'+Nm_0-1}|},$$

where we used the definition of $\mu(r,m,k)$ from Step 3. It is immediate that

$$|\nu' - \nu''| \geqslant 1,$$

since $|w_1| \neq |w_2|$. We are ready for the final estimate. From the above, we arrive at the following inequality

$$\begin{aligned}|w_1\lambda(w_1) - w_2\lambda(w_2)| &\geqslant |\lambda_{m'+km_0+j}|\,\big||w_1| - |w_2|\big| = |\lambda_{m'+km_0+j}|\mu_1(m')|\nu' - \nu''| \\ &\geqslant |\lambda_{m'}|\mu_1(m')|\,| = |\lambda_{m'}| \sum_{N=1}^{k_0} \frac{c_2}{|\lambda_{m'+Nm_0-1}|} > 2c_0,\end{aligned}$$

where the last inequality is a result of property (2.4). This completes the proof of the lemma. ∎

LEMMA 4.5. *Let $m \geqslant n_0$, $m \in \mathbb{N}$, $r \in [r_0, R_0]$, θ', $\theta'' \in [\theta_0, \theta_T]$ and $v_1 < v_2$, where $v_1 \in \{m, m+1, \ldots, m+m_0k_0-1\}$, $v_2 \in \mathbb{N}$. Also, let $\varepsilon_1, \varepsilon_2$ be two non-negative real numbers so that $0 \leqslant \varepsilon_2 < \varepsilon_1 < c_4/m$. We consider the numbers $r_1 := r + \varepsilon_1$ and $r_2 := r + \varepsilon_2$ and define the discs $B(1) := B + r_1 e^{2\pi i\theta'}\lambda_{v_1}$, $B(2) := B + r_2 e^{2\pi i\theta''}\lambda_{v_2}$. Then $B(1) \cap B(2) = \emptyset$.*

Proof. By property (2.6) we have

$$\frac{m}{m + k_0m_0} > \frac{1}{2}$$

or equivalently

$$m + k_0m_0 < 2m. \tag{4.8}$$

We also have

$$m \leqslant v_1 \leqslant m + m_0k_0 - 1, \tag{4.9}$$

by our hypothesis. Hence, by (4.8), (4.9) it follows that

$$v_1 < 2m. \tag{4.10}$$

Combining (4.10) with the definition of c_4 we get

$$\frac{c_4}{m} < \frac{r_0c_3}{v_1}. \tag{4.11}$$

By (4.11) and our hypothesis we arrive at the following inequality

$$\varepsilon_1 - \varepsilon_2 < \frac{r_0c_3}{v_1}$$

or equivalently

$$2r_0c_3 - v_1(\varepsilon_1 - \varepsilon_2) > r_0c_3. \tag{4.12}$$

Since $v_1 < v_2$ and $(|\lambda_n|)$ is strictly increasing we have

$$v_1\Big(\Big|\frac{\lambda_{v_2}}{\lambda_{v_1}}\Big| - 1\Big) \geqslant v_1\Big(\Big|\frac{\lambda_{v_1+1}}{\lambda_{v_1}}\Big| - 1\Big)$$

and in view of property (2.5) (recall that $v_1 \geqslant m \geqslant n_0$)

$$rv_1\Big(\Big|\frac{\lambda_{v_2}}{\lambda_{v_1}}\Big| - 1\Big) > 2rc_3 \geqslant 2r_0c_3. \tag{4.13}$$

By (4.12), (4.13) we get

$$rv_1\Big(\Big|\frac{\lambda_{v_2}}{\lambda_{v_1}}\Big| - 1\Big) - v_1(\varepsilon_1 - \varepsilon_2) > 2r_0c_3 - v_1(\varepsilon_1 - \varepsilon_2) > r_0c_3. \tag{4.14}$$

Since $c_4 := r_0c_3/2$, inequality (4.14) combined with property (2.7) gives

$$rv_1\Big(\Big|\frac{\lambda_{v_2}}{\lambda_{v_1}}\Big| - 1\Big) - v_1(\varepsilon_1 - \varepsilon_2) > \frac{v_1}{|\lambda_{v_1}|}2c_0,$$

or equivalently

$$r\Big(\Big|\frac{\lambda_{v_2}}{\lambda_{v_1}}\Big| - 1\Big) - (\varepsilon_1 - \varepsilon_2) > \frac{1}{|\lambda_{v_1}|}2c_0.$$

Adding on the left hand side of the previous inequality the positive term

$$\varepsilon_2\Big(\frac{|\lambda_{v_2}|}{|\lambda_{v_1}|} - 1\Big),$$

we get that

$$(r+\varepsilon_2)\Big(\Big|\frac{\lambda_{v_2}}{\lambda_{v_1}}\Big| - 1\Big) - (\varepsilon_1 - \varepsilon_2) > \frac{1}{|\lambda_{v_1}|}2c_0.$$

Multiplying both sides of the above inequality by $|\lambda_{v_1}|$ we arrive at

$$(r+\varepsilon_2)|\lambda_{v_2}| - (r+\varepsilon_1)|\lambda_{v_1}| > 2c_0,$$

which implies that the disks $B(1)$, $B(2)$ are disjoint. ∎

LEMMA 4.6. *Fix a positive integer m with $m \geqslant n_0$. Then the family $\mathcal{B}_m$, defined by*

$$\mathcal{B}_m := \{B_w : w \in P_m\} \cup \{B\},$$

consists of pairwise disjoint disks.

Proof. According to Lemma 4.1, we have that $B \cap B_w = \emptyset$ for every $w \in P_m$. So let us fix $w_1, w_2 \in P_m$ with $w_1 \neq w_2$. We have to show that $B_{w_1} \cap B_{w_2} = \emptyset$. If $|w_1| = |w_2|$ according to Lemma 4.3, the conclusion follows. So, let us assume that $|w_1| \neq |w_2|$. Now we look at $\lambda(w_1), \lambda(w_2)$. If $|\lambda(w_1)| = |\lambda(w_2)|$, and keeping in mind that $|\lambda_n| = |\lambda_{n'}|$ if and only if $\lambda_n = \lambda_{n'}$, then according to Lemma 4.4, the corresponding disks B_{w_1}, B_{w_2} are disjoint. What remains to be dealt with is case $|\lambda(w_1)| \neq |\lambda(w_2)|$. Without loss of generality, assume that $|w_1| < |w_2|$. We shall consider the following two cases.

Case 1. $|\lambda(w_1)| < |\lambda(w_2)|$. Then according to Lemma 4.2 we conclude that $B_{w_1} \cap B_{w_2} = \emptyset$.

Case 2. $|\lambda(w_1)| > |\lambda(w_2)|$. According the definition of partition P_m we have that P_m is a union of partitions of order 2, so there exist pairs (r_1, m_1), (r_2, m_2) for certain $r_1, r_2 > 0$ and m_1, m_2 positive integers so that $w_1 \in P_2^{r_1,m_1}$ and $w_2 \in P_2^{r_2,m_2}$. If $(r_1, m_1) \neq (r_2, m_2)$, as the proof of Claim 1 in Lemma 4.4 and Lemma 3.5 shows, it follows that $|\lambda(w_1)| < |\lambda(w_2)|$, which is a contradiction. Therefore, w_1, w_2 belong to the same partition of order 2, say $P_2^{r',m'}$. In order to apply Lemma 4.5, we introduce the following "strange" notation:

$$r_1 := |w_2|, \quad r_2 := |w_1|.$$

Since $w_1 \in P_2^{r',m'}$, we have that $r_1 = |w_2| = r' + \varepsilon_1$ for some $0 \leqslant \varepsilon_1 < +\infty$. By a similar reasoning, we have that $r_2 = |w_1| = r' + \varepsilon_2$ for a certain positive number ε_2. Observe that $\varepsilon_1 > 0$ because $|w_1| < |w_2|$. Recall that $|w| < r' + c_4/m'$ for every $w \in P_2^{r',m'}$; see Step 4. On the other hand,

$$M^{r',m'} := \max\{|w| : w \in P_2^{r',m'}\},$$

by Step 5. Hence, we get

$$|w_1| = r_2 = r' + \varepsilon_2 \leqslant M^{r',m'} < r' + \frac{c_4}{m'} \quad \text{and} \quad |w_2| = r_1 = r' + \varepsilon_1 \leqslant M^{r',m'} < r' + \frac{c_4}{m'},$$

from which it follows that

$$\varepsilon_1 < \frac{c_4}{m'}. \tag{4.15}$$

The inequality $|w_1| < |w_2|$ implies that $\varepsilon_2 < \varepsilon_1$. From the last and (4.15) we conclude that

$$0 < \varepsilon_1 - \varepsilon_2 \leqslant \varepsilon_1 < \frac{c_4}{m'}.$$

We also have

$$\lambda(w_1) = \lambda_{v_2}, \quad \lambda(w_1) = \lambda_{v_1},$$

for some positive integers v_1, v_2 with $v_1, v_2 \geqslant m'$, $v_1 \leqslant m' + k_0 m_0 - 1$ and $v_1 < v_2$. Since $w_2 = r_1 e^{2\pi i\theta'}$, $w_1 = r_2 e^{2\pi i\theta''}$ for a certain $\theta', \theta'' \in [\theta_0, \theta_T)$, we apply Lemma 4.5 and the desired result follows. This completes the proof of the lemma. ∎

5. PROOF OF LEMMA 2.5

Let us fix some $j_1, s_1, k_1 \in \mathbb{N}$. We will prove that the set $\bigcup\limits_{m=1}^{\infty} E(m, j_1, s_1, k_1)$ is dense in $(\mathcal{H}(\mathbb{C}), \mathcal{T}_u)$.

For simplicity we write $p_{j_1} = p$. Consider a fixed $g \in \mathcal{H}(\mathbb{C})$, a compact set $C \subseteq \mathbb{C}$ and $\varepsilon_0 > 0$. We seek $f \in \mathcal{H}(\mathbb{C})$ and a positive integer m_1 so that

$$f \in E(m_1, j_1, s_1, k_1) \quad \text{and} \tag{5.1}$$

$$\sup_{z \in C} |f(z) - g(z)| < \varepsilon_0. \tag{5.2}$$

Fix $R_1 > 0$ sufficiently large so that $C \cup \{z \in \mathbb{C} : |z| \leqslant k_1\} \subset \{z \in \mathbb{C} : |z| \leqslant R_1\}$. Choose $0 < \delta_0 < 1$ so that

$$\text{if } |z| \leqslant R_1 \text{ and } |z - w| < \delta_0 \text{ then } |p(z) - p(w)| < \frac{1}{2s_1}. \tag{5.3}$$

We set

$$B := \{z \in \mathbb{C} : |z| \leqslant R_1 + \delta_0\}, \quad c_0 := R_1 + \delta_0, \quad c_1 := \frac{4\pi(R_1 + \delta_0)}{\delta_0},$$
$$c_2 := \frac{\delta_0}{2R_0}, \quad m_0 := \left[\frac{R_0}{r_0} c_1\right] + 1, \quad k_0 := \left[\frac{2(R_1 + \delta_0)}{c_2}\right] + 1,$$
$$c_3 \text{ any, fixed, positive number} \quad \text{and} \quad c_4 := \frac{r_0 c_3}{2}.$$

Fix a natural number n_0 so that all properties (2.1)–(2.8) hold for every $n \geqslant n_0$ with respect to the above fixed quantities. Let us also fix a positive integer $m \geqslant n_0$. After that, on the basis of the fixed numbers $r_0, R_0, \theta_0, \theta_T, c_0, c_1, c_2, c_3, k_0, m_0$ and the natural number m we define the set L_m as follows:

$$L_m := B \cup \Big(\bigcup\nolimits_{w \in P_m} B_w\Big),$$

where the discs B_w, $w \in P_m$ are constructed in Subsection 4.1. By Lemma 4.6, the disks in the family $\mathcal{B}_m$ are pairwise disjoint. Therefore the compact set L_m has connected complement. This property is needed in order to apply Mergelyan's theorem later. We now define the function h on the compact set L_m by

$$h(z) = \begin{cases} g(z) & z \in B, \\ p(z - w\lambda(w)) & z \in B_w, w \in P_m. \end{cases}$$

By Mergelyan's theorem [35] there exists an entire function f (in fact a polynomial) so that

$$\sup_{z \in L_m} |f(z) - h(z)| < \min\left\{\frac{1}{2s_1}, \varepsilon_0\right\}. \tag{5.4}$$

By the definition of h and (5.4), it follows that

$$\begin{aligned} \sup_{z \in C} |f(z) - g(z)| &\leqslant \sup_{z \in B} |f(z) - g(z)| = \sup_{z \in B} |f(z) - h(z)| \\ &\leqslant \sup_{z \in L_m} |f(z) - h(z)| < \varepsilon_0 \end{aligned} \tag{5.5}$$

which implies the desired inequality (5.2).

It remains to show (5.1).

Let $a \in S$. We can write $a = re^{2\pi i\theta}$ for a certain $r \in [r_0, R_0]$ and $t \in [\theta_0, \theta_T]$. Since $P_m = \bigcup P_0^{r',m'}$, consider all the r' that appear in the previous union and order them as follows: $r_0 < r_1 < \cdots < r_N \leqslant R_0$ for a certain $N \in \mathbb{N}$. Then either there exists a unique $\nu \in \{0, 1, \ldots, N-1\}$ so that $r_\nu \leqslant r < r_{\nu+1}$ or $r_N \leqslant r \leqslant R_0$. Define

$$\begin{aligned} &r_1 := r_\nu, \quad r_2 := r_{\nu+1}, \quad \text{if } r_\nu \leqslant r < r_{\nu+1}, \quad \text{and} \\ &r_1 := r_N, \quad r_2 := R_0, \quad \text{if } r_N \leqslant r \leqslant R_0. \end{aligned}$$

Observe that in either case we have $r_1 \leqslant r \leqslant r_2$.

Consider now all the partitions with height r_1 and order 0, $P_0^{r_1,m'}$ that appear in P_m. By the construction of P_m, either there exists a unique m' so that the partition $P_0^{r_1,m'}$ appears in P_m, in other words there exists a unique partition of order 0 with height r_1, or there exist exactly two different partitions of order 0 and height r_1, say $P_0^{r_1,m'}$, $P_0^{r_1,m''}$. In the latter case, we consider the partition with the biggest density, for which we use again the symbol $P_0^{r_1,m'}$.

In the above paragraph, we fixed a partition of order 0 and height r_1, $P_0^{r_1,m'}$. The positive integer m' reflects the density of the partition and remember that, in Step 1,

$$\Delta_{m'} := \{\theta_0^{(m')}, \theta_1^{(m')}, \ldots, \theta_{\nu_{m'}}^{(m')}\}.$$

It now follows that either there exists a unique $j \in \{1, 2, \ldots, \nu_{m'} - 1\}$ so that

$$\theta_j^{(m')} \leqslant \theta < \theta_{j+1}^{(m')} \quad \text{or} \quad \theta_{\nu_{m'}}^{(m')} \leqslant \theta \leqslant \theta_T.$$

Then we define

$$\begin{aligned} &\theta_1 := \theta_j^{(m')}, \quad \theta_2 := \theta_{j+1}^{(m')}, \quad \text{if } \theta_j^{(m')} \leqslant \theta < \theta_{j+1}^{(m')} \quad \text{and} \\ &\theta_1 := \theta_{\nu_{m'}}^{(m')}, \quad \theta_2 := \theta_T, \quad \text{if } \theta_{\nu_{m'}}^{(m')} \leqslant \theta \leqslant \theta_T. \end{aligned}$$

Let us also define

$$w_0 := r_1 e^{2\pi i \theta_1} \in P_m.$$

We will now prove that for every $z \in \mathbb{C}$ with $|z| \leqslant R_1$ we have $z + a\lambda(w_0) \in B_{w_0}$. Remember that $B_{w_0} := B + w_0\lambda(w_0)$. We have $B_{w_0} = \overline{D}(w_0\lambda(w_0), R_1 + \delta_0)$. Thus, it suffices to prove that

$$|(z + a\lambda(w_0)) - w_0\lambda(w_0)| < R_1 + \delta_0, \quad \text{for } |z| \leqslant R_1. \tag{5.6}$$

For $|z| \leqslant R_1$ we have

$$|z + a\lambda(w_0) - w_0\lambda(w_0)| \leqslant R_1 + |\lambda(w_0)|\,|re^{2\pi i\theta} - r_1 e^{2\pi i\theta_1}|. \tag{5.7}$$

By (5.7), in order to prove (5.6) it suffices to prove that

$$|\lambda(w_0)|\,|re^{2\pi i\theta} - r_1 e^{2\pi i\theta_1}| < \delta_0. \tag{5.8}$$

We have now:

$$\begin{aligned} |re^{2\pi i\theta} - r_1 e^{2\pi i\theta_1}| &\leqslant |r_1 - r_2| + R_0|e^{2\pi i\theta_1} - e^{2\pi i\theta_2}| \\ &\leqslant \frac{\delta_0}{2R_0} \cdot \frac{1}{|\lambda(w_0)|} + R_0 2\sin(\pi(\theta_2 - \theta_1)) \end{aligned}$$

$$< \frac{\delta_0}{2R_0} \cdot \frac{1}{|\lambda(w_0)|} + 2R_0\pi(\theta_2 - \theta_1)$$
$$< \frac{\delta_0}{2R_0} \cdot \frac{1}{|\lambda(w_0)|} + 2\pi R_0 \cdot \frac{\delta_0}{4\pi R_0} \cdot \frac{1}{|\lambda(w_0)|} = \frac{\delta_0}{2|\lambda(w_0)|}\Big(\frac{1}{R_0}+1\Big).$$

So

$$|\lambda(w_0)|\,|re^{2\pi i\theta} - r_1 e^{2\pi i\theta_1}| < \frac{\delta_0}{2}\Big(\frac{1}{R_0}+1\Big) < \delta_0,$$

because $R_0 > 1$, which implies (5.8). For z with $|z| \leqslant R_1$ we have

$$\begin{aligned} |f(z+a\lambda(w_0)) - p(z)| &\leqslant |f(z+a\lambda(w_0)) - p(z+\lambda(w_0)(re^{2\pi i\theta} - r_1e^{2\pi i\theta_1}))| \\ &\quad + |p(z+\lambda(w_0)(re^{2\pi i\theta} - r_1e^{2\pi i\theta_1})) - p(z)|. \end{aligned} \tag{5.9}$$

Previously, we proved that for every $|z| \leqslant R_1$ we have $z + a\lambda(w_0) \in B_{w_0}$. Thus, by the definition of h and (5.4) we have

$$|f(z+a\lambda(w_0)) - p(z+\lambda(w_0)(re^{2\pi i\theta} - r_1e^{2\pi i\theta_1}))| < \frac{1}{2s_1}. \tag{5.10}$$

By (5.8) and (5.3) for $|z| \leqslant R_1$ we have

$$|p(z+\lambda(w_0)(re^{2\pi i\theta} - r_1e^{2\pi i\theta_1})) - p(z)| < \frac{1}{2s_1}. \tag{5.11}$$

By (5.9), (5.10) and (5.11) we get

$$\sup_{|z|\leqslant R_1} |f(z+a\lambda(w_0)) - p(z)| < \frac{1}{s_1}.$$

So

$$\sup_{|z|\leqslant k_1} |f(z+a\lambda(w_0)) - p(z)| < \frac{1}{s_1}. \tag{5.12}$$

Setting

$$m_1 := \max\{n \in \mathbb{N} : \lambda_n = \lambda(w) \text{ for some } w \in P_m\},$$

we have that for every $a \in S$ there exists $w_0 \in P_m$ such that $\lambda(w_0) = \lambda_n$ for a certain $n \in \mathbb{N}$ with $n \leqslant m_1$ and (5.12) holds. Clearly the last implies that $f \in E(m_1, j_1, s_1, k_1)$, (5.1) holds and the proof of Lemma 2.5 is complete. ∎

6. PROOF OF LEMMA 2.3

By Mergelyan's theorem it easily follows that

$$U := \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} E(m,j,s,k) \subseteq \bigcap_{a\in S} HC(\{T_{\lambda_n a}\}).$$

We have to show the reverse inclusion. For every polynomial p of one complex variable with coefficients in $\mathbb{Q}+i\mathbb{Q}$ define the set

$$\mathcal{U}(p):=\Big\{f\in\mathcal{H}(\mathbb{C})\mid \forall\, a\in S\,\exists\,(m_n)\subset\mathbb{N}:\ \forall r>0 \lim_{n\to+\infty}\sup_{|z|\leqslant r}|f(z+\lambda_{m_n}a)-p(z)|=0\Big\}.$$

Let p_j, $j=1,2,\dots$ be an enumeration of all polynomials of one complex variable with coefficients in $\mathbb{Q}+i\mathbb{Q}$. We see easily that

$$\bigcap_{a\in S} HC(\{T_{\lambda_n a}\})=\bigcap_{j=1}^{\infty}\mathcal{U}(p_j). \tag{6.1}$$

For $x>0$ and $n,j\in\mathbb{N}$ define the set

$$V(x,n,j):=\Big\{f\in\mathcal{H}(\mathbb{C}):\forall a\in S\,\exists\, m\in\mathbb{N}, m\leqslant n \text{ with } \sup_{|z|\leqslant x}|f(z+\lambda_m a)-p_j(z)|<\frac{1}{x}\Big\}.$$

We shall show that the following holds:

$$\mathcal{U}(p_j)\subseteq\bigcap_{x>0}\bigcup_{n=1}^{\infty}V(x,n,j). \tag{6.2}$$

Let $f\in\mathcal{H}(\mathbb{C})$, $x_0>0$, $j_0,m_0\in\mathbb{N}$ and consider the set

$$V_f(j_0,x_0,m_0):=\Big\{a\in S:\sup_{|z|\leqslant x_0}|f(z+\lambda_{m_0}a)-p_{j_0}(z)|<\frac{1}{x_0}\Big\}.$$

We first show that $V_f(j_0,x_0,m_0)$ is open in S. Let $a_0\in V_f(j_0,x_0,m_0)$ and take (a_ν) a sequence in S so that $a_\nu\to a_0$. We have

$$\begin{aligned}\sup_{|z|\leqslant x_0}|f(z+\lambda_{m_0}a_\nu)-p_{j_0}(z)| &\leqslant \sup_{|z|\leqslant x_0}|f(z+\lambda_{m_0}a_0)-p_{j_0}(z)|\\ &\quad+\sup_{|z|\leqslant x_0}|f(z+\lambda_{m_0}a_\nu)-f(z+\lambda_{m_0}a_0)|\end{aligned} \tag{6.3}$$

for every $\nu=1,2,\dots$.

The function $\varphi:S\times\overline{D(0,x_0)}\to\mathbb{C}$ defined by $\varphi(a,z)=z+\lambda_{m_0}a$ is continuous, where the set $S\times\overline{D(0,x_0)}$ is endowed with the product topology,

$$\rho:(S\times\overline{D(0,x_0)})\times(S\times\overline{D(0,x_0)})\to\mathbb{R}^+\quad \rho((\beta,z_1),(\gamma,z_2))=\sqrt{|\beta-\gamma|^2+|z_1-z_2|^2},$$
$$\beta,\gamma\in S,\ z_1,z_2\in\overline{D(0,x_0)}.$$

Setting

$$\varepsilon_0:=\frac{1}{x_0}-\sup_{|z|\leqslant x_0}|f(z+\lambda_{m_0}a_0)-p_{j_0}(z)|,$$

we observe that $\varepsilon_0>0$ since $a_0\in V_f(j_0,x_0,m_0)$. By the uniform continuity of $f\circ\varphi$ on $S\times\overline{D(0,x_0)}$, there exists $\delta_0>0$ so that for each $x,y\in S\times\overline{D(0,x_0)}$, $\rho(x,y)<\delta_0$ it holds $|(f\circ\varphi)(x)-(f\circ\varphi)(y)|<\varepsilon_0$. Since $a_\nu\to a_0$, there exists $n_0\in\mathbb{N}$ such that $|a_\nu-a_0|<\delta_0$ for each $\nu\in\mathbb{N}$, $\nu\geqslant n_0$. Now for every $z\in\overline{D(0,x_0)}$ and $\nu\geqslant n_0$,

$\nu \in \mathbb{N}$, we have $\rho((a_\nu, z), (a_0, z)) = \sqrt{|a_\nu - a_0|^2 + |z - z|^2} = |a_\nu - a_0| < \delta_0$. So $|(f \circ \varphi)(a_\nu, z) - (f \circ \varphi)(a_0, z)| < \varepsilon_0, \nu \geqslant n_0$, which in turn implies

$$\sup_{|z| \leqslant x_0} |f(z + \lambda_{m_0} a_0) - p_{j_0}(z)| + \sup_{|z| \leqslant x_0} |f(z + \lambda_{m_0} a_\nu) - f(z + \lambda_{m_0} a_0)| < \frac{1}{x_0} \tag{6.4}$$

for $\nu \geqslant n_0$. In view of (6.3) and (6.4) there exists $n_0 \in \mathbb{N}$ so that for every $\nu \geqslant n_0$, $a_\nu \in V_f(j_0, x_0, m_0)$. From the last we conclude that the set $V_f(j_0, x_0, m_0)$ is open.

Thus, for every $f \in \mathcal{H}(\mathbb{C}), j, m \in \mathbb{N}$ and every $x > 0$ the set $V_f(j, x, m)$ is open in S.

Take $g \in \mathcal{U}(p_{j_0})$. Then for each $a \in S$ there exists a subsequence $(\lambda_{m_n(a)})$ of (λ_n) (that depends on a), so that for every $r > 0$

$$\sup_{|z| \leqslant r} |g(z + \lambda_{m_n(a)} a) - p_{j_0}(z)| \to 0 \quad \text{as } n \to +\infty.$$

In particular we get $\sup_{|z| \leqslant x_0} |g(z + \lambda_{m_n(a)} a) - p_{j_0}(z)| \to 0$, as $n \to +\infty$. Thus, for $\varepsilon = 1/x_0$ we have that for every $a \in S$ there exists $n_a \in \mathbb{N}$ (that depends on a) so that for each $n \geqslant n_a$, $n \in \mathbb{N}$, it holds $\sup_{|z| \leqslant x_0} |g(z + \lambda_{m_n(a)} a) - p_{j_0}(z)| < 1/x_0$. Therefore, the set

$$\mathcal{N}(j_0, x_0, g) := \left\{ n \in \mathbb{N} \mid \exists\, a \in S : \sup_{|z| \leqslant x_0} |g(z + \lambda_n a) - p_{j_0}(z)| < \frac{1}{x_0} \right\}$$

is non-empty. It is obvious by the above definitions that

$$V_g(j_0, x_0, m) \subset S \quad \text{for each } m \in \mathbb{N}. \tag{6.5}$$

Let a certain $a \in S$. Then there exists $n \in \mathcal{N}(j_0, x_0, g)$ so that $a \in V_g(j_0, x_0, n)$. Hence we get

$$S \subseteq \bigcup_{n \in \mathcal{N}(j_0, x_0, g)} V_g(j_0, x_0, n). \tag{6.6}$$

Now, (6.5) and (6.6) imply $S = \bigcup_{n \in \mathcal{N}(j_0, x_0, g)} V_g(j_0, x_0, n)$, so the family $V_g(j_0, x_0, n)$, $n \in \mathcal{N}(j_0, x_0, g)$ is an open covering of S. Since S is a compact set, there exists a finite subset $A \subset \mathcal{N}(j_0, x_0, g)$, $A = \{\nu_1, \nu_2, \ldots, \nu_{m_0}\}$ so that $S = \bigcup_{n=1}^{m_0} V_g(j_0, x_0, \nu_n)$. Let $\ell_0 := \max A$. Then for each $a \in S$, there exists $n \in \mathbb{N}$, $n \leqslant \ell_0$ so that

$$\sup_{|z| \leqslant x_0} |g(z + \lambda_n a) - p_{j_0}(z)| < \frac{1}{x_0}.$$

It follows that $\mathcal{U}(p_{j_0}) \subset V(x_0, \ell_0, j_0)$ for arbitrary $x_0 > 0$, from which we get

$$\mathcal{U}(p_{j_0}) \subset \bigcap_{x > 0} \bigcup_{n=1}^{\infty} V(x, n, j_0). \tag{6.7}$$

Thus (6.2) holds for every $j = 1, 2, \ldots$. It is obvious that

$$\bigcap_{x>0} \bigcup_{n=1}^{\infty} V(x, n, j_0) \subset \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} V(m, n, j_0). \tag{6.8}$$

By (6.2) and (6.8) we get

$$\mathcal{U}(p_j) \subset \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} V(m, n, j) \quad \text{for every } j = 1, 2, \ldots. \tag{6.9}$$

So

$$\bigcap_{j=1}^{\infty} \mathcal{U}(p_j) \subset \bigcap_{j=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} V(m, n, j). \tag{6.10}$$

By (6.1) and (6.10) we have

$$\bigcap_{a \in S} HC(\{T_{\lambda_n a}\}) \subset \bigcap_{j=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} V(m, n, j), \tag{6.11}$$

and now it is plain that

$$U \subset \bigcap_{j=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} V(m, n, j). \tag{6.12}$$

We consider the following families of sets

$$\mathcal{D}_1 := \Big\{ \bigcup_{n=1}^{\infty} V(m, n, j), m, j \in \mathbb{N} \Big\} \quad \text{and} \quad \mathcal{D}_2 := \Big\{ \bigcup_{m=1}^{\infty} E(s, j, k, m), s, j, k \in \mathbb{N} \Big\}.$$

Clearly $\mathcal{D}_1 \subseteq \mathcal{D}_2$. Thus

$$\bigcap_{E \in \mathcal{D}_2} E \subset \bigcap_{V \in \mathcal{D}_1} V. \tag{6.13}$$

Let $E = E(s, j, k, m)$ for some $s, j, k, m \in \mathbb{N}$. Then $V(\ell, m, j) \subset E$ for $\ell = \max\{s, k\}$. Hence, for every $E \in \mathcal{D}_2$ there exists $\Gamma \in \mathcal{D}_1$ such that $\Gamma \subset E$. If we set $\widetilde{\mathcal{D}} = \{\Gamma \in \mathcal{D}_1 \mid \exists\, E \in \mathcal{D}_2 : \Gamma \subset E\}$, it follows that $\bigcap_{\Gamma \in \widetilde{\mathcal{D}}} \Gamma \subset \bigcap_{E \in \mathcal{D}_2} E$. But then

$$\bigcap_{V \in \mathcal{D}_1} V \subset \bigcap_{\Gamma \in \widetilde{\mathcal{D}}} \Gamma \subset \bigcap_{E \in \mathcal{D}_2} E. \tag{6.14}$$

By (6.12), (6.13) and (6.14) we have

$$U = \bigcap_{j=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} V(m, n, j). \tag{6.15}$$

Now (6.11) and (6.15) yield

$$\bigcap_{a \in S} HC(\{T_{\lambda_n a}\}) \subset U \tag{6.16}$$

and the proof of Lemma 2.3 is complete. ∎

The above lemma holds with the same proof for every compact subset $K \subseteq \mathbb{C} \setminus \{0\}$ instead of S and for every sequence of non-zero complex numbers (λ_n) so that $\lambda_n \to \infty$ as $n \to +\infty$.

7. FINAL STEP OF THE PROOF OF THEOREM 1.2

To conclude the proof of Theorem 1.2, we need the following three elementary lemmas.

LEMMA 7.1. *Let (λ_n) be a sequence of non-zero complex numbers such that $\lambda_n \to \infty$ as $n \to +\infty$. Suppose that $\limsup_{n\to+\infty} |\lambda_{n+1}/\lambda_n| = 1$. Then for any fixed positive numbers M_1, M_2, there exists a subsequence (μ_n) of (λ_n) with the following properties:*
(i) $|\mu_{n+1}| - |\mu_n| > M_1$, *for every* $n = 1, 2, \ldots$;
(ii) $|\mu_{n+1}/\mu_n| \to 1$ *as* $n \to +\infty$;
(iii) $\liminf_{n\to+\infty} (n(|\mu_{n+1}/\mu_n| - 1)) > M_2$.

Proof. We prove this lemma in three steps.

Step 1. We construct a subsequence (θ_n) of (λ_n) so that $(|\theta_n|)$ is strictly increasing and $|\theta_{n+1}/\theta_n| \to 1$ as $n \to +\infty$.

Step 2. We construct a subsequence (k_n) of (θ_n) so that $|k_{k+1}| - |k_n| > M_1$ $\forall n = 1, 2, \ldots$ and $|k_{n+1}/k_n| \to 1$ as $n \to +\infty$.

Step 3. Finally, we construct a subsequence (μ_n) of (k_n) which has the three properties (i), (ii) and (iii) of the lemma.

Proof of Step 1. We set $\theta_1 := \lambda_1$. Let $n_1 \geqslant 2$ be the smallest natural number so that $|\lambda_{n_1}| > |\lambda_1|$. Define $\theta_2 := \lambda_{n_1}$. Suppose now that we have inductively constructed the numbers $\lambda_{n_1}, \lambda_{n_2}, \ldots, \lambda_{n_k}$ for a certain $k \geqslant 2$, where $|\lambda_{n_{i+1}}| > |\lambda_{n_i}|$ and n_{i+1} is the smallest natural number so that $n_{i+1} > n_i$ and $|\lambda_{n_{i+1}}| > |\lambda_{n_i}|$ for every $i = 1, 2, \ldots, k-1$. Set $\theta_{i+1} = \lambda_{n_i}$ for $i = 1, \ldots, k$. Next we consider the number $\lambda_{n_{k+1}}$, where n_{k+1} is the smallest natural number with $n_{k+1} \geqslant n_k + 1$, and so that $|\lambda_{n_{k+1}}| > |\lambda_{n_k}|$, and we set $\theta_{k+2} = \lambda_{n_{k+1}}$.

So, we have constructed a subsequence (θ_n) of (λ_n) so that the sequence $(|\theta_n|)$ is strictly increasing. For every $k \in \mathbb{N}$ we have

$$1 < \left|\frac{\lambda_{n_{k+1}}}{\lambda_{n_k}}\right| \leqslant \left|\frac{\lambda_{n_{k+1}}}{\lambda_{n_{k+1}-1}}\right|$$

and by our assumptions on (λ_n) we conclude that $|\theta_{n+1}|/|\theta_n| \to 1$. ∎

Proof of Step 2. Now we construct a subsequence of (θ_n) as follows. We set $k_1 := \theta_1$. Let v_1 be the smallest natural number so that $v_1 \geqslant 2$ and $|\theta_{v_1}| > |k_1| + M_1$. Set now $k_2 := \theta_{v_1}$. Suppose that we have inductively constructed the numbers $\theta_1, \theta_{v_1}, \ldots, \theta_{v_m}$ for a certain $m \geqslant 2$, where v_{i+1} is the smallest natural number so that $|\theta_{v_{i+1}}| > |\theta_{v_i}| + M_1$ and $v_{i+1} > v_i$ for each $i = 1, \ldots, m-1$. Then set $k_{i+1} = \theta_{v_i}$ for $i = 1, \ldots, m$. Next we consider the smallest natural number $v_{m+1} \geqslant v_m + 1$ so that $|\theta_{v_{m+1}}| > |\theta_{v_m}| + M_1$ and we set $k_{m+2} = \theta_{v_{m+1}}$.

Therefore we have constructed a subsequence (k_n) of (θ_n) where $|k_{n+1}| > |k_n| + M_1$ for each $n = 1, 2, \ldots$. For every $m = 1, 2, \ldots$ it holds that $|\theta_{v_m}| \leqslant |\theta_{v_{m+1}-1}| \leqslant |\theta_{v_m}| + M_1$, which implies

$$1 \leqslant \Big|\frac{\theta_{v_{m+1}-1}}{\theta_{v_m}}\Big| \leqslant 1 + \frac{M_1}{|\theta_{v_m}|}. \tag{7.1}$$

On the other hand we have

$$1 < \Big|\frac{\theta_{v_{m+1}}}{\theta_{v_{m+1}-1}}\Big|, \quad m = 1, 2, \ldots \quad \text{and} \tag{7.2}$$

$$\lim_{n \to +\infty} \Big|\frac{\theta_{n+1}}{\theta_n}\Big| = 1. \tag{7.3}$$

By (7.1), (7.2) and (7.3) we conclude that $|k_{n+1}|/|k_n| \to 1$ as $n \to +\infty$. ∎

Proof of Step 3. We construct inductively a subsequence (μ_n) of (k_n) as follows. Set $\mu_1 := k_1$. Let σ_1 be the smallest natural number so that $\sigma_1 \geqslant 2$ and $|k_{\sigma_1}| > |k_1|(1 + M_2/1)$ and then define $\mu_2 := k_{\sigma_1}$. After, let σ_2 be the smallest natural number so that $\sigma_2 \geqslant \sigma_1 + 1$, $k_{\sigma_2} \geqslant \mu_2 + 1$ and $|k_{\sigma_2}| > |k_{\sigma_1}| \cdot (1 + M_2/2)$ and define $\mu_3 := k_{\sigma_2}$. In this way, we construct inductively a subsequence (μ_n) of (k_n) so that for every $n = 2, 3, \ldots$ the natural number σ_n is the smallest with the following properties: $k_{\sigma_n} \geqslant \mu_n + 1$, $\sigma_n \geqslant \sigma_{n-1} + 1$,

$$|\mu_{n+1}| \geqslant |\mu_n|\Big(1 + \frac{M_2}{n}\Big), \tag{7.4}$$

and $\mu_{n+1} = k_{\sigma_n}$. ∎

As a consequence of the above construction we get

$$1 \leqslant \Big|\frac{k_{\sigma_{n+1}-1}}{k_{\sigma_n}}\Big| < 1 + \frac{M_2}{n+1}, \quad n = 1, 2, \ldots \tag{7.5}$$

$$1 < \Big|\frac{k_{\sigma_{n+1}}}{k_{\sigma_{n+1}-1}}\Big|, \quad n = 1, 2, \ldots \tag{7.6}$$

$$\Big|\frac{k_{n+1}}{k_n}\Big| \to 1 \quad \text{as}\, n \to +\infty. \tag{7.7}$$

By (7.5), (7.6) and (7.7) we conclude that $|\mu_{n+1}|/|\mu_n| \to 1$ as $n \to +\infty$ and the sequence (μ_n) has all the desired properties. This completes the proof the lemma. ∎

LEMMA 7.2. *Let (λ_n) be a sequence of non-zero complex numbers so that $\lambda_n \to \infty$ as $n \to +\infty$. Suppose that*

$$\limsup_{n \to +\infty} \Big|\frac{\lambda_{n+1}}{\lambda_n}\Big| \leqslant 1 + \varepsilon$$

for some $\varepsilon > 0$. Then for every pair (M_1, M_2) of positive numbers there exists a subsequence (μ_n) of (λ_n) with the following properties:

(i) $|\mu_1| > M_1$;
(ii) $|\mu_{n+1}| - |\mu_n| > M_1$, $n = 1, 2, \dots$;
(iii) $\limsup\limits_{n\to+\infty} |\mu_{n+1}/\mu_n| \leqslant 1 + \varepsilon$;
(iv) $\liminf\limits_{n\to+\infty} (n(|\mu_{n+1}/\mu_n| - 1)) > M_2$.

Proof. The proof is almost identical to the proof of Lemma 7.1. The only difference is that, whenever needed, instead of $\limsup\limits_{n\to+\infty} |\lambda_{n+1}/\lambda_n| = 1$ we use $\limsup\limits_{n\to+\infty} |\lambda_{n+1}/\lambda_n| \leqslant 1 + \varepsilon$. ∎

By Lemma 7.2, along with elementary considerations, we obtain the following lemma, whose proof is left to the interested reader.

LEMMA 7.3. *Let $\Lambda := (\lambda_n)$ be a fixed sequence of non-zero complex numbers such that $\lambda_n \to \infty$ as $n \to +\infty$. Then $i(\Lambda) = 1$ if and only if for every positive number σ_j, $j = 1, 2, 3, 4, 5$ and positive integers m_0, k_0 with $m_0 \geqslant [\sigma_1] + 1$, $k_0 \geqslant [\sigma_3] + 1$ there exist a subsequence (μ_n) of (λ_n) and a positive integer n_0 so that for every $n \geqslant n_0$ the following five properties hold:*
(i) $|\mu_n| \cdot \sum\limits_{k=0}^{m_0-1} (1/|\mu_{n+k}|) > \sigma_1$;
(ii) $|\mu_{n+1}| - |\mu_n| > \sigma_2$;
(iii) $|\mu_n| \cdot \sum\limits_{i=1}^{k_0} (1/|\mu_{n+im_0-1}|) > \sigma_3$;
(iv) $n(|\mu_{n+1}/\mu_n| - 1) > \sigma_4$;
(v) $n/|\mu_n| < \sigma_5$.

Proof of Theorem 1.2. A careful inspection of the proof of Proposition 2.2 shows that the conclusion of Proposition 2.2 holds whenever the sequence (λ_n) has a subsequence μ_n that satisfies the properties (2.1)–(2.8) in Subsection 2.1. It readily follows that

$$\bigcap_{a\in S} HC(\{T_{\lambda_n a}\})$$

is G_δ and dense subset of $(\mathcal{H}(\mathbb{C}), \mathcal{T}_{\mathrm{u}})$. Then, applying Baire's category theorem once more, and referring to the discussion after the statement of Theorem 2.1, we conclude the proof of Theorem 1.2. ∎

8. EXAMPLES OF SEQUENCES $\Lambda := (\lambda_n)$ WITH $i(\Lambda) = 1$

Let $\Lambda = (\lambda_n)$ be a sequence of non-zero complex numbers. Define the set

$$\mathcal{B}(\Lambda) := \Big\{ a \in [0, +\infty] : \exists (\mu_n) \subset \Lambda \,\text{with}\, a = \limsup_n \Big| \frac{\mu_{n+1}}{\mu_n} \Big| \Big\}.$$

Observe now that, by definition, $i(\Lambda) = \inf \mathcal{B}(\Lambda)$ and whenever $\lambda_n \to \infty$ then $\mathcal{B}(\Lambda) \subset [1, +\infty]$. We shall present four distinct classes of sequences $\Lambda := (\lambda_n)$ satisfying the property $i(\Lambda) = 1$ in order to illustrate our main result, Theorem 1.2.

8.1. EXAMPLES WITH $\lambda_n \to \infty$ AND $|\lambda_{n+1}|/|\lambda_n| \to 1$. A sample of sequences satisfying the previously mentioned properties is: n, n^2, $p(n)$ where p is a non-constant complex polynomial, $\log n$, $n^\beta \log n$, $\beta > 0$, $n^\gamma / \log(n+1)$, $\gamma > 2$ etc. Of course, one can assign fixed unimodular numbers with arbitrary arguments to each term of the above sequences and still satisfy the desired properties i.e. $e^{i\theta_n} n^2$, $e^{i\theta_n} \log n$ for $\theta_n \in \mathbb{R}$, etc.

A more interesting example is the sequence e^{n^c}, for $0 < c < 1$, which has super-polynomial growth. Observe that the case $c = 1$, is borderline for the validity of Theorem 1.2. Indeed, as we already mentioned in the Introduction,

$$\bigcap_{a \in \{z : |z| = 1\}} HC(\{T_{e^n a}\}) = \emptyset,$$

through the main result in [25].

A last family of sequences, satisfying the above properties, which we would like to mention is the following: $e^{n/\log n}$, $e^{n/\log\log n}$, etc. Note that such sequences grow faster than any sequence of the form e^{n^c}, $0 < c < 1$.

8.2. EXAMPLES WITH $\lambda_n \to \infty$, THE LIMIT $\lim_{n \to +\infty} |\lambda_{n+1}/\lambda_n|$ DOES NOT EXIST, BUT $\limsup_n |\lambda_{n+1}/\lambda_n| = 1$. There is a plethora of sequences exhibiting such behavior. For instance, set $\lambda_1 = 1$. We shall define the sequence (λ_n) inductively according to the following rule. If for a certain $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ so that $\lambda_k = n^2$, then define

$$\lambda_{k+i} := n + i - 1 \quad \text{for every } i = 1, 2, \ldots, n^2 + n + 2.$$

It is easy to show that the sequence (λ_n) has the desired properties.

8.3. EXAMPLES WITH $\lambda_n \to \infty$, $\limsup_n |\lambda_{n+1}/\lambda_n| > 1$ AND $\limsup_n |\mu_{n+1}/\mu_n| = 1$ FOR SOME SUBSEQUENCE (μ_n) OF (λ_n). Take $\lambda_{2n+1} = n$, $\lambda_{2n} = 2^n$ for $n = 1, 2, \ldots$ or more generally, fix a sequence of positive numbers γ_n satisfying $\gamma_n \to \infty$, $\gamma_{n+1}/\gamma_n \to 1$, consider a strictly increasing sequence (m_n) of positive integers with $m_n > n$ for every $n = 1, 2, \ldots$ and then define $\lambda_{m_n} = \gamma_n$ so that it is correct for $\{\rho_1 < \rho_2 < \cdots\} := \mathbb{N} \setminus \{m_n : n = 1, 2, \ldots\}$ define λ_{ρ_n} to be any positive number so that $\lambda_{\rho_n} \to +\infty$ and $\lambda_{\rho_{n+1}}/\lambda_{\rho_n} \to c$ for a certain $c \in (1, +\infty]$.

8.4. EXAMPLES WITH $\lambda_n \to \infty$, $\inf \mathcal{B}(\Lambda) \notin \mathcal{B}(\Lambda)$ AND $i(\Lambda) = 1$. In all the above examples, we have that $\inf \mathcal{B}(\Lambda) \in \mathcal{B}(\Lambda)$. This means that the above infimum becomes minimum. We shall now differentiate from this situation by exhibiting examples of $\Lambda = (\lambda_n)$ such that $\lambda_n \to \infty$, $i(\Lambda) = 1$ and for every subsequence (μ_n) of (λ_n) we have $\limsup_n |\mu_{n+1}/\mu_n| > 1$. To produce such an example is not an easy task, as it requires a considerable amount of work, though elementary, concerning a particular representation of positive integers involving powers of 10. Therefore, we omit the details and we just state the following lemma without proof.

LEMMA 8.1. *For every positive integer* $n \geqslant 11$ *there exists a unique trio* (ν, k, j) *with* $\nu \in \mathbb{N} \setminus \{1\}$, $k \in \{1, 2, \dots, \nu\}$, $j \in \{1, 2, \dots, 10^k\}$ *such that*

$$n = \frac{10}{9}\Big(\frac{10}{9}(10^{\nu-1} - 1) - \nu + 10^{k-1}\Big) + j.$$

Define now the sequence (λ_n) by

$$\lambda_n = \Big(1 + \frac{1}{k}\Big)^{(\nu-k+1)10^k+j} \quad \text{for } n \geqslant 11,$$

where for every given positive integer n with $n \geqslant 11$, the numbers ν, k, j are uniquely determined by Lemma 8.1. It turns out, after a lengthy argument, that the sequence (λ_n) has the desired properties.

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