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GRADED C^*-ALGEBRAS, GRADED K-THEORY, AND TWISTED P-GRAPH C^*-ALGEBRAS

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ABSTRACT. We develop methods for computing graded K-theory of C^*-algebras as defined in terms of Kasparov theory. We establish graded versions of Pimsner's six-term exact sequences for graded Hilbert bimodules whose left action is injective and by compacts, and a graded Pimsner–Voiculescu sequence. We introduce the notion of a twisted P-graph C^*-algebra and establish connections with graded C^*-algebras. Specifically, we show how a functor from a P-graph into the group of order two determines a grading of the associated C^*-algebra. We apply our graded version of Pimsner's exact sequence to compute the graded K-theory of a graph C^*-algebra carrying such a grading.

KEYWORDS: *KK-theory, graded K-theory, C^*-algebra, P-graph, twisted C^*-algebra, graded C^*-algebra.*

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1. INTRODUCTION

This paper has two objectives. The first is to develop techniques for computing graded K-theory of C^*-algebras as defined in terms of Kasparov theory, with a view to expanding on Haag's computation of graded K-theory of Cuntz algebras [15], [14]. The second is to introduce twisted P-graph C^*-algebras, generalising [3], [4], [36], and use them to study connections between $\mathbb{Z}_2$-gradings of C^*-algebras, and the twisted k-graph C^*-algebras studied in [24], [26]. The idea is that $\mathbb{Z}_2$-valued functors on P-graphs determine gradings of the associated C^*-algebras. The twisted C^*-algebras associated to $\{-1,1\}$-valued 2-cocycles on cartesian products of P-graphs can then be used to model graded tensor products of graded C^*-algebras.

We begin by discussing graded K-theory for C^*-algebras. Though K-theory for graded Banach algebras has been extensively studied by Karoubi (see, for example, [19]), the modern literature on complex graded C^*-algebras essentially begins with the work of Kasparov [20], [21] on KK-theory. Various definitions of graded K-theory for graded C^*-algebras have been used in the literature (see [5], [6], [13], [14], [15], [38], [42] to name but a few). We take as our definition of $K_0^{\mathrm{gr}}(A)$ the Kasparov group $KK(\mathbb{C}, A)$ for the graded C^*-algebra A, and likewise define $K_1^{\mathrm{gr}}(A) := KK(\mathbb{C}, A \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1)$ where $\mathbb{C}\mathrm{liff}_1$ is the first complex Clifford algebra (this reduces to the usual K-theory for C^*-algebras if A is trivially graded). We establish that perturbing the grading of a C^*-algebra A by conjugation by an odd self-adjoint unitary in $\mathcal{M}(A)$ does not alter the graded K-theory of A. In particular, we show that the graded K-theory of the crossed product of a C^*-algebra A by its grading automorphism is identical to the graded K-theory of $A \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1$.

To help compute graded K-theory in examples, we revisit the work of Pimsner in [32] to show that his six-term exact sequences in KK-theory for Cuntz–Pimsner algebras are also valid, with appropriate modifications, for graded C^*-algebras. Unlike Pimsner, we restrict to Hilbert bimodules in which the left action is both compact and injective. We obtain a six-term exact sequence in graded K-theory, which in turn gives a Pimsner–Voiculescu sequence for graded crossed products by $\mathbb{Z}$.

We next develop substantial classes of graded C^*-algebras to which we can apply our theorems. We introduce twisted P-graph C^*-algebras by straightforward generalisation of the notion of a twisted k-graph algebra. We establish a number of fundamental structure results for these C^*-algebras, including a version of the gauge-invariant uniqueness theorem, to help us make identifications between these C^*-algebras and key examples later in the paper. We prove that if P has the form $\mathbb{N}^k \times F$ where F is a countable abelian group, then every P-graph is a crossed product of a k-graph by an action of the group F, in a sense analogous to that studied in [9].

We next discuss how a functor from a P-graph to $\mathbb{Z}_2$ induces gradings of the associated twisted C^*-algebras. We show that if Z_2 denotes a copy of the order-two group $\mathbb{Z}_2$, regarded as a $\mathbb{Z}_2$-graph with one vertex, then $C^*(Z_2)$, under the grading induced by the degree functor, is isomorphic to $\mathbb{C}\mathrm{liff}_1$. More generally, we consider the situation where $P = \mathbb{N}^k \times \mathbb{Z}_2^l$. Any P-graph Λ carries both a natural functor δ_Λ taking values in $\mathbb{Z}_2$, and a natural $\{-1, 1\}$-valued 2-cocycle c_Λ. We establish a universal description of the graded twisted C^*-algebra determined by this functor and cocycle.

These threads come together when we study graded tensor products in terms of cartesian products of P-graphs. We prove that if Λ is a P-graph, Γ is a Q-graph, and we consider the associated graded, twisted C^*-algebras for the functors and cocycles described in the preceding paragraph, then the graded tensor product (the C^*-algebras are nuclear, so there is only one possible C^*-algebraic tensor product) is isomorphic to the graded twisted C^*-algebra of the $(P \times Q)$-graph $\Lambda \times \Gamma$ under its own natural cocycle and functor. Combining this with the results of previous sections, we show that the higher complex Clifford algebras can be realised as graded twisted P-graph C^*-algebras for appropriate P, and also that graded tensor products of graded twisted P-graph C^*-algebras with $\mathbb{C}\mathrm{liff}_1$ can be realised as graded twisted $(P \times \mathbb{Z}_2)$-graph algebras.

We apply our graded Pimsner sequence to the C^*-algebras of row-finite 1-graphs E with no sources under gradings of the sort discussed above. The result is an elegant generalisation of the well-known formula for the K-theory of the C^*-algebra of a row-finite directed graph with no sources: if A_δ denotes the $E^0 \times E^0$ matrix with $A_\delta(v,w) = \sum_{e \in vE^1w} (-1)^{\delta(e)}$, then the graded K-groups of $C^*(E)$ are the cokernel and kernel of $1 - A_\delta^{\mathrm{t}}$. This recovers Haag's formulas for the graded K-theory of Cuntz algebras [14]. If $\delta(e) = 1$ for all e (this corresponds to the grading of $C^*(E)$ coming from the order-two element of the gauge action), then A_δ is the negative of the usual adjacency matrix A_E, and so $K_*^{\mathrm{gr}}(C^*(E))$ is given by the cokernel and kernel of $1 + A_E^{\mathrm{t}}$. We also apply our results to compute the graded K-theory of certain crossed-products of graph algebras by $\mathbb{Z}_2$. Our examples and results lead us to conjecture that the graded K_0-group of a C^*-algebra can be described along the lines of the standard picture of ungraded K_0, as a group generated by equivalence classes of graded projective modules.

We begin by collecting relevant background in Section 2. In Section 3 we introduce graded K-theory in terms of Kasparov theory, and establish some fundamental results about it. In Section 4 we establish graded versions of Pimsner's six-term exact sequences for Hilbert bimodules with injective left actions by compacts (see Theorem 4.4) and apply them to obtain a graded Pimsner–Voiculescu sequence for crossed products by $\mathbb{Z}$ (Corollary 4.7). In Section 5 we introduce twisted P-graph C^*-algebras and establish the basic structure theory for them that we will need later in the paper. In Section 6, we discuss gradings of P-graph C^*-algebras induced by functors on the underlying P-graphs. In Section 7, we establish our main results about graded tensor products of graded P-graph C^*-algebras: Theorem 7.1 shows that for appropriate gradings and twisting cocycles, we have $C^*(\Lambda, c_\Lambda) \mathbin{\widehat{\otimes}} C^*(\Gamma, c_\Gamma) \cong C^*(\Lambda \times \Gamma, c_{\Lambda\times\Gamma})$. In Section 8, we apply our results from Section 4 to calculate graded K-theory for graph C^*-algebras (Lemma 8.2). We apply this lemma and our graded Pimsner–Voiculescu sequence to some illustrative examples. We conclude in Section 9 by formulating our conjecture about the structure of the graded K_0 group.

2. BACKGROUND

NOTATION. We will denote the cyclic group of order n by $\mathbb{Z}_n$. We frequently regard $\mathbb{Z}_2 = \{0,1\}$ as a ring, so we always use additive notation for the group operation, and make any identification of $\mathbb{Z}_2$ with $\{-1,1\} \subseteq \mathbb{T}$ explicit. We typically denote the multiplication operation in the ring $\mathbb{Z}_2$ by $\cdot$.

GRADED C^*-ALGEBRAS. Let A be a C^*-algebra. A *grading* of A is an automorphism α of A such that $\alpha^2 = 1$ (α is sometimes referred to as the grading automorphism). We define

$$A_0 := \{a \in A : \alpha(a) = a\} \quad \text{and} \quad A_1 := \{a \in A : \alpha(a) = -a\}.$$

So A_0, A_1 are closed, linear, self-adjoint subspaces of A and satisfy $A_iA_j \subset A_{i+j}$ for $i, j \in \mathbb{Z}_2$. We have $A = A_0 \oplus A_1$ as a Banach space. Note that A_0 is a C^*-subalgebra of A. Elements of A_0 are called *even* and elements of A_1 are called *odd*. To calculate A_0 and A_1, it is helpful to note that

$$A_0 = \Big\{\tfrac{a+\alpha(a)}{2} : a \in A\Big\} \quad \text{and} \quad A_1 = \Big\{\tfrac{a-\alpha(a)}{2} : a \in A\Big\}. \tag{2.1}$$

If $a \in A_i$ then we say that a is *homogeneous* of *degree i* and write $\partial a = i$; in particular ∂a is an element of the group $\mathbb{Z}_2$. If α is the identity map on A then $A_0 = A$ and $A_1 = \{0\}$. The resulting grading is called the *trivial grading*. Since a C^*-algebra may admit several different gradings (we discuss explicit examples of this in Examples 8.5 below), we shall frequently write a C^*-algebra A with grading α as the pair (A, α).

A graded C^*-algebra is *inner-graded* if there is a self-adjoint unitary (called a grading operator) $U \in \mathcal{M}(A)$ such that $\alpha(a) = UaU$ for all $a \in A$. In Section 14.1 of [2] Blackadar calls an inner-grading *even*. A *graded homomorphism* $\pi : (A, \alpha) \to (B, \beta)$ between graded C^*-algebras is a homomorphism from A to B which intertwines the gradings (i.e. $\pi \circ \alpha = \beta \circ \pi$).

Given a graded C^*-algebra (A, α_A) and homogeneous elements $a, b \in A$, the *graded commutator* $[a, b]^{\mathrm{gr}}$ is defined as $[a, b]^{\mathrm{gr}} = ab - (-1)^{\partial a \cdot \partial b} ba$. This formula extends to arbitrary a and b by bilinearity. In particular, if $a \in A_1$, then

$$[a, b]^{\mathrm{gr}} = ab - \alpha_A(b)a \quad \text{and} \quad [b, a]^{\mathrm{gr}} = ba - a\alpha_A(b). \tag{2.2}$$

If A is trivially graded, then $[a, b]^{\mathrm{gr}}$ reduces to the usual commutator $[a, b] = ab - ba$.

There is a graded tensor product operation for graded C^*-algebras, defined as follows. Let (A, α), (B, β) be graded C^*-algebras, and $A \odot B$ be their algebraic tensor product. This becomes a $*$-algebra when endowed with multiplication and involution given by

$$(a \mathbin{\widehat{\otimes}} b)(a' \mathbin{\widehat{\otimes}} b') = (-1)^{\partial b \cdot \partial a'} aa' \mathbin{\widehat{\otimes}} bb' \quad \text{and} \quad (a \mathbin{\widehat{\otimes}} b)^* = (-1)^{\partial a \cdot \partial b} a^* \mathbin{\widehat{\otimes}} b^*$$

for homogeneous elements $a, a' \in A$ and $b, b' \in B$. We decorate the $\odot$ symbol with a hat, $\widehat{\odot}$, to indicate that we are using this $*$-algebra structure on the algebraic tensor product. We write $A \mathbin{\widehat{\otimes}} B$ for the closure of the image of $A \mathbin{\widehat{\odot}} B$ in $\mathcal{B}(H \mathbin{\widehat{\otimes}} K)$ under the tensor-product representation $\pi_H \otimes \pi_K$ of faithful graded representations $\pi_H : A \to \mathcal{B}(H)$ and $\pi_K : B \to \mathcal{B}(K)$. (If either A or B is nuclear then this agrees with the maximal norm.)

The grading automorphism of $A \mathbin{\widehat{\otimes}} B$ is $\alpha \mathbin{\widehat{\otimes}} \beta$. So $a \mathbin{\widehat{\otimes}} b$ is homogeneous of degree $\partial a + \partial b$ (the addition takes place in $\mathbb{Z}_2$) if a and b are both homogeneous.

Writing $A_i \mathbin{\widehat{\otimes}} B_j$ for the closed linear span in $A \mathbin{\widehat{\otimes}} B$ of $\{a \mathbin{\widehat{\otimes}} b : a \in A_i \text{ and } b \in B_j\}$, we have

$$(A \mathbin{\widehat{\otimes}} B)_0 = A_0 \mathbin{\widehat{\otimes}} B_0 + A_1 \mathbin{\widehat{\otimes}} B_1 \quad \text{and} \quad (A \mathbin{\widehat{\otimes}} B)_1 = A_0 \mathbin{\widehat{\otimes}} B_1 + A_1 \mathbin{\widehat{\otimes}} B_0.$$

It is straightforward to show that the graded tensor product operation is associative (modulo the natural isomorphism $(a \mathbin{\widehat{\otimes}} b) \mathbin{\widehat{\otimes}} c \mapsto a \mathbin{\widehat{\otimes}} (b \mathbin{\widehat{\otimes}} c)$). We have $A \mathbin{\widehat{\otimes}} B \cong B \mathbin{\widehat{\otimes}} A$ as graded C^*-algebras. For this and other basic facts about graded C^*-algebras we refer the reader to Section 14 of [2].

It may aid intuition to observe that for unital graded C^*-algebras A, B, under this grading we have $[a \mathbin{\widehat{\otimes}} 1_B, 1_A \mathbin{\widehat{\otimes}} b]^{\mathrm{gr}} = 0 = [1_A \mathbin{\widehat{\otimes}} b, a \mathbin{\widehat{\otimes}} 1_B]^{\mathrm{gr}}$ for all $a \in A$, and $b \in B$.

An example of a graded C^*-algebra that we shall use very frequently is $M_{2n}(\mathbb{C})$ with grading automorphism $\alpha(\theta_{i,j}) = (-1)^{i-j}\theta_{i,j}$. This is an inner grading, as it is implemented by the grading operator $U \in M_{2n}(\mathbb{C})$ given by $U_{i,j} = (-1)^i \delta_{ij}$. We often write $\widehat{M}_{2n}(\mathbb{C})$ to emphasise that we are using this grading.

CLIFFORD ALGEBRAS OVER $\mathbb{C}$. We refer to Section 14 of [2] (see also Section 2 of [20]). Following Examples 14.1.2(b) of [2] the C^*-algebra $A = \mathbb{C} \oplus \mathbb{C}$ has a grading automorphism α given by $\alpha(z, w) = (w, z)$. So $(\mathbb{C} \oplus \mathbb{C})_0 = \{(z, z) : z \in \mathbb{C}\}$ and $(\mathbb{C} \oplus \mathbb{C})_1 = \{(z, -z) : z \in \mathbb{C}\}$. This graded C^*-algebra is called the first (complex) *Clifford algebra*, and we denote it by $\mathbb{C}\mathrm{liff}_1$ with this grading α implicit. As a graded C^*-algebra $\mathbb{C}\mathrm{liff}_1$ is generated by the odd self-adjoint unitary $u = (1, -1)$, because, for $(z, w) \in \mathbb{C}\mathrm{liff}_1$, we can write

$$(z, w) = \frac{1}{2}(z + w, z + w) + \frac{1}{2}(z - w, w - z) = \frac{1}{2}(z + w)1 + \frac{1}{2}(z - w)u.$$

Note that $\mathbb{C}\mathrm{liff}_1$ is not inner-graded (because it is abelian), and is isomorphic to the group C^*-algebra $C^*(\mathbb{Z}_2)$ with grading given by the dual action of $\widehat{\mathbb{Z}}_2 \cong \mathbb{Z}_2$.

The higher complex Clifford algebras are defined inductively: $\mathbb{C}\mathrm{liff}_{n+1} = \mathbb{C}\mathrm{liff}_n \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1$ for $n \geqslant 1$. It is straightforward to show that $\mathbb{C}\mathrm{liff}_2 \cong \widehat{M}_2(\mathbb{C})$ as graded C^*-algebras (see also Example 6.3(iii)). Observe that $\mathbb{C}\mathrm{liff}_n$ is generated by n mutually anticommuting odd self-adjoint unitaries.

GRADED HILBERT MODULES. Suppose that B is a graded C^*-algebra. A graded (right) Hilbert B-module is a Hilbert B-module X together with a decomposition of X as a direct sum of two closed subspaces X_0 and X_1 compatible with the grading of B in the sense that $X_i \cdot B_j \subset X_{i+j}$ and $\langle X_i, X_j \rangle \subset B_{i+j}$ (the graded components X_i need not be Hilbert submodules). We define the grading operator α_X on X on homogeneous elements by $\alpha_X(x) = (-1)^j x$ if $x \in X_j$. This α_X is not necessarily an adjointable operator. Given a graded Hilbert B-module X we write X^{op} for the same Hilbert B-module with the grading components switched (so $\alpha_{X^{\mathrm{op}}} = -\alpha_X$). The grading operator on X induces a grading $\widetilde{\alpha}_X$ on $\mathcal{L}(X)$ given by

$$\widetilde{\alpha}_X(T) = \alpha_X \circ T \circ \alpha_X \quad \text{for all } T \in \mathcal{L}(X). \tag{2.3}$$

Under this induced grading, T is homogeneous of degree j if and only if $TX_k \subseteq X_{j+k}$ for $j,k \in \mathbb{Z}_2$. For $\xi,\eta \in X$ we write $\theta_{\xi,\eta}$ for the generalised compact operator $\theta_{\xi,\eta}(\zeta) = \xi \cdot \langle \eta, \zeta\rangle_B$. The grading $\widetilde{\alpha}$ of $\mathcal{L}(X)$ restricts to a grading of $\mathcal{K}(X)$ satisfying $\widetilde{\alpha}_X(\theta_{\xi,\eta}) = \theta_{\alpha_X(\xi),\alpha_X(\eta)}$.

Naturally B may be regarded as a graded Hilbert B-module B_B with inner product $\langle a, b\rangle_B = a^*b$, right action given by multiplication, and grading operator α_B. We write $\mathcal{H}_B$ for the graded Hilbert B-module obtained as the direct sum of countably many copies of B_B. We define $\widehat{\mathcal{H}}_B := \mathcal{H}_B \oplus \mathcal{H}_B^{\mathrm{op}}$. If X is a graded Hilbert B-module, then $X \oplus \widehat{\mathcal{H}}_B \cong \widehat{\mathcal{H}}_B$ by Kasparov's stabilisation theorem (see Theorem 14.6.1 of [2]).

A graded Hilbert B-module which is a finitely generated projective module will be called a graded projective B-module throughout the paper. If X is a graded projective B-module, then $\mathcal{K}(X) = \mathcal{L}(X)$ and so in particular $1_X \in \mathcal{K}(X)$. Kasparov's stabilisation theorem implies that every graded projective B-module X is isomorphic to $p\widehat{\mathcal{H}}_B$ for some even projection $p \in \mathcal{K}(\widehat{\mathcal{H}}_B)$. Moreover $p\widehat{\mathcal{H}}_B$ is a graded projective B-module for any such projection. Given even projections $p, q \in \mathcal{K}(\widehat{\mathcal{H}}_B)$, we have $p\widehat{\mathcal{H}}_B \cong q\widehat{\mathcal{H}}_B$ if and only if there is an even partial isometry $v \in \mathcal{K}(\widehat{\mathcal{H}}_B)$ such that $p = v^*v$ and $q = vv^*$.

C^*-CORRESPONDENCES AND CUNTZ–PIMSNER ALGEBRAS. Here, we briefly recap the notion of a C^*-correspondence and of the associated Cuntz–Pimsner algebra and its Toeplitz extension. For a detailed introduction to C^*-correspondences, see [27], [35]. For more background on Cuntz–Pimsner algebras, see [32] and Section 8 of [33].

Given C^*-algebras A and B, an *A-B-correspondence* X is a pair (ϕ, X) consisting of a right-Hilbert B-module X and a homomorphism $\phi : A \to \mathcal{L}(X)$. We regard ϕ as implementing a left action of A on X by adjointable operators, so we often write $\phi(a)x = a \cdot x$ for $a \in A$ and $x \in X$. We say that X is *full* if $\overline{\operatorname{span}}\{\langle \xi, \eta\rangle_B : \xi, \eta \in X\} = B$. Any right-Hilbert B-module X may be regarded as a $\mathbb{C}$-B correspondence, and we write ℓ for the canonical left action of $\mathbb{C}$ by scalar multiplication. We say that X is countably generated if there is a sequence $(x_i)_{i=1}^\infty$ in X such that $X = \overline{\operatorname{span}}\{x_i \cdot b : i \geqslant 1, b \in B\}$. Note that B_B is a countably generated correspondence if B is σ-unital (so in particular if B is separable).

If B is a C^*-algebra then there is an isomorphism of $\mathcal{M}(B)$ onto $\mathcal{L}(B_B)$ that carries a multiplier m to the operator of left-multiplication by m on A. So any homomorphism $\phi : A \to \mathcal{M}(B)$ determines an A-B-correspondence structure on B_B. We denote this correspondence by ${}_\phi B$. The isomorphism of $\mathcal{M}(B)$ onto $\mathcal{L}(B_B)$ carries B onto $\mathcal{K}(B_B)$, so the left action of A on ${}_\phi B$ is by compacts if and only if ϕ takes values in B.

If (ϕ, X) is an A-B-correspondence and (ψ, Y) is a B-C-correspondence, then the internal tensor product $X \otimes_\psi Y$ is formed as follows: define $[\cdot,\cdot]_C$ on the algebraic tensor product $X \odot Y$ by sesquilinear extension of the formula $[x \odot y, x' \odot y']_C := \langle y, \psi(\langle x, x'\rangle_B)y'\rangle_C$. Let $N = \{\xi \in X \odot Y : [\xi,\xi]_C = 0\}$. Then $X \otimes_\psi Y$ is defined to be the completion of $(X \odot Y)/N$ in the norm determined by the inner-product $\langle \xi + N, \eta + N\rangle_C = [\xi, \eta]_C$. For $x \in X$ and $y \in Y$, we write $x \otimes y$ for $(x \odot y) + N \in X \otimes_\psi Y$. There is then a homomorphism $\widetilde{\psi} : \mathcal{L}(X) \to \mathcal{L}(X \otimes_\psi Y)$ given by

(2.4) $$\widetilde{\psi}(T)(x \otimes y) = (Tx) \otimes y \quad \text{for all } x \in X, y \in Y \text{ and } T \in \mathcal{L}(X).$$

In particular, $\widetilde{\psi} \circ \phi$ is a homomorphism of A into $\mathcal{L}(X \otimes_\psi Y)$, making $X \otimes_\psi Y$ into an A-C-correspondence with

$$a \cdot (x \otimes y) = \widetilde{\psi}(\phi(a))(x \otimes y) = (a \cdot x) \otimes y.$$

When the actions ϕ, ψ are clear from context, we frequently write $X \otimes_B Y$ instead of $X \otimes_\psi Y$. If $\phi : A \to B$ and $\psi : B \to C$ are homomorphisms and ψ is nondegenerate, then ${}_\phi B \otimes_C {}_\psi C \cong {}_{\psi \circ \phi} C$ under an isomorphism taking $b \otimes c$ to $\psi(b)c$.

If X is an A-A correspondence, then we can form its tensor powers $X^{\bigotimes n}$ given by $X^{\bigotimes 0} := A$, $X^{\bigotimes 1} := X$ and $X^{\bigotimes (n+1)} := X \otimes_A X^{\bigotimes n}$. The *Fock space* of X is the completion $\mathcal{F}_X$ of the algebraic direct sum $\bigoplus_{n=0}^{\infty} X^{\bigotimes n}$ in the norm coming from the inner product $\langle \bigoplus_n x_n, \bigoplus_n y_n\rangle_A = \sum_n \langle x_n, y_n\rangle_A$. This $\mathcal{F}_X$ is a C^*-correspondence over A with respect to the pointwise actions. Observe that $\mathcal{F}_X$ is full (and hence is a $\mathcal{K}(\mathcal{F}_X)$-$A$ imprimitivity bimodule) even if X is not, but that the left action of A on $\mathcal{F}_X$ is not by compacts even if the action of A on X is (unless X is the zero module).

A *representation* of X in a C^*-algebra B is a pair (ψ, π) where $\psi : X \to B$ is a linear map, $\pi : A \to B$ is a homomorphism, and we have $\psi(a \cdot \xi) = \pi(a)\psi(\xi)$, $\psi(\xi \cdot a) = \psi(x)\pi(a)$ and $\pi(\langle \xi, \eta\rangle_A) = \psi(\xi)^*\psi(\eta)$ for all $\xi, \eta \in X$ and $a \in A$. There is a universal C^*-algebra $\mathcal{T}_X$, called the *Toeplitz algebra* of X, generated by a representation (i_X, i_A) of X. There is also a representation (L_1, L_0) of X in $\mathcal{L}(\mathcal{F}_X)$ such that $L_0(a)\rho = a \cdot \rho$ for $a \in A$ and such that, for $\xi \in \mathcal{F}_X$, we have $L_1(\xi)\rho = \xi \otimes \rho$ for $\rho \in \bigcup_{n \geqslant 1} X^{\bigotimes n}$ and $L_1(\xi)a = \xi \cdot a$ for $a \in X^{\bigotimes 0} = A$. The universal property of $\mathcal{T}_X$ gives a homomorphism $L_1 \times L_0 : \mathcal{T}_X \to \mathcal{L}(\mathcal{F}_X)$ satisfying $(L_1 \times L_0) \circ i_X = L_1$ and $(L_1 \times L_0) \circ i_A = L_0$. Pimsner proves Proposition 3.3 of [32] that $L_1 \times L_0$ is injective.

To describe the Cuntz–Pimsner algebra of X, we will restrict attention to the situation where the left action of A on X is injective and by compacts. As discussed on page 202 of [32], given a representation (ψ, π) of X in B there is a homomorphism $\psi^{(1)} : \mathcal{K}(X) \to B$ such that $\psi^{(1)}(\theta_{\xi,\eta}) = \psi(\xi)\psi(\eta)^*$ for all $\xi, \eta \in X$. We say that the representation (ψ, π) is *Cuntz–Pimsner covariant*, or just covariant, if $\psi^{(1)}(\phi(a)) = \pi(a)$ for all $a \in A$. The *Cuntz–Pimsner algebra* $\mathcal{O}_X$ of X is the universal C^*-algebra generated by a covariant representation (j_X, j_A) of X; so it coincides with the quotient of $\mathcal{T}_X$ by the ideal generated by elements of the form $i_A(a) - i_X^{(1)}(\phi(a))$, $a \in A$. Under our hypotheses, $\mathcal{K}(\mathcal{F}_X) \subseteq \mathcal{T}_X$ and is generated as an ideal by $\{L_0(a) - L_1^{(1)}(\phi(a)) : a \in A\}$, and so $\mathcal{O}_X \cong \mathcal{T}_X/\mathcal{K}(\mathcal{F}_X)$.

If A is a C^*-algebra and α is an automorphism of A, then there is an isomorphism of the Cuntz–Pimsner algebra of the A-A correspondence $X := {}_\alpha A$ onto the crossed product $A \rtimes_\alpha \mathbb{Z}$ that intertwines $i_A : A \to \mathcal{O}_X$ with the canonical inclusion $\iota : A \hookrightarrow A \rtimes_\alpha \mathbb{Z}$, and carries $i_X(a) \in \mathcal{O}_X$ to $U\iota(a)$, where $U \in \mathcal{M}(A \rtimes_\alpha \mathbb{Z})$ is the unitary generator of the copy of $\mathbb{Z}$. There is a corresponding isomorphism of $\mathcal{T}_X$ onto the natural Toeplitz extension of the crossed product (that is, Stacey's endomorphism crossed-product of A by α [44]).

ELEMENTS OF KASPAROV THEORY. We introduce the elements of Kasparov theory needed for our work on graded K-theory later. For more background, see [2].

Let A, B be C^*-algebras, and let X be an A-B-correspondence. Given gradings α_A of A and α_B of B, a grading operator on X is a map $\alpha_X : X \to X$ such that $\alpha_X^2 = 1$, $\alpha_X(a \cdot x \cdot b) = \alpha_A(a) \cdot \alpha_X(x) \cdot \alpha_B(b)$ for all a, x, b, and $\alpha_B(\langle x, y\rangle_B) = \langle \alpha_X(x), \alpha_X(y)\rangle_B$ for all x, y. We call the pair (X, α_X) (or just X if the grading operator is understood from context) a graded A-B-correspondence (see, for example, [16]).

Given graded C^*-algebras A, B, C, a graded A-B-correspondence (X, α_X) and a graded B-C-correspondence (Y, α_Y), there is a well-defined grading operator $\alpha_X \mathbin{\widehat{\otimes}} \alpha_Y$ on $X \mathbin{\widehat{\otimes}}_B Y$ characterised by $(\alpha_X \mathbin{\widehat{\otimes}} \alpha_Y)(x \mathbin{\widehat{\otimes}} y) = \alpha_X(x) \mathbin{\widehat{\otimes}} \alpha_Y(y)$; note that if $\phi : A \to B$ is a graded homomorphism of C^*-algebras, then ${}_\phi B$ is a graded Hilbert module.

Recall that a C^*-algebra is said to be *σ-unital* if it has a countable approximate identity (or, equivalently, if it has a strictly positive element, see 12.3 of [2]). If (A, α_A) and (B, α_B) are σ-unital graded C^*-algebras, then a *Kasparov A-B-module* is a quadruple (X, ϕ, F, α_X) consisting of a countably generated A-B-correspondence, (ϕ, X), a grading operator α_X on X, and an odd element $F \in \mathcal{L}(X)$ that is odd with respect to the grading $\widetilde{\alpha}_X$ described at (2.3) in the sense that $F \circ \alpha_X = -\alpha_X \circ F$ and satisfies

$$(F - F^*)\phi(a) \in \mathcal{K}(X), \quad (F^2 - 1)\phi(a) \in \mathcal{K}(X), \quad \text{and} \quad [F, \phi(a)]^{\mathrm{gr}} \in \mathcal{K}(X) \quad \text{for all } a \in A.$$

Observe that since F is odd graded, we have $[\phi(a), F]^{\mathrm{gr}} = \phi(a)F - F\phi(\alpha_A(A))$ by (2.2). We say that (X, ϕ, F, α_X) is a *degenerate* Kasparov module if

$$(F - F^*)\phi(a) = 0, \quad (F^2 - 1)\phi(a) = 0, \quad \text{and} \quad [F, \phi(a)]^{\mathrm{gr}} = 0 \quad \text{for all } a \in A.$$

We say that graded Kasparov modules (X, ϕ, F, α_X) and (Y, ψ, G, α_Y) are unitarily equivalent if there is a unitary $U \in \mathcal{L}(X, Y)$ of degree zero (in the sense that $\alpha_Y U = U\alpha_X$) such that $UF = GU$ and $U\phi(a) = \psi(a)U$ for all $a \in A$; that is, U intertwines the left A-actions.

In what follows, $C([0,1])$ always has the trivial grading. Fix a C^*-algebra B. For each $t \in [0,1]$, define $\epsilon_t : C([0,1]) \mathbin{\widehat{\otimes}} B \to B$ by $\epsilon_t(f \mathbin{\widehat{\otimes}} b) = f(t)b$. A *homotopy* of Kasparov (A, α_A)-(B, α_B)-modules from $(X_0, \phi_0, F_0, \alpha_{X_0})$ to $(X_1, \phi_1, F_1, \alpha_{X_1})$ is a Kasparov A-$(C([0,1]) \mathbin{\widehat{\otimes}} B)$-module (X, ϕ, F, α_X) such that, for $t \in \{0, 1\}$, and with $\widetilde{\epsilon}_t : \mathcal{L}(X) \to \mathcal{L}(X \mathbin{\widehat{\otimes}}{}_B B_B)$ as described in (2.4), there is a unitary equivalence between the module

$$(X \mathbin{\widehat{\otimes}}_{\epsilon_t}{}_B B_B, \widetilde{\epsilon}_t \circ \phi, \widetilde{\epsilon}_t(F), \alpha_X \mathbin{\widehat{\otimes}} \alpha_B)$$

and $(X_t, \phi_t, F_t, \alpha_{X_t})$. We write $(X_0, \phi_0, F_0, \alpha_{X_0}) \sim_{\mathrm{h}} (X_1, \phi_1, F_1, \alpha_{X_1})$ and say that these two Kasparov modules are *homotopy equivalent* if there exists a homotopy from $(X_0, \phi_0, F_0, \alpha_{X_0})$ to $(X_1, \phi_1, F_1, \alpha_{X_1})$. It is implicit in both [20] and [2] that homotopy equivalence is an equivalence relation on Kasparov modules, and indeed Proposition 18.5.3 of [2] combined with the fact that $\sim_{\mathrm{oh}}$ is defined in Definition 17.2.4 of [2] to be an equivalence relation shows indirectly that homotopy is an equivalence relation amongst Kasparov A-B-modules provided that A is separable and B is σ-unital. But, as the anonymous referee points out, it is not explicitly proved in [20], [2] that $\sim_{\mathrm{h}}$ is an equivalence relation, so we have included a proof in Appendix A for completeness.

We write $KK(A, B)$ for the collection of all equivalence classes of Kasparov A-B-modules under $\sim_{\mathrm{h}}$. This $KK(A, B)$ forms an abelian group with addition given by direct sum:

$$[X, \phi, F, \alpha_X] + [Y, \psi, G, \alpha_Y] = [X \oplus Y, \phi \oplus \psi, F \oplus G, \alpha_X \oplus \alpha_Y],$$

and identity element equal to the class of the trivial module $[B_B, 0, 0, \mathrm{id}_B]$; this class coincides with the class of any degenerate Kasparov A-B-module. As detailed in the proof of Proposition 17.3.3 in [2], the (additive) inverse of a class in $KK(A, B)$ is given by

$$-[X, \phi, F, \alpha_X] = [X, \phi \circ \alpha_A, -F, -\alpha_X]. \tag{2.5}$$

Let (A, α_A) and (B, α_B) be graded C^*-algebras, let (ϕ, X) be an A-B-correspondence, and suppose that α_X is a grading operator on X. If $G, H \in \mathcal{L}(X)$ are operators for which (X, ϕ, G, α_X) and (X, ϕ, H, α_X) are both Kasparov A-B-modules, then an *operator homotopy* between these Kasparov modules is a norm-continuous map $t \mapsto F_t$ from $[0, 1]$ to $\mathcal{L}(X)$ such that (X, ϕ, F_t, α_X) is a Kasparov module for each t, $F_0 = G$ and $F_1 = H$. An operator homotopy is a special case of a homotopy in the following sense: the space $\overline{X} := C([0, 1], X)$ is a graded A-$C([0, 1], B)$-correspondence with left action given by $(\overline{\phi}(a)(x))(t) = \phi(a)x(t)$ and grading operator $(\alpha_{\overline{X}}(x))(t) = \alpha_X(x(t))$. Moreover, there is an operator $\overline{F} \in \mathcal{L}(C([0, 1], X))$ given by $\overline{F}(x)(t) = F_t(x(t))$, and then $(\overline{X}, \overline{\phi}, \overline{F}, \alpha_{\overline{X}})$ is a Kasparov A-$C([0, 1], B)$-module. Identifying $C([0, 1], B)$ with $B \mathbin{\widehat{\otimes}} C([0, 1])$ in the canonical way, we see that $(\overline{X}, \overline{\phi}, \overline{F}, \alpha_{\overline{X}})$ is a homotopy from (X, ϕ, G, α_X) to (X, ϕ, H, α_X).

There is a category whose objects are σ-unital graded C^*-algebras while the morphisms from A to B are homotopy classes of Kasparov A-B-modules. The composition in this category is called the *Kasparov product*, denoted $\widehat{\otimes}_B : KK(A, B) \times KK(B, C) \to KK(A, C)$. The identity morphism for the object (A, α_A) is the class of $(A_A, \mathrm{id}, 0, \alpha_A)$. Given a Kasparov A-B-module (X, ϕ, F, α_X) and a Kasparov B-C-module (Y, ψ, G, α_Y), the Kasparov product

$$[X, \phi, F, \alpha_X] \mathbin{\widehat{\otimes}}_B [Y, \psi, G, \alpha_Y]$$

has the form

$$[X \widehat{\otimes}_\psi Y, \widetilde{\psi} \circ \phi, H, \alpha_X \widehat{\otimes} \alpha_Y]$$

for an appropriate choice of operator H; the details are formidable in general, but we will not need them here. For us it will suffice to consider Kasparov products in which one of the factors has the form $[B_B, \phi, 0, \alpha_B]$ for some graded homomorphism $\phi : (A, \alpha_A) \to (B, \alpha_B)$ of C^*-algebras.

In detail, suppose that $\phi : (A, \alpha_A) \to (B, \alpha_B)$ is a graded homomorphism of graded C^*-algebras. Since $B \cong \mathcal{K}(B_B)$ via the map $b \mapsto (a \mapsto ba)$, the quadruple $(B_B, \phi, 0, \alpha_B)$ is a Kasparov A-B-module. If (X, ψ, F, α_X) is a Kasparov B-C-module, then $(X, \psi \circ \phi, F, \alpha_X)$ is also a Kasparov module, whose class in $KK(A, C)$ we denote by $\phi^*[X, \psi, F, \alpha_X]$. Proposition 18.7.2(b) of [2] shows that

$$[B_B, \phi, 0, \alpha_B] \widehat{\otimes}_B [X, \psi, F, \alpha_X] = \phi^*[X, \psi, F, \alpha_X].$$

Likewise, if (Y, ψ, G, α_Y) is a Kasparov C-A-module, then $(Y \widehat{\otimes}_\phi B_B, \psi \widehat{\otimes} 1, G \widehat{\otimes} 1, \alpha_Y \widehat{\otimes} \alpha_B)$ is a Kasparov C-B-module whose class we denote by $\phi_*[Y, \psi, G, \alpha_Y]$, and Proposition 18.7.2(b) of [2] shows that

$$[Y, \psi, G, \alpha_Y] \widehat{\otimes}_A [B_B, \phi, 0, \alpha_B] = \phi_*[Y, \psi, G, \alpha_Y].$$

Observe that if (A, α) is a graded C^*-algebra, then the discussion above shows that $KK(A, A)$ is a ring under the Kasparov product, with multiplicative identity

$$[\mathrm{id}_A] := [A, \mathrm{id}_A, 0, \alpha_A].$$

By definition of the additive inverse (see (2.5)), the tuple $(A, \alpha_A, 0, -\alpha_A)$ is a Kasparov module, and

$$[A, \alpha_A, 0, -\alpha_A] = -[\mathrm{id}_A]. \tag{2.6}$$

3. GRADED K-THEORY OF C^*-ALGEBRAS

In this section, we consider graded K-theory for C^*-algebras. There does not appear to be a universally-accepted definition of graded K-theory for C^*-algebras in the literature to date. We have chosen to take Kasparov's KK-theory [20] as the basis for our definition (see Definition 17.3.1 of [2]). We establish some basic properties of graded K-theory; in particular, that both taking graded tensor products with $\mathbb{C}\mathrm{liff}_1$, and taking crossed products by $\mathbb{Z}_2$ with respect to a suitable grading, interchange graded K-groups.

The following definition is used implicitly in [14], [15].

DEFINITION 3.1. Let A be a σ-unital C^*-algebra and let α be a grading automorphism of A. We define the *graded K-theory of A* as follows: $K_0^{\mathrm{gr}}(A, \alpha) := KK(\mathbb{C}, A)$ and $K_1^{\mathrm{gr}}(A, \alpha) := KK(\mathbb{C}, A \widehat{\otimes} \mathbb{C}\mathrm{liff}_1)$. When α is understood from context we often write $K_i^{\mathrm{gr}}(A)$ for $K_i^{\mathrm{gr}}(A, \alpha)$.

REMARK 3.2. From the above definition and results from Kasparov theory, we see that K_j^{gr} is covariantly functorial, continuous with respect to direct limits, and invariant under graded Morita equivalence. For $j \geqslant 2$ we define $K_j^{\mathrm{gr}}(A) = KK(\mathbb{C}, A \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_j)$. The functors K_j^{gr} satisfy Bott periodicity (see Remark 3.3 below). So we write $K_*^{\mathrm{gr}}(A) = (K_0^{\mathrm{gr}}(A), K_1^{\mathrm{gr}}(A))$ as we do for ungraded K-theory.

Up to isomorphism Definition 3.1 coincides with the usual definition of K-theory for σ-unital, trivially graded C^*-algebras A (see 18.5.4 of [2]). Furthermore, if A is inner graded, then $K_*^{\mathrm{gr}}(A) \cong K_*(A)$ (see 14.5.1 and 14.5.2 of [2]).

REMARK 3.3. By definition $K_0^{\mathrm{gr}}(A \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1) = KK(\mathbb{C}, A \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1) = K_1^{\mathrm{gr}}(A)$. We also have a natural isomorphism $K_1^{\mathrm{gr}}(A \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1) \cong K_0^{\mathrm{gr}}(A)$. To see this, recall that by Corollary 17.8.8 of [2], we have a natural isomorphism $KK(\mathbb{C}, A \mathbin{\widehat{\otimes}} \widehat{M}_2(\mathbb{C})) \cong KK(\mathbb{C}, A)$. Using this at the last step, we calculate

$$K_1^{\mathrm{gr}}(A \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1) = KK(\mathbb{C}, A \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1 \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1) \cong KK(\mathbb{C}, A \mathbin{\widehat{\otimes}} \widehat{M}_2(\mathbb{C})) = K_0^{\mathrm{gr}}(A)$$

as claimed.

EXAMPLE 3.4. Since $\mathbb{C}$ is trivially graded we have $K_*^{\mathrm{gr}}(\mathbb{C}) = K_*(\mathbb{C}) = (\mathbb{Z}, 0)$. From Remark 3.3 we have $K_i^{\mathrm{gr}}(A \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1) = K_{i+1}^{\mathrm{gr}}(A)$, and it is easy to show that $\mathbb{C} \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1 \cong \mathbb{C}\mathrm{liff}_1$. Hence putting $A = \mathbb{C}$ we have

$$K_i^{\mathrm{gr}}(\mathbb{C}\mathrm{liff}_1) = K_{i+1}^{\mathrm{gr}}(\mathbb{C}) = K_{i+1}(\mathbb{C}) = \begin{cases} 0 & \text{if } i = 0, \\ \mathbb{Z} & \text{if } i = 1. \end{cases}$$

Since $\mathbb{C}\mathrm{liff}_1 \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1 = \mathbb{C}\mathrm{liff}_2$, the preceding paragraph applied with $A = \mathbb{C}\mathrm{liff}_1$ gives $K_i^{\mathrm{gr}}(\mathbb{C}\mathrm{liff}_2) \cong K_i(\mathbb{C})$. Repeating this procedure gives $K_i^{\mathrm{gr}}(\mathbb{C}\mathrm{liff}_n) \cong K_{i+n}(\mathbb{C})$. So $K_i^{\mathrm{gr}}(\mathbb{C}\mathrm{liff}_n) \cong \mathbb{Z}$ if $i+n$ is even and it is trivial otherwise.

Before moving on to some tools for computing graded K-theory, it is helpful to relate it to our intuition for ordinary K-theory. We think of $K_0(A)$ as a group generated by equivalence classes of projections in $A \otimes \mathcal{K}$ so that, in particular, $[v^*v] = [vv^*]$ whenever v is a partial isometry. The following example indicates that in graded K-theory similar relations hold for *homogeneous* partial isometries in graded C^*-algebras, but with an additional dependence on the parity of the partial isometry in question. We discuss this further in Section 9.

EXAMPLE 3.5. Let A be a graded C^*-algebra with grading automorphism α and suppose that v is an odd partial isometry in $\mathcal{K}(\widehat{\mathcal{H}}_A)$. We obtain graded Kasparov modules as follows: let $p = v^*v$ and $q = vv^*$. We let α denote the grading operator on $\widehat{\mathcal{H}}_A$ and observe that if $p \in \mathcal{K}(\widehat{\mathcal{H}}_A)$ is a homogeneous projection then α restricts to a grading operator on the graded submodule $p\widehat{\mathcal{H}}_A \subseteq \widehat{\mathcal{H}}_A$ denoted $\alpha|_{p\widehat{\mathcal{H}}_A}$. Recall that ℓ denotes the action of $\mathbb{C}$ by scalar multiplication on any Hilbert module. We can form the Kasparov modules $(p\widehat{\mathcal{H}}_A, \ell, 0, \alpha|_{p\widehat{\mathcal{H}}_A})$ and

$(q\widehat{\mathcal{H}}_A, \ell, 0, \alpha|_{q\widehat{\mathcal{H}}_A})$. We claim that the classes $[p\widehat{\mathcal{H}}_A]_K$ and $[q\widehat{\mathcal{H}}_A]_K$ of these Kasparov modules satisfy $[p\widehat{\mathcal{H}}_A]_K = -[q\widehat{\mathcal{H}}_A]_K$ in $KK(\mathbb{C}, A)$. These are Kasparov modules because $\mathcal{K}(p\widehat{\mathcal{H}}_A)$ is isomorphic to $p\mathcal{K}(\widehat{\mathcal{H}}_A)p$ which is unital with unit p, and so all adjointables are compact. In particular the zero operator $F = 0$ trivially has the property that $F^2 - 1$, $F^* - F$ and $[F, \ell(a)]^{\mathrm{gr}}$ are compact for all $a \in \mathbb{C}$.

To see that $[p\widehat{\mathcal{H}}_A]_K = -[q\widehat{\mathcal{H}}_A]_K$, observe that $\operatorname{Ad} v$ implements an isomorphism $p\widehat{\mathcal{H}}_A \to q\widehat{\mathcal{H}}_A$. This isomorphism is odd in the sense that $\alpha|_{q\widehat{\mathcal{H}}_A} = \operatorname{Ad} v \circ (-\alpha|_{p\widehat{\mathcal{H}}_A})$. We claim that

$$(p\widehat{\mathcal{H}}_A \oplus q\widehat{\mathcal{H}}_A, \ell, 0, \alpha|_{p\widehat{\mathcal{H}}_A} \oplus \alpha|_{q\widehat{\mathcal{H}}_A}) \tag{3.1}$$

is operator homotopic to a degenerate Kasparov module. To see this, note that the module

$$\Big(p\widehat{\mathcal{H}}_A \oplus q\widehat{\mathcal{H}}_A, \ell, \left(\begin{smallmatrix} 0 & v^* \\ v & 0 \end{smallmatrix}\right), \alpha|_{p\widehat{\mathcal{H}}_A} \oplus \alpha|_{q\widehat{\mathcal{H}}_A}\Big) \tag{3.2}$$

is a degenerate Kasparov module because $F_v := \left(\begin{smallmatrix} 0 & v^* \\ v & 0 \end{smallmatrix}\right)$ is self-adjoint, commutes with $\ell(\mathbb{C})$ and satisfies $F_v^2 = \mathrm{id}$. Since the straight-line path from 0 to F_v implements an operator homotopy from (3.1) to (3.2), we conclude that (3.1) represents $0_{KK(\mathbb{C},A)}$ as required.

We now discuss how crossed products by $\mathbb{Z}_2$ relate to graded K-theory. Let A be a graded C^*-algebra with grading automorphism α. Let v be an odd self-adjoint unitary in $\mathcal{M}(A)$ and define $\widetilde{\alpha} = \alpha \circ \operatorname{Ad} v$. Since α commutes with $\operatorname{Ad} v$, this makes A bi-graded in the sense that it admits two commuting gradings by $\mathbb{Z}_2$, or equivalently a grading by $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let A_{jk} denote the bihomogeneous elements of degree j with respect to α and of degree k with respect to $\operatorname{Ad} v$; that is, $a \in A_{jk}$ if and only if $\alpha(a) = (-1)^j a$ and $\operatorname{Ad} v(a) = (-1)^k a$. So if $a \in A_{jk}$, then $\widetilde{\alpha}(a) = (-1)^{j+k} a$. Let $\widetilde{A}$ denote the C^*-algebra A graded by $\widetilde{\alpha}$.

In the following statement and proof, indices in $\mathbb{Z}_2$ are denoted j, k, l, and i is reserved for the imaginary number $i = \sqrt{-1}$.

THEOREM 3.6. *With notation as above there is an isomorphism of graded C^*-algebras $\phi : A \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1 \to \widetilde{A} \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1$ such that*

$$\phi(a_{jk} \mathbin{\widehat{\otimes}} u^l) = a_{jk}(\mathrm{i}v)^k \mathbin{\widehat{\otimes}} u^l,$$

where $a_{jk} \in A_{jk}$ for $j, k, l \in \mathbb{Z}_2$, and u is the odd generator of Cliff_1.

Proof. First, we check that ϕ preserves the grading: the element $a_{jk} \mathbin{\widehat{\otimes}} u^l$ is $(j+l)$-graded in $A \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1$ (with A graded by α). The element $a_{jk}(\mathrm{i}v)^k \mathbin{\widehat{\otimes}} u^l$ is homogeneous of degree $(j+k+k)+l = j+l$ in $\widetilde{A} \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1$ with $\widetilde{A}$ graded by $\widetilde{\alpha}$. In $A \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1$ we compute $(a_{jk} \mathbin{\widehat{\otimes}} u^l)(a'_{j'k'} \mathbin{\widehat{\otimes}} u^{l'}) = (-1)^{lj'} a_{jk}a'_{j'k'} \mathbin{\widehat{\otimes}} u^{l+l'}$. Applying

ϕ to both sides, and computing in $\widetilde{A} \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1$ (with $\widetilde{A}$ graded by $\widetilde{\alpha}$) we obtain

$$\begin{aligned}\phi(a_{jk} \mathbin{\widehat{\otimes}} u^l)\phi(a'_{j'k'} \mathbin{\widehat{\otimes}} u^{l'}) &= (a_{jk}(\mathrm{i}v)^k \mathbin{\widehat{\otimes}} u^l)(a'_{j'k'}(\mathrm{i}v)^{k'} \mathbin{\widehat{\otimes}} u^{l'}) \\ &= (-1)^{l(j'+k'+k')} a_{jk}(\mathrm{i}v)^k a'_{j'k'}(\mathrm{i}v)^{k'} \mathbin{\widehat{\otimes}} u^{l+l'}.\end{aligned}$$

Since $va'_{j'k'}v = (-1)^{k'} a'_{j'k'}$, we have $(\mathrm{i}v)^k a'_{j'k'}(\mathrm{i}v)^{k'} = (-1)^{k\cdot k'} a'_{j'k'}(\mathrm{i}v)^k(\mathrm{i}v)^{k'}$. Since k, k' belong to the ring $\mathbb{Z}_2$, we have $i^k i^{k'} = (-1)^{k\cdot k'} i^{k+k'}$. So the factors of $(-1)^{k\cdot k'}$ cancel, and we obtain

$$\begin{aligned}\phi(a_{jk} \mathbin{\widehat{\otimes}} u^l)\phi(a'_{j'k'} \mathbin{\widehat{\otimes}} u^{l'}) &= (-1)^{lj'} a_{jk}a'_{j'k'}(\mathrm{i}v)^{k+k'} \mathbin{\widehat{\otimes}} u^{l+l'} \\ &= \phi((-1)^{lj'} a_{jk}a'_{j'k'} \mathbin{\widehat{\otimes}} u^{l+l'}) \\ &= \phi((a_{jk} \mathbin{\widehat{\otimes}} u^l)(a'_{j'k'} \mathbin{\widehat{\otimes}} u^{l'})).\end{aligned}$$

The result follows because A is spanned by its bihomogeneous elements. ∎

Recall that $\widehat{M}_2(\mathbb{C})$ denotes the algebra of 2×2 complex matrices with the standard inner grading (so diagonal elements are even and off diagonal elements are odd). Similarly, let $\widehat{\mathcal{K}}$ denote the C^*-algebra of compact operators with the standard inner grading. We define $\widehat{M}_2(A) := A \mathbin{\widehat{\otimes}} \widehat{M}_2(\mathbb{C}) \cong A \mathbin{\widehat{\otimes}} \mathrm{Cliff}_2 \cong A \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1 \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1$.

COROLLARY 3.7. *Continuing with the notation of Theorem* 3.6, *the isomorphism* $\phi : A \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1 \to \widetilde{A} \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1$ *induces a natural isomorphism* $\widehat{M}_2(A) \cong \widehat{M}_2(\widetilde{A})$, *which in turn induces a natural isomorphism* $A \mathbin{\widehat{\otimes}} \widehat{\mathcal{K}} \cong \widetilde{A} \mathbin{\widehat{\otimes}} \widehat{\mathcal{K}}$. *In particular,* $K_*^{\mathrm{gr}}(A)$ *is naturally isomorphic to* $K_*^{\mathrm{gr}}(\widetilde{A})$.

Proof. By Theorem 3.6, we have a natural isomorphism $A \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1 \cong \widetilde{A} \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1$. Hence,

$$\widehat{M}_2(A) \cong A \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1 \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1 \cong \widetilde{A} \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1 \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1 \cong \widehat{M}_2(\widetilde{A})$$

by the associativity of the graded tensor product. The second assertion then follows from the canonical isomorphism $\widehat{\mathcal{K}} \cong \mathcal{K} \mathbin{\widehat{\otimes}} \widehat{M}_2(\mathbb{C})$. The final statement follows from the stability of Kasparov theory. ∎

Let B be a graded C^*-algebra with grading automorphism β. By Proposition 14.5.4 of [2] we have $B \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1 \cong B \rtimes_\beta \mathbb{Z}_2$ as C^*-algebras, and the grading on $B \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1$ is determined by the automorphism $\alpha := (\beta \times 1) \circ \widehat{\beta}$ where $\widehat{\beta}$ is the grading determined by the dual action on $B \rtimes_\beta \mathbb{Z}_2$ under this identification. Now let u be the canonical self-adjoint unitary generator of Cliff_1 and let $v := 1 \mathbin{\widehat{\otimes}} u \in \mathcal{M}(B \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1)$. Then v is also an odd self-adjoint unitary (with respect to the grading α); moreover, we have $\mathrm{Ad}\, v = \beta \times 1$.

COROLLARY 3.8. *With notation as above, if we endow* $B \rtimes_\beta \mathbb{Z}_2$ *with the grading associated to the dual action, then* $K_i^{\mathrm{gr}}(B \rtimes_\beta \mathbb{Z}_2)$ *is naturally isomorphic to* $K_i^{\mathrm{gr}}(B \mathbin{\widehat{\otimes}} \mathrm{Cliff}_1) = K_{i+1}^{\mathrm{gr}}(B)$ *for* $i = 0, 1$.

The proof follows immediately from the final assertion of Corollary 3.7 with $A := B \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1$ and $\alpha := (\beta \times 1) \circ \widehat{\beta}$.

REMARK 3.9. With notation as in the above corollary, observe that since the canonical embedding $B \to B \rtimes_\beta \mathbb{Z}_2$ may be regarded as a graded homomorphism when B is given the trivial grading and $B \rtimes_\beta \mathbb{Z}_2$ is given the dual grading, there is a natural homomorphism

$$K_i(B) \to K_i^{\mathrm{gr}}(B \rtimes_\beta \mathbb{Z}_2) \cong K_{i+1}^{\mathrm{gr}}(B) \quad \text{for } i = 0, 1.$$

EXAMPLE 3.10. Let β be the grading automorphism of $C(\mathbb{T})$ defined by the formula $\beta(f)(z) = f(\overline{z})$. Then there is a short exact sequence of graded C^*-algebras:

$$0 \to C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1 \xrightarrow{\iota} C(\mathbb{T}) \xrightarrow{\epsilon} \mathbb{C} \oplus \mathbb{C} \to 0$$

where $C_0(\mathbb{R})$ and $\mathbb{C} \oplus \mathbb{C}$ are trivially graded and $\epsilon(f) = f(1) \oplus f(-1)$ for all $f \in C(\mathbb{T})$. Hence by Theorem 1.1 of [42] we have a six-term exact sequence:

$$\begin{tikzcd}
K_0^{\mathrm{gr}}(C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1) \arrow[r, "\iota_*"] & K_0^{\mathrm{gr}}(C(\mathbb{T})) \arrow[r, "\epsilon_*"] & K_0^{\mathrm{gr}}(\mathbb{C} \oplus \mathbb{C}) \arrow[d] \\
K_1^{\mathrm{gr}}(\mathbb{C} \oplus \mathbb{C}) \arrow[u] & K_1^{\mathrm{gr}}(C(\mathbb{T})) \arrow[l, "\epsilon_*"] & K_1^{\mathrm{gr}}(C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1). \arrow[l, "\iota_*"]
\end{tikzcd}$$

Since $K_0^{\mathrm{gr}}(C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1) \cong \mathbb{Z}$, $K_0^{\mathrm{gr}}(\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{Z}^2$ and $K_1^{\mathrm{gr}}(C_0(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1) = K_1^{\mathrm{gr}}(\mathbb{C} \oplus \mathbb{C}) = 0$, we obtain $K_*^{\mathrm{gr}}(C(\mathbb{T})) = (\mathbb{Z}^3, 0)$ (cf. p. 105 of [14]). It follows by Corollary 3.8 and Remark 3.3 that $K_*^{\mathrm{gr}}(C(\mathbb{T}) \rtimes_\beta \mathbb{Z}_2) = (0, \mathbb{Z}^3)$. Note that $C(\mathbb{T}) \rtimes_\beta \mathbb{Z}_2$ is isomorphic to the C^*-algebra of the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$. Under the isomorphism $C(\mathbb{T}) \rtimes_\beta \mathbb{Z}_2 \cong C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$, the dual grading $\widehat{\beta}$ becomes the canonical grading on $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ determined by requiring that both self-adjoint unitary generators be odd. Note that $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ is the universal unital C^*-algebra generated by two projections (see [34]).

REMARK 3.11. Let α be the grading of $C_0(\mathbb{R})$ given by $\alpha(f)(x) = f(-x)$ for $f \in C_0(\mathbb{R})$. A computation similar to the above shows that $K_*^{\mathrm{gr}}(C_0(\mathbb{R})) \cong (\mathbb{Z}^2, 0)$.

4. PIMSNER'S EXACT SEQUENCES FOR GRADED C^*-ALGEBRAS

The main result of this section, Theorem 4.4, shows how to compute the graded K-theory of the Cuntz–Pimsner algebra of a graded C^*-correspondence over a nuclear, σ-unital C^*-algebra. We obtain from this theorem a graded version of the Pimsner–Voiculescu six-term exact sequence for crossed products in Corollary 4.7.

To prove our main theorem we follow Pimsner's computation of the KK-theory of the Cuntz–Pimsner algebra $\mathcal{O}_X$ in Section 4 of [32], keeping track of the gradings.

SET UP. For the duration of this section we fix a graded, σ-unital, nuclear C^*-algebra (A, α_A), and a graded A-A-correspondence (X, α_X) in the sense of Section 2.

We assume that the left action $\varphi : A \to \mathcal{L}(X)$ is injective, by compacts (i.e. $\varphi(A) \subseteq \mathcal{K}(X)$) and essential in the sense that $\overline{\varphi(A)X} = X$. It is not clear that all of these hypotheses are required for our arguments (for example, Pimsner does not require that the left action should be by compacts or injective in [32]), but they simplify the discussion and cover the examples that interest us most.

Recall that there is an induced grading $\widetilde{\alpha}_X$ of $\mathcal{L}(X)$ given by $\widetilde{\alpha}_X(T) = \alpha_X \circ T \circ \alpha_X$.

Let $[X] \in KK(A, A)$ denote the class of the Kasparov module $(X, \varphi, 0, \alpha_X)$.

LEMMA 4.1. *With notation as above, if α_A is trivial, then $\alpha_X \in \mathcal{L}(X)$, and it is an even self-adjoint unitary with respect to $\widetilde{\alpha}_X$. Let*

$$X_0 := \overline{\operatorname{span}}\{x + \alpha_X(x) : x \in A\} \quad \textit{and} \quad X_1 := \overline{\operatorname{span}}\{x - \alpha_X(x) : x \in X\}.$$

Then $X \cong X_0 \oplus X_1$ as A-A-correspondences, and in $KK(A, A)$, we have $[X] = [X_0] - [X_1]$.

Proof. Since α_X is idempotent and α_A is trivial, for all $\xi, \eta \in X$ we have

$$\langle \alpha_X(\xi), \eta \rangle_A = \langle \alpha_X(\xi), \alpha_X^2(\eta) \rangle_A = \alpha_A(\langle \xi, \alpha_X(\eta) \rangle_A) = \langle \xi, \alpha_X(\eta) \rangle_A.$$

Hence α_X is a self-adjoint unitary in $\mathcal{L}(X)$ and since $\widetilde{\alpha}_X(\alpha_X) = \alpha_X \circ \alpha_X \circ \alpha_X = \alpha_X$ it follows that α_X is even. Since α_A is trivial, for $a, b \in A$, we have

$$a \cdot (x \pm \alpha_X(x)) \cdot b = a \cdot x \cdot b \pm a \cdot \alpha_X(x) \cdot b = a \cdot x \cdot b \pm \alpha_X(a \cdot x \cdot b),$$

so $A \cdot X_i, X_i \cdot A \subseteq X_i$ for $i = 0, 1$.

For $\xi \in X_0$ and $\eta \in X_1$, we have

$$\langle \xi, \eta \rangle_A = \langle \alpha_X(\xi), \eta \rangle_A = \langle \xi, \alpha_X(\eta) \rangle_A = \langle \xi, -\eta \rangle_A = -\langle \xi, \eta \rangle_A.$$

So $X_0 \perp X_1$, giving $X \cong X_0 \oplus X_1$ as right-Hilbert A-modules. Since α_A is trivial, if $\varphi : A \to \mathcal{K}(X)$ is the homomorphism defining the left action, then $\varphi(A)X_j \subseteq X_j$ for $j = 0, 1$. So $X \cong X_0 \oplus X_1$ as C^*-correspondences. We write $\varphi_j : A \to \mathcal{K}(X_j)$ for the homomorphism $a \mapsto \varphi(a)|_{X_j}$.

We now have

$$(X, \varphi, 0, \alpha_X) \cong \Big(X_0 \oplus X_1, \varphi_0 \oplus \varphi_1, 0, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\Big)$$

as graded Kasparov modules. The class of the right-hand side is the Kasparov sum of $[X_0, \varphi_0, 0, \mathrm{id}]$ and $[X_1, \varphi_1, 0, -\mathrm{id}]$. Since $\alpha_A = \mathrm{id}_A$ we have $\varphi_1 = \varphi_1 \circ \alpha_A$, and so $[X_1, \varphi_1, 0, -\mathrm{id}]$ is precisely the inverse of $[X_1] = [X_1, \varphi_1, 0, \mathrm{id}]$ in $KK(A, A)$ as described at (2.5), and the result follows. ∎

Whether or not A is trivially graded, computing in $X_A \mathbin{\widehat{\otimes}}_A X_A$, for $\xi, \eta \in X$, and $a \in A$, we have

$$\alpha_X(\xi \cdot a) \mathbin{\widehat{\otimes}} \alpha_X(\eta) = \alpha_X(\xi) \mathbin{\widehat{\otimes}} \alpha_X(a \cdot \eta),$$

and using this we see that there is an isometric idempotent operator $\alpha_X \mathbin{\widehat{\otimes}} \alpha_X : X \mathbin{\widehat{\otimes}}_A X \to X \mathbin{\widehat{\otimes}}_A X$ characterised by $\xi \mathbin{\widehat{\otimes}} \eta \mapsto \alpha_X(\xi) \mathbin{\widehat{\otimes}} \alpha_X(\eta)$. So α_X induces isometric operators $\alpha_X^{\otimes n} : X^{\otimes n} \to X^{\otimes n}$. We regard the $X^{\otimes n}$ as graded A-A-correspondences with respect to these operators. When we want to emphasise this grading, we write $X^{\widehat{\otimes} n}$ for the tensor-product module. Under this grading, if $\xi_1, \ldots, \xi_n$ are homogeneous, say $\xi_k \in X_{j_k}$, then $\xi_1 \mathbin{\widehat{\otimes}}_A \cdots \mathbin{\widehat{\otimes}}_A \xi_n$ is homogeneous with degree $\sum\limits_k j_k$. When convenient we write $X^{\widehat{\otimes} 0}$ for A.

If A is trivially graded, then Lemma 4.1 shows that each $\alpha_X^{\widehat{\otimes} n}$ is a self-adjoint unitary.

Let $\mathcal{F}_X := \bigoplus\limits_{n=0}^{\infty} X^{\widehat{\otimes} n}$ be the Fock space of X [32]. Then $\mathcal{F}_X$ is a C^*-correspondence over A. We write φ^∞ for the homomorphism $A \to \mathcal{L}(\mathcal{F}_X)$ implementing the diagonal left action.

The operator $\alpha_X^\infty := \bigoplus\limits_{n=0}^{\infty} \alpha_X^{\widehat{\otimes} n}$ is a grading of $\mathcal{F}_X$ and the induced grading on $\mathcal{L}(\mathcal{F}_X)$ restricts to gradings $\alpha_{\mathcal{K}}$ and $\alpha_{\mathcal{T}}$ of $\mathcal{K}(\mathcal{F}_X)$ and $\mathcal{T}_X$ respectively. These satisfy

$$\alpha_{\mathcal{K}}(\theta_{\xi,\eta}) = \theta_{\alpha_X^{\widehat{\otimes} n}(\xi), \alpha_X^{\widehat{\otimes} m}(\eta)} \quad \text{and} \quad \alpha_{\mathcal{T}}(T_\xi) = T_{\alpha_X^{\widehat{\otimes} n}(\xi)}$$

for $\xi \in X^{\widehat{\otimes} n}$ and $\eta \in X^{\widehat{\otimes} m}$. Since these gradings are compatible with the inclusion $\mathcal{K}(\mathcal{F}_X) \hookrightarrow \mathcal{T}_X$, they induce a grading $\alpha_{\mathcal{O}}$ on $\mathcal{O}_X \cong \mathcal{T}_X / \mathcal{K}(\mathcal{F}_X)$.

If A is trivially graded, then Lemma 4.1 shows that $\alpha_{\mathcal{T}}$ and $\alpha_{\mathcal{K}}$ are inner gradings; but $\alpha_{\mathcal{O}}$ need not be, as we shall see later.

PIMSNER'S SIX-TERM EXACT SEQUENCE IN KK-THEORY. Let $i_A : A \to \mathcal{T}_X$ denote the canonical inclusion homomorphism. Then i_A determines a Kasparov class

$$[i_A] = [\mathcal{T}_X, i_A, 0, \alpha_{\mathcal{T}}] \in KK(A, \mathcal{T}_X). \tag{4.1}$$

Pimsner constructs a class in $KK(\mathcal{T}_X, A)$ as follows: let $\pi_0 : \mathcal{T}_X \to \mathcal{L}(\mathcal{F}_X)$ denote the canonical representation determined by $\pi_0(T_\xi)\rho := \xi \mathbin{\widehat{\otimes}}_A \rho$ for $\xi \in X$ and $\rho \in X^{\widehat{\otimes} n}$. One checks, using the universal property of $\mathcal{T}_X$, that there is a second representation $\pi_1 : \mathcal{T}_X \to \mathcal{L}(\mathcal{F}_X)$ such that for $\rho \in X^{\widehat{\otimes} n} \subseteq \mathcal{F}_X$,

$$\pi_1(T)\rho = \begin{cases} \pi_0(T)\rho & \text{if } n \geqslant 1, \\ 0 & \text{if } n = 0. \end{cases}$$

Arguing as in Lemma 4.2 of [32], we see that $\pi_0(T) - \pi_1(T) \in \mathcal{K}(\mathcal{F}_X)$ for all $T \in \mathcal{T}_X$. The operator $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is odd-graded with respect to the grading operator

$\overline{\alpha}_X^\infty := \left(\begin{smallmatrix} \alpha_X^\infty & 0 \\ 0 & -\alpha_X^\infty \end{smallmatrix}\right)$, and so for $T \in \mathcal{T}_X$, using the formula (2.2), we compute the graded commutator:

$$\begin{aligned}\left[\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \pi_0 & 0 \\ 0 & \pi_1 \circ \alpha_{\mathcal{T}} \end{smallmatrix}\right)(T)\right]^{\mathrm{gr}} &= \left(\begin{smallmatrix} 0 & \pi_1 \circ \alpha_{\mathcal{T}}(T) \\ \pi_0(T) & 0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 & \pi_0(\alpha_{\mathcal{T}}(T)) \\ \pi_1 \circ \alpha_{\mathcal{T}}(\alpha_{\mathcal{T}}(T)) & 0 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 0 & (\pi_1 - \pi_0) \circ \alpha_{\mathcal{T}}(T) \\ (\pi_0 - \pi_1)(T) & 0 \end{smallmatrix}\right),\end{aligned}$$

which is compact. Hence we obtain a Kasparov module

$$M := \Big(\mathcal{F}_X \oplus \mathcal{F}_X, \pi_0 \oplus \pi_1 \circ \alpha_{\mathcal{T}}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_X^\infty & 0 \\ 0 & -\alpha_X^\infty \end{smallmatrix}\right)\Big).$$

Since the essential subspace of $\mathcal{F}_X \oplus \mathcal{F}_X$ for $\pi_0 \oplus \pi_1 \circ \alpha_T$ is complemented, replacing $\mathcal{F}_X \oplus \mathcal{F}_X$ with the essential subspace for $\pi_0 \oplus \pi_1 \circ \alpha_{\mathcal{T}}$, and adjusting the Fredholm operator accordingly yields a module representing the same class (see Proposition 18.3.6 of [2]). Hence, writing $P : \mathcal{F}_X \to \mathcal{F}_X \ominus A = \bigoplus_{n=1}^{\infty} X^{\widehat{\otimes} n}$ for the projection onto the orthogonal complement of the 0^{th} summand, we have

$$[M] = \Big[\mathcal{F}_X \oplus (\mathcal{F}_X \ominus A), \pi_0 \oplus \pi_1 \circ \alpha_{\mathcal{T}}, \left(\begin{smallmatrix} 0 & 1 \\ P & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_X^\infty & 0 \\ 0 & -\alpha_X^\infty \end{smallmatrix}\right)\Big] \in KK(\mathcal{T}_X, A). \tag{4.2}$$

In the ungraded case, the classes $[M] \in KK(\mathcal{T}_X, A)$ and $[i_A] \in KK(A, \mathcal{T}_X)$ described at (4.2) and (4.1) are denoted α and β in Section 4 of [32].

THEOREM 4.2 (cf. Theorem 4.4 of [32]). *Under the setup described at the beginning of the section (we do not assume that α_A is trivial), and with notation as above, the pair $[M]$ and $[i_A]$ are mutually inverse. That is, $[i_A] \widehat{\otimes}_{\mathcal{T}_X} [M] = [\mathrm{id}_A]$ and $[M] \widehat{\otimes}_A [i_A] = [\mathrm{id}_{\mathcal{T}_X}]$. In particular A and $\mathcal{T}_X$ are KK-equivalent as graded C^*-algebras.*

Proof. Since the class $[i_A]$ is induced by a homomorphism of C^*-algebras, we can compute the products $[i_A] \widehat{\otimes}_{\mathcal{T}_X} [M]$ and $[M] \widehat{\otimes}_A [i_A]$ using Proposition 18.7.2 of [2].

As discussed in Pimsner's proof, Proposition 18.7.2(b) of [2] implies that the product $[i_A] \widehat{\otimes}_{\mathcal{T}_X} [M]$ is equal to $(i_A)^*[M]$, and so, using the representative (4.2) of $[M]$, we obtain

$$[i_A] \widehat{\otimes}_{\mathcal{T}_X} [M] = \Big[\mathcal{F}_X \oplus (\mathcal{F}_X \ominus A), (\pi_0 \oplus \pi_1 \circ \alpha_{\mathcal{T}}) \circ i_A, \left(\begin{smallmatrix} 0 & 1 \\ P & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_X^\infty & 0 \\ 0 & -\alpha_X^\infty \end{smallmatrix}\right)\Big].$$

We have

$$\begin{aligned}&\Big(\mathcal{F}_X \oplus (\mathcal{F}_X \ominus A), (\pi_0 \oplus \pi_1 \circ \alpha_{\mathcal{T}}) \circ i_A, \left(\begin{smallmatrix} 0 & 1 \\ P & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_X^\infty & 0 \\ 0 & -\alpha_X^\infty \end{smallmatrix}\right)\Big) \oplus (A, \alpha_A, 0, -\alpha_A) \\ &\qquad \cong \Big(\mathcal{F}_X \oplus \mathcal{F}_X, (\pi_0 \oplus \pi_0 \circ \alpha_{\mathcal{T}}) \circ i_A, \left(\begin{smallmatrix} 0 & P \\ P & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_X^\infty & 0 \\ 0 & -\alpha_X^\infty \end{smallmatrix}\right)\Big).\end{aligned} \tag{4.3}$$

The operator $F = \left(\begin{smallmatrix} 0 & P \\ P & 0 \end{smallmatrix}\right)$ satisfies $F^2 = 1$, $F = F^*$. For $a \in A$, we have

$$\begin{aligned} F(\pi_0 \oplus \pi_0 \circ \alpha_{\mathcal{T}})(i_A(a)) &= \left(\begin{smallmatrix} 0 & P \\ P & 0 \end{smallmatrix}\right) \begin{pmatrix} \pi_0(i_A(a)) & 0 \\ 0 & \pi_0(i_A(\alpha_A(a))) \end{pmatrix} \\ &= \begin{pmatrix} \pi_0(i_A(\alpha_A(a))) & 0 \\ 0 & \pi_0(i_A(a)) \end{pmatrix} \left(\begin{smallmatrix} 0 & P \\ P & 0 \end{smallmatrix}\right) \\ &= (\pi_0 \oplus \pi_0 \circ \alpha_{\mathcal{T}})(i_A(\alpha_A(a)))F. \end{aligned}$$

So $[F, (\pi_0 \oplus \pi_0 \circ \alpha_{\mathcal{T}})(i_A(a))]^{\mathrm{gr}} = 0$ by (2.2). Hence (4.3) is a degenerate Kasparov module, and hence represents the zero class. By (2.6), we have $[A, \alpha_A, 0, -\alpha_A] = -[\mathrm{id}_A]$, so we have $[i_A] \mathbin{\widehat{\otimes}}_{\mathcal{T}_X} [M] - [\mathrm{id}_A] = 0_{KK(A,A)}$ giving $[i_A] \mathbin{\widehat{\otimes}}_{\mathcal{T}_X} [M] = [\mathrm{id}_A]$.

For the reverse composition, Proposition 18.7.2(a) of [2] shows that $[M] \mathbin{\widehat{\otimes}}_A [i_A]$ is equal to $(i_A)_*[M]$, which is represented by

$$\left((\mathcal{F}_X \oplus \mathcal{F}_X) \mathbin{\widehat{\otimes}}_A \mathcal{T}_X, (\pi_0 \oplus \pi_1 \circ \alpha_{\mathcal{T}}) \mathbin{\widehat{\otimes}} 1_{\mathcal{T}_X}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \begin{pmatrix} \alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} & 0 \\ 0 & -\alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} \end{pmatrix} \right).$$

We write π_0' and π_1' for $\pi_0 \mathbin{\widehat{\otimes}} 1_{\mathcal{T}_X}$ and $(\pi_1 \circ \alpha_{\mathcal{T}}) \mathbin{\widehat{\otimes}} 1_{\mathcal{T}_X}$. Since X is essential as a left A-module, we have $A \mathbin{\widehat{\otimes}}_A \mathcal{T}_X \cong \mathcal{T}_X$ as graded A-$\mathcal{T}_X$-correspondences, so the grading $\alpha_{\mathcal{T}}$ implements a left action of $\mathcal{T}_X$ on $A \mathbin{\widehat{\otimes}}_A \mathcal{T}_X$. We regard this as an action τ of $\mathcal{T}_X$ on $\mathcal{F}_X \mathbin{\widehat{\otimes}}_A \mathcal{T}_X$ that acts nontrivially only on the 0^{th} summand. Consider the Kasparov module

$$\left((\mathcal{F}_X \oplus \mathcal{F}_X) \mathbin{\widehat{\otimes}}_A \mathcal{T}_X, (0 \oplus \tau), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \begin{pmatrix} \alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} & 0 \\ 0 & -\alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} \end{pmatrix} \right).$$

The essential subspace of the action $0 \oplus \tau$ is equal to the copy of $A \mathbin{\widehat{\otimes}}_A \mathcal{T}_X$ in the graded submodule $(0 \oplus \mathcal{F}_X) \mathbin{\widehat{\otimes}}_A \mathcal{T}_X$ of $(\mathcal{F}_X \oplus \mathcal{F}_X) \mathbin{\widehat{\otimes}}_A \mathcal{T}_X$. Moreover, the restriction of $0 \oplus \tau$ to this submodule is just $\alpha_{\mathcal{T}}$. Hence

$$\left[(\mathcal{F}_X \oplus \mathcal{F}_X) \mathbin{\widehat{\otimes}}_A \mathcal{T}_X, (0 \oplus \tau), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \begin{pmatrix} \alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} & 0 \\ 0 & -\alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} \end{pmatrix} \right] = [\mathcal{T}_X, \alpha_{\mathcal{T}}, 0, -\alpha_{\mathcal{T}}] = -[\mathrm{id}_{\mathcal{T}_X}]$$

by (2.6).

Therefore, using that the essential subspaces of $\pi_0' \oplus \pi_1'$ and $0 \oplus \tau$ are one another's orthogonal complements, we see that

$$\begin{aligned} (i_A)_*[M] - [1_{\mathcal{T}_X}] &= \left[(\mathcal{F}_X \oplus \mathcal{F}_X) \mathbin{\widehat{\otimes}}_A \mathcal{T}_X, \pi_0' \oplus \pi_1', \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \begin{pmatrix} \alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} & 0 \\ 0 & -\alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} \end{pmatrix} \right] \\ &\quad + \left[(\mathcal{F}_X \oplus \mathcal{F}_X) \mathbin{\widehat{\otimes}}_A \mathcal{T}_X, (0 \oplus \tau), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \begin{pmatrix} \alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} & 0 \\ 0 & -\alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} \end{pmatrix} \right] \\ &= \left[(\mathcal{F}_X \oplus \mathcal{F}_X) \mathbin{\widehat{\otimes}}_A \mathcal{T}_X, \pi_0' \oplus (\pi_1' + \tau), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \begin{pmatrix} \alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} & 0 \\ 0 & -\alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} \end{pmatrix} \right]. \end{aligned} \tag{4.4}$$

We claim there is a homotopy of graded homomorphisms $\pi_t' : \mathcal{T}_X \to \mathcal{L}(\mathcal{F}_X \mathbin{\widehat{\otimes}}_A \mathcal{T}_X)$ from $\pi_0' \circ \alpha_{\mathcal{T}}$ to $\pi_1' + \tau$ such that for each $t \in [0, 1]$,

$$\left((\mathcal{F}_X \oplus \mathcal{F}_X) \mathbin{\widehat{\otimes}}_A \mathcal{T}_X, \pi_0' \oplus \pi_t', \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \begin{pmatrix} \alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} & 0 \\ 0 & -\alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} \end{pmatrix} \right) \tag{4.5}$$

is a Kasparov module. To see this, we invoke the universal property of $\mathcal{T}_X$. For each t, following Pimsner, define a linear map $\psi_t : X \to \mathcal{L}(\mathcal{F}_X \mathbin{\widehat{\otimes}}_A \mathcal{T}_X)$ by

$$\psi_t(\xi) = (\cos(\pi t/2)(\pi_0'(\alpha_{\mathcal{T}}(T_\xi)) - \pi_1'(T_\xi)) + \sin(\pi t/2)\tau(T_\xi)) + \pi_1'(T_\xi).$$

Recall that $\varphi^\infty : A \to \mathcal{L}(\mathcal{F}_X)$ denotes the homomorphism given by the diagonal left action of A. We write $\widetilde{\varphi}^\infty$ for $\varphi^\infty \widehat{\otimes}_A 1_{\mathcal{T}_X}$. We aim to prove that $(\widetilde{\varphi}^\infty \circ \alpha_A, \psi_t)$ is a Toeplitz representation of X for each $t \in [0,1]$. Since each ψ_t is a convex combination of bimodule maps, we see that

$$\widetilde{\varphi}^\infty(\alpha_A(a))\psi_t(\xi) = \psi_t(a \cdot \xi) \quad \text{and} \quad \psi_t(\xi)\widetilde{\varphi}^\infty(\alpha_A(a)) = \psi_t(\xi \cdot a)$$

for all a, ξ, t. Next we check that ψ_t is compatible with the inner product. Note that for all $\xi, \eta, \zeta \in X$ the operators $(\pi'_0(\alpha_{\mathcal{T}}(T_\xi)) - \pi'_1(T_\xi))$, $\tau(T_\eta)$ and $\pi'_1(T_\zeta)$ have mutually orthogonal ranges in $\mathcal{F}_X \widehat{\otimes}_A \mathcal{T}_X$ (the same observation is made in Pimsner's argument, and the only difference between his operators and ours is post-composition with $\alpha_{\mathcal{T}}$). Given $\xi, \eta \in X$ and $t \in [0,1]$ we have

$$\begin{aligned}
&\psi_t(\xi)^*\psi_t(\eta) \\
&\quad = ((\cos(\pi t/2)(\pi'_0(\alpha_{\mathcal{T}}(T_\xi)) - \pi'_1(T_\xi)) + \sin(\pi t/2)\tau(T_\xi)) + \pi'_1(T_\xi))^* \\
&\qquad\quad (\cos(\pi t/2)(\pi'_0(\alpha_{\mathcal{T}}(T_\eta)) - \pi'_1(T_\eta)) + \sin(\pi t/2)\tau(T_\eta)) + \pi'_1(T_\eta)) \\
&\quad = ((\cos(\pi t/2)(\pi'_0(\alpha_{\mathcal{T}}(T_\xi)) - \pi'_1(T_\xi)) + \sin(\pi t/2)\tau(T_\xi)))^* \\
&\qquad\quad (\cos(\pi t/2)(\pi'_0(\alpha_{\mathcal{T}}(T_\eta)) - \pi'_1(T_\eta)) + \sin(\pi t/2)\tau(T_\eta)) + \pi'_1(T_\xi)^*\pi'_1(T_\eta).
\end{aligned}$$

Write $\widetilde{P}$ for the projection onto $(\mathcal{F}_X \widehat{\otimes}_A \mathcal{T}_X) \ominus (A \widehat{\otimes}_A \mathcal{T}_X)$. Since π'_1 is a homomorphism, we have

$$\pi'_1(T_\xi)^*\pi'_1(T_\eta) = \pi'_1(T_\xi^* T_\eta) = \pi'_1(\langle \xi, \eta \rangle_A) = \widetilde{P}\widetilde{\varphi}^\infty(\alpha_A(\langle \xi, \eta \rangle_A))\widetilde{P}.$$

For $\zeta, \zeta' \in X$ the range of $\tau(T_\zeta)$ is contained in $A \widehat{\otimes}_A \mathcal{T}_X \subseteq \mathcal{F}_X \widehat{\otimes}_A \mathcal{T}_X$, which is orthogonal to the range of $((\pi'_0 \circ \alpha_{\mathcal{T}}) - \pi'_1)(T_{\zeta'})$. Also, $((\pi'_0 \circ \alpha_{\mathcal{T}}) - \pi'_1)(T_\zeta) = \pi'_0(\alpha_{\mathcal{T}}(T_\zeta))(1 - \widetilde{P})$. Using these two points, and resuming our computation of $\psi_t(\xi)^*\psi_t(\eta)$ from above, we have

$$\begin{aligned}
&(\cos(\pi t/2)(\pi'_0(\alpha_{\mathcal{T}}(T_\xi)) - \pi'_1(T_\xi)) + \sin(\pi t/2)\tau(T_\xi))^* \\
&\qquad (\cos(\pi t/2)(\pi'_0(\alpha_{\mathcal{T}}(T_\eta)) - \pi'_1(T_\eta)) + \sin(\pi t/2)\tau(T_\eta)) \\
&\quad = \cos(\pi t/2)^2(\pi'_0(\alpha_{\mathcal{T}}(T_\xi))(1-\widetilde{P}))^*(\pi'_0(\alpha_{\mathcal{T}}(T_\eta))(1-\widetilde{P})) + \sin(\pi t/2)^2\tau(T_\xi)^*\tau(T_\eta) \\
&\quad = (1 - \widetilde{P})(\cos(\pi t/2)^2\widetilde{\varphi}^\infty(\alpha_A(\langle \xi, \eta \rangle_A)) + \sin(\pi t/2)^2\widetilde{\varphi}^\infty(\alpha_A(\langle \xi, \eta \rangle_A)))(1 - \widetilde{P}) \\
&\quad = (1 - \widetilde{P})\widetilde{\varphi}^\infty(\alpha_A(\langle \xi, \eta \rangle_A))(1 - \widetilde{P}).
\end{aligned}$$

Since $\widetilde{P}$ commutes with the range of $\widetilde{\varphi}^\infty$, we have $\psi_t(\xi)^*\psi_t(\eta) = \widetilde{\varphi}^\infty(\alpha_A(\langle \xi, \eta \rangle_A))$ and so $(\widetilde{\varphi}^\infty \circ \alpha_A, \psi_t)$ is a Toeplitz representation of X for each $t \in [0,1]$. Thus the universal property of $\mathcal{T}_X$ ensures that there exists a homomorphism $\pi'_t : \mathcal{T}_X \to \mathcal{L}(\mathcal{F}_X \widehat{\otimes}_A \mathcal{T}_X)$ such that $\pi'_t(T_\xi) = \psi_t(\xi)$ for $\xi \in X$ and $\pi'_t(a) = \widetilde{\varphi}^\infty(\alpha_A(a))$ for $a \in A$.

For $t \in [0,1]$ and $\xi \in X$, we have

$$\pi'_t(T_\xi) - \pi'_1(T_\xi) = (\cos(\pi t/2)(\pi'_0(\alpha_{\mathcal{T}}(T_\xi)) - \pi'_1(T_\xi)) + \sin(\pi t/2)\tau(T_\xi)).$$

The kernel of this operator contains $\widetilde{P}(\mathcal{F}_X \mathbin{\widehat{\otimes}}_A \mathcal{T}_X)$, and since A acts compactly on $(1-\widetilde{P})(\mathcal{F}_X \mathbin{\widehat{\otimes}}_A \mathcal{T}_X)$, the Cohen factorisation theorem ensures that $\pi'_t(T_\xi) - \pi'_1(T_\xi) \in \mathcal{K}(\mathcal{F}_X \mathbin{\widehat{\otimes}}_A \mathcal{T}_X)$. So for each t, the homomorphism π'_t is a compact perturbation of the homomorphism π'_1, which determines a Kasparov module, and therefore (4.5) is a Kasparov module for each t as claimed.

The claim shows that the class (4.4) is equal to the class of

$$\Big((\mathcal{F}_X \oplus \mathcal{F}_X) \mathbin{\widehat{\otimes}}_A \mathcal{T}_X, \pi'_0 \oplus \pi'_0 \circ \alpha_{\mathcal{T}}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} & 0 \\ 0 & -\alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} \end{smallmatrix}\right)\Big).$$

This is a degenerate Kasparov module (just calculate directly that $F^2 = 1$, $F^* = F$ and $[F, (\pi'_0 \oplus \pi'_0 \circ \alpha_{\mathcal{T}})(T)]^{\mathrm{gr}} = 0$ for all $T \in \mathcal{T}_X$), so it represents the zero class. Hence $(i_A)_*[M] = [\mathrm{id}_{\mathcal{T}_X}]$. ∎

Let $\iota : \mathcal{K}(\mathcal{F}_X) \to \mathcal{L}(\mathcal{F}_X)$ denote the canonical inclusion. Then $(\mathcal{F}_X, \iota, 0, \alpha_X^\infty)$ is a Kasparov module and we have $[\mathcal{F}_X, \iota, 0, \alpha_X^\infty] \in KK(\mathcal{K}(\mathcal{F}_X), A)$. As Pimsner points out, this is the KK-equivalence given by the equivalence bimodule $\mathcal{F}_X$. Let $j : \mathcal{K}(\mathcal{F}_X) \to \mathcal{T}_X$ be the natural inclusion.

LEMMA 4.3 (cf. Lemma 4.7 of [32]). *With notation as above we have*

$$[j] \mathbin{\widehat{\otimes}}_{\mathcal{T}_X} [M] = [\mathcal{F}_X, \iota, 0, \alpha_X^\infty] \mathbin{\widehat{\otimes}}_A ([\mathrm{id}_A] - [X])$$

in $KK(\mathcal{K}(\mathcal{F}_X), A)$.

Proof. By Proposition 18.7.2(b) of [2] we have $[j] \mathbin{\widehat{\otimes}}_{\mathcal{T}_X} [M] = j^*[M]$. Using the representation (4.2) of $[M]$ we therefore obtain

$$[j] \mathbin{\widehat{\otimes}}_{\mathcal{T}_X} [M] = \Big[\mathcal{F}_X \oplus (\mathcal{F}_X \ominus A), (\pi_0 \oplus \pi_1 \circ \alpha_{\mathcal{T}}) \circ j, \left(\begin{smallmatrix} 0 & 1 \\ P & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_X^\infty & 0 \\ 0 & -\alpha_X^\infty \end{smallmatrix}\right)\Big].$$

Since $\pi_0 \circ j(\mathcal{K}(\mathcal{F}_X)) \subseteq \mathcal{K}(\mathcal{F}_X)$ and similarly for π_1, the straight-line path from $\left(\begin{smallmatrix} 0 & 1 \\ P & 0 \end{smallmatrix}\right)$ to 0 gives an operator homotopy, so

$$\begin{aligned} [j] \mathbin{\widehat{\otimes}}_{\mathcal{T}_X} [M] &= \Big[\mathcal{F}_X \oplus (\mathcal{F}_X \ominus A), (\pi_0 \oplus \pi_1 \circ \alpha_{\mathcal{T}}) \circ j, 0, \left(\begin{smallmatrix} \alpha_X^\infty & 0 \\ 0 & -\alpha_X^\infty \end{smallmatrix}\right)\Big] \\ &= [\mathcal{F}_X, \pi_0 \circ j, 0, \alpha_X^\infty] + [\mathcal{F}_X \ominus A, \pi_1 \circ j \circ \alpha_{\mathcal{K}}, 0, -\alpha_X^\infty] \\ &= [\mathcal{F}_X, \iota, 0, \alpha_X^\infty] + [\mathcal{F}_X \ominus A, \pi_1 \circ j \circ \alpha_{\mathcal{K}}, 0, -\alpha_X^\infty]. \end{aligned}$$

We have $\mathcal{F}_X \mathbin{\widehat{\otimes}}_A X \cong \mathcal{F}_X \ominus A$ as right-Hilbert modules, and this isomorphism carries $\pi_0 \mathbin{\widehat{\otimes}} 1_X$ to π_1, and hence $(\pi_0 \circ j) \mathbin{\widehat{\otimes}} 1_X$ to $\pi_1 \circ j$. So

$$(\mathcal{F}_X \ominus A, \pi_1 \circ j \circ \alpha_{\mathcal{K}}, 0, -\alpha_X^\infty) \cong (\mathcal{F}_X \mathbin{\widehat{\otimes}}_A X, (\pi_0 \circ j \circ \alpha_{\mathcal{K}}) \mathbin{\widehat{\otimes}} 1_X, 0, -\alpha_X^\infty \mathbin{\widehat{\otimes}} \alpha_X).$$

The right-hand side represents $[\mathcal{F}_X, \iota \circ \alpha_{\mathcal{K}}, 0, -\alpha_X^\infty] \mathbin{\widehat{\otimes}}_A [X]$. Equation (2.5) implies that $[\mathcal{F}_X, \iota \circ \alpha_{\mathcal{K}}, 0, -\alpha_X^\infty] = -[\mathcal{F}_X, \iota, 0, \alpha_X^\infty]$. Thus

$$[j] \mathbin{\widehat{\otimes}}_{\mathcal{T}_X} [M] = [\mathcal{F}_X, \iota, 0, \alpha_X^\infty] - ([\mathcal{F}_X, \iota, 0, \alpha_X^\infty] \mathbin{\widehat{\otimes}}_A [X]) = [\mathcal{F}_X, \iota, 0, \alpha_X^\infty] \mathbin{\widehat{\otimes}}_A ([\mathrm{id}_A] - [X])$$

as claimed. ∎

Finally, we obtain two six-term exact sequences as in Theorem 4.9 of [32]. For the purposes of computing graded K-theory, we are most interested in the first of the two sequences, and in the situation where $B = \mathbb{C}$; but both could be useful in general. We write $i : A \to \mathcal{O}_X$ for the canonical inclusion.

THEOREM 4.4 (cf. Theorem 4.9 of [32]). *Let A and B be graded σ-unital C^*-algebras and let X be a graded correspondence over A such that the left action is injective and by compacts. Continue with notation as above. If either A or B is nuclear, then we have a six-term exact sequence as follows:*

$$\begin{tikzcd}
KK_0(B,A) \arrow[r, "\widehat{\otimes}_A([\mathrm{id}_A]-[X])"] & KK_0(B,A) \arrow[r, "i_*"] & KK_0(B,\mathcal{O}_X) \arrow[d] \\
KK_1(B,\mathcal{O}_X) \arrow[u] & KK_1(B,A) \arrow[l, "i_*"'] & KK_1(B,A). \arrow[l, "\widehat{\otimes}_A([\mathrm{id}_A]-[X])"']
\end{tikzcd} \tag{4.6}$$

If A is nuclear, then we also have a six-term sequence as follows:

$$\begin{tikzcd}
KK_0(A,B) \arrow[d] & KK_0(A,B) \arrow[l, "([\mathrm{id}_A]-[X])\widehat{\otimes}_A"'] & KK_0(\mathcal{O}_X,B) \arrow[l, "i^*"'] \\
KK_1(\mathcal{O}_X,B) \arrow[r, "i^*"] & KK_1(A,B) \arrow[r, "([\mathrm{id}_A]-[X])\widehat{\otimes}_A"] & KK_1(A,B). \arrow[u]
\end{tikzcd} \tag{4.7}$$

These sequences are, respectively, contravariantly and covariantly natural in B. They are also natural in the other variable in the following sense: if C is a graded C^*-algebra, and Y_C is a graded correspondence over C whose left action is injective and by compacts, and if $\theta_A : A \to C$ and $\theta_X : X \to Y$ constitute a graded morphism of C^*-correspondences, then θ_A and the induced homomorphism $(\theta_A \times \theta_X) : \mathcal{O}_X \to \mathcal{O}_Y$ induce morphisms of exact sequences from (4.6) *for (A,X) to* (4.6) *for (C,Y) and from* (4.7) *for (C,Y) to* (4.7) *for (A,X).*

Proof. We just prove exactness of the first diagram: the second follows from a similar argument to the one given for the first when A is nuclear.

Suppose that A is nuclear. Then so is $\mathcal{T}_X$ (see, for example, Theorem 6.3 of [37]) and so the quotient map $q : \mathcal{T}_X \to \mathcal{O}_X$ has a completely positive splitting. Hence Theorem 1.1 of [42] applied to the graded short exact sequence $0 \to \mathcal{K}(\mathcal{F}_X) \xrightarrow{j} \mathcal{T}_X \xrightarrow{q} \mathcal{O}_X \to 0$ yields homomorphisms $\delta : KK_i(B,\mathcal{O}_X) \to KK_{i+1}(B,\mathcal{K}(\mathcal{F}_X))$ for which the following six-term sequence is exact:

$$\begin{tikzcd}
KK_0(B,\mathcal{K}(\mathcal{F}_X)) \arrow[r, "j_*"] & KK_0(B,\mathcal{T}_X) \arrow[r, "q_*"] & KK_0(B,\mathcal{O}_X) \arrow[d, "\delta"] \\
KK_1(B,\mathcal{O}_X) \arrow[u, "\delta"] & KK_1(B,\mathcal{T}_X) \arrow[l, "q_*"'] & KK_1(B,\mathcal{K}(\mathcal{F}_X)). \arrow[l, "j_*"']
\end{tikzcd} \tag{4.8}$$

If instead B is nuclear, we must argue differently to obtain the sequence (4.8). If B is nuclear, then $KK_{\mathrm{nuc}}(\cdot, B)$ coincides with $KK(\cdot, B)$ as discussed immediately following Definition 2.1 of [43]. Hence Proposition 2.7 of [43] once again shows that the six-term sequence (4.8) is exact.

With the sequence (4.8) in hand, the remainder of the argument is the same regardless of which of A or B is nuclear. Define maps

$$\delta' : KK_*(B, \mathcal{O}_X) \to KK_{*+1}(B, A)$$

by $\delta' = (\cdot \mathbin{\widehat{\otimes}} [\mathcal{F}_X, \iota, 0, \alpha_X^\infty]) \circ \delta$, and consider the following diagram:

$$\begin{tikzcd}
KK_0(B,A) \arrow[rr, "\widehat{\otimes}({[\mathrm{id}_A]-[X]})"] & & KK_0(B,A) \arrow[rr, "i_*"] \arrow[d, bend right, "(i_A)_*"'] & & KK_0(B,\mathcal{O}_X) \arrow[dl, "\mathrm{id}"'] \arrow[ddd, "\delta'"] \\
& KK_0(B,\mathcal{K}(\mathcal{F}_X)) \arrow[ul, "\widehat{\otimes}{[\mathcal{F}_X,\iota,0,\alpha_X^\infty]}"'] \arrow[r, "j_*"] & KK_0(B,\mathcal{T}_X) \arrow[r, "q_*"] \arrow[u, bend right, "{[M]}"'] & KK_0(B,\mathcal{O}_X) \arrow[d, "\delta"] & \\
& KK_1(B,\mathcal{O}_X) \arrow[u, "\delta"] & KK_1(B,\mathcal{T}_X) \arrow[l, "q_*"] \arrow[d, bend right, "{[M]}"'] & KK_1(B,\mathcal{K}(\mathcal{F}_X)) \arrow[l, "j_*"] \arrow[dr, "\widehat{\otimes}{[\mathcal{F}_X,\iota,0,\alpha_X^\infty]}"] & \\
KK_1(B,\mathcal{O}_X) \arrow[uuu, "\delta'"] \arrow[ur, "\mathrm{id}"] & & KK_1(B,A) \arrow[ll, "i_*"] \arrow[u, bend right, "(i_A)_*"'] & & KK_1(B,A). \arrow[ll, "\widehat{\otimes}({[\mathrm{id}_A]-[X]})"]
\end{tikzcd}$$

The left-hand and right-hand squares commute by definition of the maps δ'. Lemma 4.3 implies that the top left and bottom right squares commute. Since $q \circ i_A = i$ as homomorphisms, we have $q_* \circ (i_A)_* = i_*$, and so the top right and bottom left squares commute as well. Since all the maps linking the inner rectangle to the outer rectangle are isomorphisms, it follows that the outer rectangle is exact as required.

Naturality follows from naturality of Pimsner's exact sequences, which in turn follows from naturality of the KK-functor for graded C^*-algebras ([2], Section 17.8). ∎

COROLLARY 4.5. *Let (A, α) be a σ-unital, graded C^*-algebra and let (X, α_X) be a countably generated, graded correspondence over A such that the left action is injective and by compacts. Then there is a six-term exact sequence for graded K-theory as follows:*

$$\begin{tikzcd}
K_0^{\mathrm{gr}}(A,\alpha) \arrow[r, "\widehat{\otimes}_A({[\mathrm{id}_A]-[X]})"] & K_0^{\mathrm{gr}}(A,\alpha) \arrow[r, "i_*"] & K_0^{\mathrm{gr}}(\mathcal{O}_X,\alpha_{\mathcal{O}}) \arrow[d] \\
K_1^{\mathrm{gr}}(\mathcal{O}_X,\alpha_{\mathcal{O}}) \arrow[u] & K_1^{\mathrm{gr}}(A,\alpha) \arrow[l, "i_*"] & K_1^{\mathrm{gr}}(A,\alpha). \arrow[l, "\widehat{\otimes}_A({[\mathrm{id}_A]-[X]})"]
\end{tikzcd} \tag{4.9}$$

This follows from the first part of Theorem 4.4 applied with B equal to the (nuclear) C^*-algebra $\mathbb{C}$.

REMARK 4.6. With notation as above, if A is nuclear, the second part of Theorem 4.4 can be applied with B equal to the C^*-algebra $\mathbb{C}$ to obtain a six-term exact sequence for the graded K-homology of $\mathcal{O}_X$, by which we mean the groups $K^0_{\mathrm{gr}}(\mathcal{O}_X) := KK(\mathcal{O}_X, \mathbb{C})$ and $K^1_{\mathrm{gr}}(\mathcal{O}_X) := KK(\mathcal{O}_X \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1, \mathbb{C})$.

THE GRADED PIMSNER–VOICULESCU EXACT SEQUENCE. If (A, α) is a σ-unital graded C^*-algebra, and γ is an automorphism of A that is graded in the sense that it commutes with α, then functoriality of KK shows that γ induces a map γ_0 on $K^{\mathrm{gr}}_0(A, \alpha) = KK(\mathbb{C}, A)$ and γ_1 on $K^{\mathrm{gr}}_1(A, \alpha) = KK(\mathbb{C}, A \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1)$.

The crossed product $A \rtimes_\gamma \mathbb{Z}$ has two natural grading automorphisms, which we will denote by β^0 and β^1. To describe them, write $i_A : A \to A \rtimes_\gamma \mathbb{Z}$ and $i_{\mathbb{Z}} : \mathbb{Z} \to \mathcal{UM}(A \rtimes_\gamma \mathbb{Z})$ for the canonical inclusions. Then for $k \in \mathbb{Z}_2$, the automorphism β^k is characterised by

$$\beta^k(i_A(a) i_{\mathbb{Z}}(n)) = (-1)^{kn} i_A(\alpha(a)) i_{\mathbb{Z}}(n). \tag{4.10}$$

So $\beta^1 = \beta^0 \circ \widehat{\gamma}_{-1}$ where $\widehat{\gamma}$ is the action of $\mathbb{T}$ on the crossed product dual to γ. The inclusion $i_A : A \to A \rtimes_\gamma \mathbb{Z}$ is a graded homomorphism with respect to α and β^k for each of $k = 0, 1$.

COROLLARY 4.7 (Graded Pimsner–Voiculescu sequence). *Let (A, α) be a σ-unital, graded C^*-algebra and γ an automorphism of A that commutes with α. Fix $k \in \{0, 1\}$ and let β^k be the grading automorphism of $A \rtimes_\gamma \mathbb{Z}$ described above. Then we obtain a six-term exact sequence in graded K-theory as follows:*

$$\begin{tikzcd}
K^{\mathrm{gr}}_0(A, \alpha) \arrow[r, "\mathrm{id} - (-\alpha_*)^k \gamma_*"] & K^{\mathrm{gr}}_0(A, \alpha) \arrow[r, "i_*"] & K^{\mathrm{gr}}_0(A \rtimes_\gamma \mathbb{Z}, \beta^k) \arrow[d] \\
K^{\mathrm{gr}}_1(A \rtimes_\gamma \mathbb{Z}, \beta^k) \arrow[u] & K^{\mathrm{gr}}_1(A, \alpha) \arrow[l, "i_*"] & K^{\mathrm{gr}}_1(A, \alpha). \arrow[l, "\mathrm{id} - (-\alpha_*)^k \gamma_*"]
\end{tikzcd} \tag{4.11}$$

The sequence is natural in the sense that if (B, κ) is another σ-unital, graded C^*-algebra, θ is an automorphism of B that commutes with κ, and $\phi : A \to B$ is a graded homomorphism that intertwines γ and θ, then ϕ and $\phi \times 1 : A \rtimes_\gamma \mathbb{Z} \to B \rtimes_\delta \mathbb{Z}$ induce a morphism of exact sequences from (4.11) *for (A, α) to* (4.11) *for (B, κ).*

Proof. Let $X := {}_\gamma A$ as a Hilbert module, endowed with the grading $(-1)^k \alpha$. Write (i_A, i_X) for the inclusions of A and X in $\mathcal{O}_X$, write $j_A : A \to A \rtimes_\gamma \mathbb{Z}$ for the canonical inclusion, and write U for the unitary element of $\mathcal{M}(A \rtimes_\gamma \mathbb{Z})$ implementing γ. Then, as pointed out in Example 3, p. 193 of [32], there is an isomorphism $\rho : \mathcal{O}_X \to A \rtimes_\gamma \mathbb{Z}$ such that $\rho(i_A(a)) = j_A(a)$ and $\rho(i_X(a)) = U j_A(a)$ for all $a \in A$. It is routine to check that this isomorphism is graded with respect to the grading β^k of $A \rtimes_\gamma \mathbb{Z}$ and the grading $\alpha_{\mathcal{O}}$ of $\mathcal{O}_X$ induced by the grading α of A and the grading $(-1)^k\alpha$ of X.

Hence Corollary 4.5 gives an exact sequence

$$\begin{tikzcd} K_0^{\mathrm{gr}}(A,\alpha) \arrow[r, "\widehat{\otimes}_A([\mathrm{id}_A]-[X])"] & K_0^{\mathrm{gr}}(A,\alpha) \arrow[r, "i_*"] & K_0^{\mathrm{gr}}(A \rtimes_\gamma \mathbb{Z}, \beta^k) \arrow[d] \\ K_1^{\mathrm{gr}}(A \rtimes_\gamma \mathbb{Z}, \beta^k) \arrow[u] & K_1^{\mathrm{gr}}(A,\alpha) \arrow[l, "i_*"] & K_1^{\mathrm{gr}}(A,\alpha). \arrow[l, "\widehat{\otimes}_A([\mathrm{id}_A]-[X])"] \end{tikzcd} \tag{4.12}$$

By definition, we have $\alpha^k_*[X] = [A_A, \gamma \circ \alpha^k, 0, (-1)^k\alpha]$. So (2.6) shows that $\alpha^k_*[X] = (-1)^k\gamma_*$. Since α, and hence α_*, has order 2, we deduce that $[X] = (-\alpha_*)^k\gamma_*$, giving the desired six-term exact sequence.

Naturality follows immediately from naturality in Theorem 4.4. ∎

5. TWISTED P-GRAPH C^*-ALGEBRAS AND ACTIONS BY COUNTABLE GROUPS

We now begin our investigation of how to use P-graphs to construct examples of graded C^*-algebras. Throughout we write $\mathbb{N}$ for the additive monoid $\{0,1,2,\dots\}$. Let F be a countable (discrete) abelian group, fix $k \geqslant 0$ and let $P := \mathbb{N}^k \times F$ regarded as a cancellative abelian monoid and let G_P denote the Grothendieck group of P. We frequently regard P as a small category with a single object and composition given by addition. Given a small category Λ, we typically write $\lambda \in \Lambda$ to mean that λ is a morphism of Λ. Following Definition 2.1 of [4], a P-graph consists of a countable small category Λ equipped with a functor $d : \Lambda \to P$ satisfying the factorisation property: if $d(\lambda) = p + q$ then there exist unique $\mu, \nu \in \Lambda$ with $d(\mu) = p$, $d(\nu) = q$ and $\lambda = \mu\nu$. We write

$$\Lambda^p := d^{-1}(p) \quad \text{for } p \in P.$$

The factorisation property ensures that Λ^0 is precisely the collection of identity morphisms of Λ; we write $r, s : \Lambda \to \Lambda^0$ for the maps induced by the codomain and domain maps — that is, $r(\lambda)$ is the identity morphism at the codomain of $\lambda \in \Lambda$, and $s(\lambda)$ is the identity morphism at the domain of $\lambda \in \Lambda$. For $X \subseteq \Lambda$ and $\mu \in \Lambda$, we define

$$X\mu := \{\lambda\mu : \lambda \in X \text{ and } s(\lambda) = r(\mu)\} \quad \text{and} \quad \mu X := \{\mu\lambda : \lambda \in X \text{ and } r(\lambda) = s(\mu)\}.$$

In particular, for $v \in \Lambda^0$, we have $Xv = X \cap s^{-1}(v)$ and $vX = X \cap r^{-1}(v)$. We say that Λ is *row-finite* if $v\Lambda^p$ is finite for every $v \in \Lambda^0$ and $p \in P$, and that it has *no sources* if $v\Lambda^p$ is nonempty for every $v \in \Lambda^0$ and $p \in P$.

There is a natural pre-order on P given by $p \leqslant q$ if $q = p + u$ for some $u \in P$. Note that $\leqslant$ need not be a partial order: if F is nontrivial, then $\leqslant$ is not antisymmetric.

EXAMPLES 5.1. (i) Let $\Omega_P = \{(p,q) \in P \times P : p \leqslant q\}$. Regarding P as a subsemigroup of $\mathbb{Z}^k \times F$, we can define $d : \Omega_P \to P$ by the expression $d(p,q) = q - p$. Identify $\Omega_P^0 := d^{-1}(0)$ with P via $(p,p) \mapsto p$, and define $r,s : \Omega_P \to \Omega_P^0$ by $r(p,q) = p$ and $s(p,q) = q$. Finally define composition by $(p,q)(q,n) = (p,n)$. Then Ω_P is a P-graph.

(ii) When P is regarded as a category with one object it becomes a P-graph with degree map the identity functor, composition the group operation, and range and source both given by the trivial map $p \mapsto 0$.

(iii) Every k-graph is an $\mathbb{N}^k$-graph.

(iv) In particular, when $P = \mathbb{N}^k$ in example (ii), we obtain the k-graph with one vertex and one path of each degree in $\mathbb{N}^k$. As is standard [24], [26], we denote this k-graph by T_k; its C^*-algebra is isomorphic to $C(\mathbb{T}^k)$.

(v) Another example we shall use frequently is the 1-graph B_n with one vertex and n distinct edges, whose C^*-algebra is the Cuntz algebra $\mathcal{O}_n$. The "B" here stands for "bouquet" and we sometimes refer to B_n as the "bouquet of n loops".

The definition of the categorical cohomology of a k-graph given in Section 3 of [26] applies to P-graphs and we use the formalism and notation from there. In detail, let A be an abelian group and Λ^{*r} the collection of composable r-tuples of elements of Λ; that is,

$$\Lambda^{*r} := \Big\{(\lambda_1, \dots, \lambda_r) \in \prod_{i=1}^{r} \Lambda : s(\lambda_i) = r(\lambda_{i+1}) \text{ for all } 1 \leqslant i < r\Big\}.$$

Then $Z^2(\Lambda, A)$, the group of normalised 2-cocycles on Λ, consists of all functions $f : \Lambda^{*2} \to A$ such that

$$f(\lambda,\mu) + f(\lambda\mu,\nu) = f(\mu,\nu) + f(\lambda,\mu\nu)$$

for all $(\lambda,\mu,\nu) \in \Lambda^{*3}$ and $f(r(\lambda),\lambda) = 0 = f(\lambda, s(\lambda))$ for all $\lambda \in \Lambda$ (cf. Lemma 3.8 of [26]). Furthermore $f_1, f_2 \in Z^2(\Lambda, A)$ are *cohomologous* if they differ by a coboundary: that is there is a map $b : \Lambda \to A$ such that $(f_1 - f_2)(\lambda,\mu) = (\delta^1 b)(\lambda,\mu) := b(\lambda) - b(\lambda\mu) + b(\mu)$ for all $(\lambda,\mu) \in \Lambda^{*2}$. As usual, when $A = \mathbb{T}$ the group operation is written multiplicatively.

The following example of a 2-cocycle on $\mathbb{Z}_2^l$ will prove important later.

EXAMPLE 5.2. Consider the finite group $\mathbb{Z}_2^l$. Let $Z^2(\mathbb{Z}_2^l, \mathbb{Z}_2)$ be the group of $\mathbb{Z}_2$-valued group 2-cocycles on $\mathbb{Z}_2^l$. (Of course, we can also regard $\mathbb{Z}_2^l$ as a $\mathbb{Z}_2^l$-graph Z_2^l with respect to the identity functor, and then this group of group 2-cocycles on $\mathbb{Z}_2^l$ is the same as the group of categorical 2-cocycles on Z_2^l.) Given a P-graph Λ and a homomorphism $\pi : A \to B$ of abelian groups, there is an induced homomorphism $\pi_* : Z^2(\Lambda, A) \to Z^2(\Lambda, B)$ given by $\pi_*(c)(\lambda,\mu) = \pi(c(\lambda,\mu))$, and this π_* carries cohomologous elements to cohomologous elements. In particular, the canonical embedding $\mathbb{Z}_2 \hookrightarrow \mathbb{T}$ given by $m \mapsto (-1)^m$ induces a map

$\kappa \mapsto c_\kappa$ from $Z^2(\mathbb{Z}_2^l, \mathbb{Z}_2)$ to $Z^2(\mathbb{Z}_2^l, \mathbb{T})$ by

$$c_\kappa(m,n) = (-1)^{\kappa(m,n)} \quad \text{for } (m,n) \in \mathbb{Z}_2^l \times \mathbb{Z}_2^l.$$

For example, consider $\kappa : \mathbb{Z}_2^l \times \mathbb{Z}_2^l \to \mathbb{Z}_2$ given by

$$\kappa(m,n) = \sum_{1 \leqslant j < i \leqslant l} m_i \cdot n_j, \quad \text{where } m_i, n_j \in \mathbb{Z}_2. \tag{5.1}$$

Then $\kappa \in Z^2(\mathbb{Z}_2^l, \mathbb{Z}_2)$. Indeed, κ is biadditive and on pairs (e_i, e_j) of generators of $\mathbb{Z}_2^l$, it satisfies $\kappa(e_i, e_j) = 1 \in \mathbb{Z}_2$ if $j < i$ and $\kappa(e_i, e_j) = 0 \in \mathbb{Z}_2$ if $i \leqslant j$.

Let σ be a permutation of $\{1, \ldots, l\}$. Define $(m^\sigma)_i = m_{\sigma(i)}$ for $m \in \mathbb{Z}_2^l$. Then $m \to m^\sigma$ is an automorphism of $\mathbb{Z}_2^l$, and then for $\kappa \in Z^2(\mathbb{Z}_2^l, \mathbb{Z}_2)$ we may form the 2-cocycle $\kappa^\sigma \in Z^2(\mathbb{Z}_2^l, \mathbb{Z}_2)$, by $\kappa^\sigma(m,n) = \kappa(m^\sigma, n^\sigma)$ for $(m,n) \in \mathbb{Z}_2^l \times \mathbb{Z}_2^l$. We then have

$$c_{\kappa^\sigma}(m,n) = \prod_{j<i} (-1)^{m_{\sigma(i)} n_{\sigma(j)}}.$$

If $b : \mathbb{Z}_2^l \to \mathbb{T}$ is a function, then $\delta^1 b$ is the associated 2-coboundary given by $\delta^1 b(m,n) = b(m) b(m+n)^{-1} b(n)$.

LEMMA 5.3. *With notation as above c_κ is cohomologous to c_{κ^σ} in $Z^2(\mathbb{Z}_2^l, \mathbb{T})$. Hence there is a map $b : \mathbb{Z}_2^l \to \mathbb{T}$ such that $c_{\kappa^\sigma} = c_\kappa \delta^1 b$.*

Proof. Let $\chi_{c_{\kappa^\sigma}} : \mathbb{Z}_2^l \times \mathbb{Z}_2^l \to \{1, -1\} \subseteq \mathbb{T}$ be the bicharacter of $\mathbb{Z}_2^l$ defined by

$$\chi_{c_{\kappa^\sigma}}(m,n) = c_{\kappa^\sigma}(m,n) c_{\kappa^\sigma}(n,m)^{-1} = c_{\kappa^\sigma}(m,n) c_{\kappa^\sigma}(n,m)$$

for all $m, n \in \mathbb{Z}_2^l$. For generators $e_i, e_j \in A$ we have

$$\chi_{c_{\kappa^\sigma}}(e_i, e_j) = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{otherwise.} \end{cases}$$

So $\chi_{c_{\kappa^\sigma}}$ does not depend on σ, and in particular $\chi_{c_{\kappa^\sigma}} = \chi_{c_{\kappa^{\mathrm{id}}}} = \chi_{c_\kappa}$. Thus Proposition 3.2 of [29] implies that c_κ and c_{κ^σ} are cohomologous (see also Lemmata 7.1 and 7.2 of [22]). The final statement follows by the definition of a coboundary (see Definition 3.2 of [26]). ∎

For $P := \mathbb{N}^k \times \mathbb{Z}_2^l$, there is a natural surjection $\rho : P \to \mathbb{Z}_2^{k+l}$ given by taking the residue of each coordinate modulo 2.

PROPOSITION 5.4. *Let Λ be a P-graph where $P = \mathbb{N}^k \times \mathbb{Z}_2^l$. With κ defined as in* (5.1), *the formula $c_\Lambda(\lambda, \mu) = c_\kappa((\rho(d(\lambda))), \rho(d(\mu)))$ for $(\lambda, \mu) \in \Lambda^{*2}$ defines a 2-cocycle $c_\Lambda \in Z^2(\Lambda, \mathbb{T})$ with values in $\{\pm 1\}$.*

Proof. Since κ is biadditive on $\mathbb{Z}_2^{k+l} \times \mathbb{Z}_2^{k+l}$ and since ρ is a homomorphism, the map $(m,n) \mapsto (-1)^{\kappa(\rho(m), \rho(n))}$ is a bicharacter of $\mathbb{N}^k \times \mathbb{Z}_2^l$. It follows immediately from the definition of κ that $c_\Lambda(r(\lambda), \lambda) = 1 = c_\Lambda(\lambda, s(\lambda))$. Since d is a functor, for a composable triple λ, μ, ν, we have

$$c_\Lambda(\lambda\mu, \nu) = c_\kappa((\rho(d(\lambda))), \rho(d(\nu))c_\kappa((\rho(d(\mu))), \rho(d(\nu))) \quad \text{and}$$
$$c_\Lambda(\lambda, \mu\nu) = c_\kappa((\rho(d(\lambda))), \rho(d(\mu))c_\kappa((\rho(d(\lambda))), \rho(d(\nu))).$$

Hence

$$\begin{aligned} c_\Lambda(\lambda, \mu)c_\Lambda(\lambda\mu, \nu) \\ &= c_\kappa((\rho(d(\lambda))), \rho(d(\mu))c_\kappa((\rho(d(\lambda))), \rho(d(\nu))c_\kappa((\rho(d(\mu))), \rho(d(\nu))) \\ &= c_\Lambda(\lambda, \mu\nu)c_\Lambda(\mu, \nu). \quad \blacksquare \end{aligned}$$

DEFINITION 5.5. Let Λ be a row-finite P-graph with no sources, and $c \in Z^2(\Lambda, \mathbb{T})$. A *Cuntz–Krieger (Λ, c)-family* in a C^*-algebra B is a function $t : \lambda \mapsto t_\lambda$ from Λ to B such that:

(CK1) $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(CK2) $t_\mu t_\nu = c(\mu, \nu)t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
(CK3) $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$; and
(CK4) $t_v = \sum_{\lambda \in v\Lambda^p} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $p \in P$.

The following lemma is more or less standard.

LEMMA 5.6. *Let Λ be a row-finite P-graph with no sources. Take $c \in Z^2(\Lambda, \mathbb{T})$. There exists a universal C^*-algebra $C^*(\Lambda, c)$ generated by a Cuntz–Krieger (Λ, c)-family $\{s_\lambda : \lambda \in \Lambda\}$.*

This follows from a standard argument (see, for example Theorem 2.10 of [28]) using that the relations force the t_λ to be partial isometries.

In what follows, a morphism of P-graphs is a functor $f : \Lambda \to \Gamma$ between P-graphs that intertwines the degree functors: that is, $d_\Gamma(f(\lambda)) = d_\Lambda(\lambda)$ for $\lambda \in \Lambda$. Let Λ be a row-finite P-graph with no sources. The path space Λ^Ω is defined to be the collection of all morphisms $x : \Omega \to \Lambda$ where $\Omega = \Omega_P$ is as in Examples 5.1 above. The collection of sets $Z(\lambda) := \{x \in \Lambda^\Omega : \lambda = x(0, d(\lambda))\}$, indexed by $\lambda \in \Lambda$, is a basis of compact open sets for a locally compact Hausdorff topology on Λ^Ω. For each $r \in P$ we define the shift map $\sigma^r : \Lambda^\Omega \to \Lambda^\Omega$ by $(\sigma^r x)(p, q) := x(p + r, q + r)$. We will need to use the path groupoid $\mathcal{G}_\Lambda$ of Λ introduced by Carlsen et al. in Section 2 of [4]. Recalling that G_P denotes the Grothendieck group of P so that for $p, q \in P$ the expression $p - q$ defines an element of G_P, we define

$$\mathcal{G}_\Lambda := \bigcup_{p,q \in P} \{(x, p - q, y) : \sigma^p(x) = \sigma^q(y)\} \subseteq \Lambda^\Omega \times G_P \times \Lambda^\Omega.$$

We identify the path space Λ^Ω with the unit space $\mathcal{G}_\Lambda^{(0)}$ via the map $x \mapsto (x, 0, x)$. For $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$, let

$$Z(\mu, \nu) := \{(x, d(\mu) - d(\nu), y) : x \in Z(\mu), y \in Z(\nu), \sigma^{d(\mu)}(x) = \sigma^{d(\nu)}(y)\}.$$

The topology on $\mathcal{G}_\Lambda$ is the one with basis $\mathcal{U}_\Lambda = \{Z(\mu,\nu) : \mu,\nu \in \Lambda, s(\mu) = s(\nu)\}$. (To see that this is a basis, one checks that if $(x,m,y) \in Z(\mu,\nu) \cap Z(\eta,\zeta)$, then $d(\mu) - d(\nu) = d(\eta) - d(\zeta) = m$, and $\alpha := x(0, d(\mu) + d(\eta))$ and $\beta = y(0, d(\nu) + d(\eta))$ satisfy $(x,m,y) \in Z(\alpha,\beta) \subseteq Z(\mu,\nu) \cap Z(\eta,\zeta)$.) This is a locally compact Hausdorff topology on $\mathcal{G}_\Lambda$ and $\mathcal{G}_\Lambda$ is étale in this topology because $s : Z(\mu,\nu) \to Z(\mu)$ is a homeomorphism for each μ. The elements of $\mathcal{U}_\Lambda$ are all compact open sets. Given $\mu,\nu \in \Lambda$ with $s(\mu) = s(\nu)$ and $p \in P$ we have $Z(\mu,\nu) = \bigsqcup_{\alpha \in s(\mu)\Lambda^p} Z(\mu\alpha, \nu\alpha)$.

PROPOSITION 5.7. *Let Λ be a row-finite P-graph with no sources and take $c \in Z^2(\Lambda, \mathbb{T})$. There is a continuous normalised $\mathbb{T}$-valued groupoid 2-cocycle ς_c on $\mathcal{G}_\Lambda$ and a surjective homomorphism $\pi : C^*(\Lambda, c) \to C^*(\mathcal{G}_\Lambda, \varsigma_c)$ such that $\pi(s_\lambda) = 1_{Z(\lambda, s(\lambda))} \neq 0$ for all $\lambda \in \Lambda$.*

Proof. We follow the argument of Section 6 in [26], which proves the analogous result in the case that Λ is a k-graph (see Theorem 6.7 of [26]). The only additional difficulty in our current setting is that the notion of minimal common extension for a pair of elements in Λ does not in general make sense in a P-graph. So we must check that we can still construct a partition of $\mathcal{G}_\Lambda$ as in Lemma 6.6 of [26] without using minimal common extensions.

We must show that there is a partition $\mathcal{Q}$ of $\mathcal{G}_\Lambda$ consisting of elements of $\mathcal{U}_\Lambda$ such that $Z(\lambda, s(\lambda)) \in \mathcal{Q}$ for all $\lambda \in \Lambda$.

For each $p \in P$, let $M_p := \{(x, p, \sigma^p(x)) : x \in \Lambda^\Omega\} \subseteq \mathcal{G}_\Lambda$. For $\mu, \nu \in \Lambda$ we have $Z(\mu,\nu) \subseteq M_p$ if and only if $\mu = \mu'\nu$ for some $\mu' \in \Lambda^p$, and otherwise $Z(\mu,\nu) \cap M_p = \emptyset$. Then $M_p = \bigsqcup_{d(\lambda)=p} Z(\lambda, s(\lambda))$, and $\mathcal{G}_\lambda \setminus M_p = \bigcup\{Z(\mu,\nu) : \nu \in \Lambda, \mu \in \Lambda s(\nu) \setminus \Lambda^p\nu\}$, so M_p is clopen in $\mathcal{G}_\Lambda$. Moreover, the M_p are pairwise disjoint, and $M := \bigsqcup_{p \in P} M_p$ satisfies

$$\mathcal{G}_\Lambda \setminus M = \bigcup\{Z(\mu,\nu) : \nu \in \Lambda \text{ and } \mu \in \Lambda s(\nu) \setminus \Lambda\nu\}. \tag{5.2}$$

So M is clopen. It remains to show that $\mathcal{G}_\Lambda \setminus M$ is a disjoint union of elements of $\mathcal{U}_\Lambda$. Since $\mathcal{U}_\Lambda$ is a countable basis, we can write the complement of M as a countable union of elements of $\mathcal{U}_\Lambda$.

Claim. Let $Z(\kappa,\lambda), Z(\mu,\nu) \in \mathcal{U}_\Lambda$. Then both $Z(\kappa,\lambda) \cap Z(\mu,\nu)$ and $Z(\kappa,\lambda) \setminus Z(\mu,\nu)$ can be expressed as a disjoint union of elements of $\mathcal{U}_\Lambda$.

To prove the claim, first note that $Z(\kappa,\lambda)$ and $Z(\mu,\nu)$ are disjoint unless $d(\kappa) - d(\lambda) = d(\mu) - d(\nu)$. So we may assume that $d(\kappa) - d(\lambda) = d(\mu) - d(\nu)$. There exist $p, q \in P$ (for example $p = d(\nu)$ and $q = d(\lambda)$) such that $d(\kappa) + p = d(\mu) + q$ and $d(\lambda) + p = d(\nu) + q$. Let

$$A := \{\alpha \in s(\lambda)\Lambda^p : (\kappa\alpha, \lambda\alpha) = (\mu\beta, \nu\beta) \text{ for some } \beta \in s(\nu)\Lambda^q\}$$

and let $B := s(\lambda)\Lambda^p \setminus A$. We have

$$Z(\kappa,\lambda) = \bigsqcup_{\alpha \in s(\kappa)\Lambda^p} Z(\kappa\alpha,\lambda\alpha) \quad \text{and} \quad Z(\mu,\nu) = \bigsqcup_{\beta \in s(\mu)\Lambda^q} Z(\mu\beta,\nu\beta). \tag{5.3}$$

For a given α, β, since $d(\kappa\alpha) = d(\mu\beta)$ the sets $Z(\kappa\alpha)$ and $Z(\mu\beta)$ are either equal or disjoint, and similarly $Z(\lambda\alpha)$ and $Z(\nu\beta)$ are either equal or disjoint. Indeed, they are equal if and only if $\alpha \in A$, and then β is the unique element of $s(\nu)\Lambda^q$ such that $(\kappa\alpha, \lambda\alpha) = (\mu\beta, \nu\beta)$. From this we obtain

$$Z(\kappa,\lambda) \cap Z(\mu,\nu) = \bigsqcup_{\alpha \in A} Z(\kappa\alpha,\lambda\alpha),$$

and then (5.3) yields

$$Z(\kappa,\lambda) \setminus Z(\mu,\nu) = \bigsqcup_{\alpha \in B} Z(\kappa\alpha,\lambda\alpha).$$

This proves the claim.

Now let $((\mu_i,\nu_i))_{i\in\mathbb{N}}$ be an enumeration of the set $\{(\mu,\nu) \in \Lambda \times \Lambda : \mu \in \Lambda s(\nu) \setminus \Lambda\nu\}$. The claim shows that for each i, the set $Z(\mu_i,\nu_i) \setminus \bigcup_{j<i} Z(\mu_j,\nu_j)$ can be expressed as a disjoint union of elements of $\mathcal{U}_\Lambda$; say $Z(\mu_i,\nu_i) \setminus \bigcup_{j<i} Z(\mu_j,\nu_j) = \bigsqcup_{k=1}^{n_i} Z(\alpha_{i,k},\beta_{i,k})$. So

$$\mathcal{Q} := \{Z(\lambda, s(\lambda)) : \lambda \in \Lambda\} \cup \{Z(\alpha_{i,k},\beta_{i,k}) : i \in \mathbb{N} \text{ and } k \leqslant N_i\}$$

is the desired partition.

The groupoid 2-cocycle ς_c is constructed as in Lemma 6.3 of [26] (there the cocycle is denoted σ_c and the partition is denoted $\mathcal{P}$). By construction of ς_c the map $\lambda \mapsto 1_{Z(\lambda,s(\lambda))}$ constitutes a Cuntz–Krieger (Λ,c)-family. Hence, there is a homomorphism $\pi : C^*(\Lambda,c) \to C^*(\mathcal{G}_\Lambda,\varsigma_c)$ such that $\pi(s_\lambda) = 1_{Z(\lambda,s(\lambda))}$. Moreover, for every $Z(\mu,\nu) \in \mathcal{U}_\Lambda$, we have

$$1_{Z(\mu,\nu)} = 1_{Z(\mu,s(\mu))} 1^*_{Z(\nu,s(\nu))} = \pi(s_\mu s_\nu^*);$$

and since the span of elements of the form $1_{Z(\mu,\nu)}$ is dense, π is surjective. ∎

If $c \in Z^2(\Lambda,\mathbb{T})$ is the trivial cocycle, then our definition of the twisted C^*-algebra $C^*(\Lambda,c)$ reduces to the definition of the C^*-algebra $C^*(\Lambda)$ (see Definition 2.4 of [4]). If $P = \mathbb{N}^k$ and c is an arbitrary cocycle, then our definition of $C^*(\Lambda,c)$ agrees with the existing definition of the twisted k-graph algebra (see Definition 5.2 of [26]). We also need to know that cohomologous cocycles yield isomorphic C^*-algebras. The proof follows that of Proposition 5.6 in [26] almost verbatim, so we just sketch it here.

LEMMA 5.8. *Let Λ be a row-finite P-graph with no sources. Suppose that $c_1, c_2 \in Z^2(\Lambda,\mathbb{T})$ are cohomologous. Then there is a map $b : \Lambda \to \mathbb{T}$ satisfying $b(v) = 1$ for all $v \in \Lambda^0$ such that there is an isomorphism $C^*(\Lambda,c_1) \cong C^*(\Lambda,c_2)$ satisfying $s_\lambda \mapsto b(\lambda)s_\lambda$ for all $\lambda \in \Lambda$.*

Proof sketch. Since c_1 and c_2 are cohomologous, there exists $b : \Lambda \to \mathbb{T}$ such that $c_1 = \delta^1 b c_2$. For $v \in \Lambda^0$ we then have $1 = c_1(v,v) = \delta^1 b(v,v) c_2(v,v) = b(v)b(v)\overline{b}(v^2)c_2(v,v) = b(v)$. For $\lambda \in \Lambda$, define $t_\lambda := b(\lambda)s_\lambda \in C^*(\Lambda, c_2)$. Then for $(\lambda,\mu) \in \Lambda^{*2}$ we have $t_\lambda t_\mu = b(\lambda)b(\mu)s_\lambda s_\mu = b(\lambda)b(\mu)c_2(\lambda,\mu)s_{\lambda\mu} = \delta^1(b)c_2(\lambda,\mu)t_{\lambda\mu} = c_1(\lambda,\mu)t_{\lambda,\mu}$. Using that $b(v) = 1$ for $v \in \Lambda^0$ to verify (CK1), one checks that $\{t_\lambda : \lambda \in \Lambda\}$ is a Cuntz–Krieger (Λ, c_1)-family in $C^*(\Lambda, c_2)$, so the universal property gives a homomorphism $C^*(\Lambda, c_1) \to C^*(\Lambda, c_2)$ satisfying $s_\lambda \mapsto b(\lambda)s_\lambda$ for all $\lambda \in \Lambda$. The same argument using that $c_2 = \delta^1(\overline{b})c_1$ yields an inverse for this homomorphism, showing that it is an isomorphism. ∎

EXAMPLES 5.9. (i) Let Ω_P be the P-graph described in Examples 5.1(i). Then $C^*(\Omega_P) \cong \mathcal{K}(\ell^2(P))$.

(ii) Let P be the P-graph from Examples 5.1(ii). Then $C^*(P)$ is isomorphic to the group C^*-algebra $C^*(G_P)$ of the Grothendieck group of P.

Let (Λ, d) be a P-graph and $c \in Z^2(\Lambda, \mathbb{T})$. Following Lemma 2.5 of [4] we describe the gauge action of $\widehat{G}_P$ on $C^*(\Lambda, c)$. For $\chi \in \widehat{G}_P$ and $\lambda \in \Lambda$ set

$$\gamma^\Lambda_\chi(s_\lambda) = \chi(d(\lambda))s_\lambda. \tag{5.4}$$

The following standard argument — which is also outlined in the discussion following Lemma 2.5 of [4], and goes back at least to Remark 2.2 of [17] — shows that the above formula defines a strongly continuous action of $\widehat{G}_P$ on $C^*(\Lambda, c)$ (we thank the anonymous referee for suggesting that we include the details). The elements $t_\lambda := \chi(d(\lambda))s_\lambda$ determine a Cuntz–Krieger (Λ, c)-family in $C^*(\Lambda, c)$, and so the universal property gives an endomorphism γ^Λ_χ of $C^*(\Lambda, c)$ satisfying (5.4). For any $\chi, \chi' \in \widehat{G}_P$, the endomorphisms $\gamma^\Lambda_\chi \circ \gamma^\Lambda_{\chi'}$ and $\gamma^\Lambda_{\chi\chi'}$ agree on generators, so are equal. Since γ^Λ_1 agrees with the identity map on generators, it is the identity map. We deduce that each γ^Λ_χ is an automorphism and that $\chi \mapsto \gamma^\Lambda_\chi$ is a group homomorphism from $\widehat{G}_P$ to $\operatorname{Aut}(C^*(\Lambda, c))$. Certainly $\chi \mapsto \gamma^\Lambda_\chi(s_\mu s_\nu^*)$ is continuous for all $\mu, \nu \in \Lambda$. It follows that $\chi \mapsto \gamma^\Lambda_\chi(a)$ is continuous for $a \in \operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda\}$ and then an $\varepsilon/3$-argument shows that γ^Λ is strongly continuous.

PROPOSITION 5.10 (Gauge invariant uniqueness theorem). *Let Λ be a row-finite P-graph with no sources, and fix $c \in Z^2(\Lambda, \mathbb{T})$. Let $t : \Lambda \to B$ be a Cuntz–Krieger (Λ, c)-family in a C^*-algebra B. Suppose that there is a strongly continuous action β of $\widehat{G}_P$ on B satisfying $\beta_\chi(t_\lambda) = \chi(d(\lambda))t_\lambda$ for all $\lambda \in \Lambda$ and $\chi \in \widehat{G}_P$. Then the induced homomorphism $\pi_t : C^*(\Lambda, c) \to B$ is injective if and only if $t_v \neq 0$ for all $v \in \Lambda^0$.*

The result follows from the same argument as in Proposition 2.7 of [4] and the observation that the fixed point algebra for the gauge action on $C^*(\Lambda, c)$ is identical to the fixed point algebra for the gauge action on $C^*(\Lambda)$ (cf. Theorem 4.2 of [25]).

COROLLARY 5.11. *With notation as in Proposition* 5.7 *the map* $\pi : C^*(\Lambda, c) \to C^*(\mathcal{G}_\Lambda, \varsigma_c)$ *is an isomorphism.*

Proof. We argue as in Corollary 7.8 of [26]. The cocycle $\widetilde{d} : \mathcal{G}_\Lambda \to G_P$ given by $\widetilde{d}(x, m, y) = m$ induces an action β of $\widehat{G}_P$ on $C^*(\mathcal{G}_\Lambda, \varsigma_c)$ satisfying $\beta_\chi(f)(x, p - q, y) = \chi(p - q) f(x, p - q, y)$ for $f \in C_c(\mathcal{G}_\Lambda)$. By construction, π intertwines this action with the gauge action. Proposition 5.7 shows that each $\pi(s_v)$ is nonzero. So the result follows from Proposition 5.10. ∎

LEMMA 5.12 (cf. Proposition 3.2 of [9]). *Let F be a countable abelian group and let Λ be a row-finite k-graph with no sources. Suppose that F acts on Λ by k-graph automorphisms $g \mapsto \rho_g$. Let $P = \mathbb{N}^k \times F$. There is a unique P-graph $\Lambda \times_\rho F$ such that:*

(i) *as a set, $\Lambda \times_\rho F = \Lambda \times F$ with degree map $d(\lambda, g) = (d(\lambda), g)$;*

(ii) *$r(\lambda, g) = (r(\lambda), 0)$, and $s(\lambda, g) = (s(\rho_{-g}(\lambda)), 0)$;*

(iii) *$(\mu, g)(\nu, h) = (\mu\rho_g(\nu), g + h)$ whenever $s(\mu) = r(\rho_g(\nu))$.*

Proof. For associativity we compute

$$\begin{aligned}((\lambda, g)(\mu, h))(\nu, k) &= (\lambda\rho_g(\mu), g + h)(\nu, k) = (\lambda\rho_g(\mu)\rho_{g+h}(\nu), g + h + k) \\ &= (\lambda, g)(\mu\rho_h(\nu), h + k) = (\lambda, g)((\mu, h)(\nu, k)).\end{aligned}$$

For the factorisation property, we use the standard notation for k-graphs that if $\lambda \in \Lambda$ and $m \leqslant n \leqslant d(\lambda)$, then $\lambda(m, n)$ is defined to be the unique element of Λ^{n-m} such that $\lambda = \lambda'\lambda(m, n)\lambda''$ for some $\lambda \in \Lambda^m$ and $\lambda'' \in \Lambda^{d(\lambda)-n}$. Suppose that $d(\lambda, g) = (m + n, h + k)$. Then $\mu = \lambda(0, m)$ and $\nu = \rho_{-h}(\lambda(m, m + n))$ satisfy $(\lambda, g) = (\mu, h)(\nu, k)$ with $d(\mu, h) = (m, h)$ and $d(\nu, k) = (n, k)$. To see that this factorisation is unique, suppose that

$$(\lambda, g) = (\mu, h')(\nu, k') = (\mu\rho_{h'}(\nu), h' + k')$$

with $d(\mu, h') = (m, h)$ and $d(\nu, k') = (n, k)$. Then $h' = h$ and $k' = k$ by definition of d. Furthermore, $\lambda = \mu\rho_h(\nu)$ where $d(\mu) = m$ and $d(\rho_h(\nu)) = d(\nu) = n$. So the factorisation property in Λ forces $\mu = \lambda(0, m)$ and $\rho_h(\nu) = \lambda(m, m + n)$, forcing $\nu = \rho_{-h}(\lambda(m, m + n))$. ∎

EXAMPLE 5.13. Let F be a countable abelian group. We can regard F as a 0-graph and then there is an action τ of F on this 0-graph by translation. So Lemma 5.12 yields an F-graph $F \times_\tau F$. It is straightforward to check that $F \times_\tau F \cong \Omega_F$ via the map $(g, h) \mapsto (g, g + h)$.

PROPOSITION 5.14. *Continue the notation of Lemma* 5.12. *Then there is an action $\widetilde{\rho} : F \to \operatorname{Aut}(C^*(\Lambda))$ such that $\widetilde{\rho}_g(s_\lambda) = s_{\rho_g(\lambda)}$.*

This follows from the universal property of $C^*(\Lambda)$ (cf. Proposition 3.1 of [9]).

THEOREM 5.15. *Continue the notation of Lemma* 5.12 *and Proposition* 5.14. *Then*

(i) *there is a unitary representation of* $u : F \to \mathcal{UM}(C^*(\Lambda \times_\rho F))$ *given by* $u(g) = \sum_{v \in \Lambda^0} s_{(v,g)}$*;*

(ii) *there is a homomorphism* $\phi : C^*(\Lambda) \to C^*(\Lambda \times_\rho F)$ *given by* $s_\lambda \mapsto s_{(\lambda,0)}$*;*

(iii) *we have* $u(g)\phi(a)u(g)^* = \widetilde{\rho}_g(a)$ *for all* $a \in C^*(\Lambda)$ *and* $g \in F$*;*

(iv) *there is an isomorphism* $\phi \times u : C^*(\Lambda) \rtimes_{\widetilde{\rho}} F \to C^*(\Lambda \times_\rho F)$ *such that* $(\phi \times u)(s_\lambda, g) = s_{(\lambda,g)}$.

This follows from the proof of Theorem 3.4 in [9] mutatis mutandis.

EXAMPLES 5.16. (i) Let B_n be the 1-graph with a single vertex v and edges $f_1, \ldots, f_n$ (see Example 5.1(v)). Let $\mathbb{Z}_n$ act on B_n by cyclicly permuting the edges. Then $C^*(B_n) \times \mathbb{Z}_n \cong C^*(B_n \times \mathbb{Z}_n)$.

(ii) Let Λ be a k-graph, let F be a countable abelian group, and $b : \Lambda \to F$ be a functor. Then the skew product graph $\Lambda \times_b F$ carries a natural F-action τ (see Remark 5.6 of [23]) given by $\tau_g(\lambda, h) = (\lambda, g + h)$. We have

$$C^*((\Lambda \times_b F) \times_\tau F) \cong C^*(\Lambda \times_b F) \rtimes_{\widetilde{\tau}} F \cong C^*(\Lambda) \otimes \mathcal{K}(\ell^2(F)).$$

THEOREM 5.17 (cf. Proposition 3.5 of [9]). *Let* $P = \mathbb{N}^k \times F$ *where* F *is a countable abelian group. Suppose that* Λ *is a* P*-graph, and let* Γ *denote the sub-*k*-graph* $d^{-1}(\mathbb{N}^k \times \{0\})$ *of* Λ*. For each* $g \in F$ *and* $v \in \Lambda^0$ *the sets* $v\Lambda^{(0,g)}$ *and* $\Lambda^{(0,g)}v$ *are singletons. Moreover, there is an action* ρ *of* F *on* Γ *such that for all* $\lambda \in \Gamma$ *and* $g \in F$*, the unique elements* $\mu \in r(\lambda)\Lambda^{(0,g)}$ *and* $\nu \in s(\lambda)\Lambda^{(0,g)}$ *satisfy* $\mu\rho_g(\lambda) = \lambda\nu$*. Furthermore,* Λ *is isomorphic to the* P*-graph* $\Gamma \times_\rho F$ *of Lemma* 5.12.

Proof. The factorisation property ensures that each vertex $v \in \Lambda^0$ has unique factorisations $v = \mu\nu$ with $\mu \in \Lambda^{(0,g)}$ and $\nu \in \Lambda^{(0,-g)}$. Hence $v\Lambda^{(0,g)} = \{\mu\}$, and similarly $\Lambda^{(0,g)}v = \{\nu\}$.

The map ρ preserves degree by definition. It remains to show that ρ_g is a functor for each g and $\rho_g \circ \rho_h = \rho_{g+h}$. Fix $\lambda_1, \lambda_2 \in \Gamma$ such that $\lambda_2\lambda_1 \in \Gamma$ and $g \in F$. Let $v_0 = s(\lambda_1)$, $v_1 = r(\lambda_1) = s(\lambda_2)$ and $v_2 = r(\lambda_2)$. Let μ_i be the unique element of $v_i\Lambda^{(0,g)}$ for $i = 0, 1, 2$. Then

$$\mu_i\rho_g(\lambda_i) = \lambda_i\mu_{i-1} \quad \text{for } i = 1, 2.$$

Combining the two equations we get

$$\mu_2\rho_g(\lambda_2)\rho_g(\lambda_1) = \lambda_2\mu_1\rho_g(\lambda_1) = \lambda_2\lambda_1\mu_0,$$

and hence $\rho_g(\lambda_2\lambda_1) = \rho_g(\lambda_2)\rho_g(\lambda_1)$ by uniqueness of factorisations. A similar argument shows that $\rho_g \circ \rho_h = \rho_{g+h}$ for all $g, h \in F$.

For $\lambda \in \Lambda^{(n,g)}$ define $\psi(\lambda) = (\lambda((0,0),(n,0)), g) \in \Gamma \times_\rho F$. Then ψ is an isomorphism of P-graphs. Its inverse is given as follows: let $(\lambda, g) \in \Gamma \times_\rho F$ and μ be the unique element in $s(\lambda)\Lambda^{(0,g)}$; then $\psi^{-1}(\lambda, g) = \lambda\mu$. ∎

COROLLARY 5.18. *Suppose that* F *is a countable abelian group. Then the only possible* F*-graphs are disjoint unions of quotients of* Ω_F *by subgroups of* F*.*

Proof. Let Λ be a F-graph. Then Theorem 5.17 applied in the case $k = 0$ shows that $\Lambda \cong \Gamma \times_\rho F$ where Γ is a 0-graph, which is just a countable set. The group F then acts on Γ, and each orbit is of the form F/H for some subgroup H of F. Since the orbits of ρ correspond to the connected components of $\Gamma \times_\rho F$ each component of Λ is isomorphic to $F/H \times_{\tau_H} F$ where τ_H is the action of F on F/H induced by translation. The result then follows from the identification of Ω_F with $F \times_\tau F$ given in Example 5.13 ∎

6. GRADINGS OF P-GRAPH C^*-ALGEBRAS INDUCED BY FUNCTORS

Let P be a finitely-generated, cancellative abelian monoid of the form $P \cong \mathbb{N}^k \times \mathbb{Z}_2^l$. We will be particularly interested in gradings of twisted P-graph C^*-algebras that arise from functors from the underlying P-graphs into $\mathbb{Z}_2$.

LEMMA 6.1. *Let Λ be a P-graph, let $\delta : \Lambda \to \mathbb{Z}_2$ be a functor and let c be a $\mathbb{T}$-valued 2-cocycle on Λ. Then there is a grading automorphism α_δ of $C^*(\Lambda, c)$ such that*

$$\alpha_\delta(s_\lambda) = (-1)^{\delta(\lambda)} s_\lambda \quad \textit{for all } \lambda \in \Lambda. \tag{6.1}$$

For $i \in \mathbb{Z}_2$, we have $C^*(\Lambda, c)_i = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \delta(\mu) - \delta(\nu) = i\}$.

Proof. The universal property of $C^*(\Lambda, c)$ yields an automorphism α_δ satisfying (6.1). For $\mu, \nu \in \Lambda$ and $j \in \mathbb{Z}_2$, we have

$$\frac{s_\mu s_\nu^* + (-1)^j \alpha_\delta(s_\mu s_\nu^*)}{2} = \begin{cases} s_\mu s_\nu^* & \text{if } \delta(\mu) - \delta(\nu) = j, \\ 0 & \text{if } \delta(\mu) - \delta(\nu) = j + 1. \end{cases}$$

By (2.1), we have $C^*(\Lambda, c)_j = \big\{(a + (-1)^j \alpha_\delta(a))/2 : a \in C^*(\Lambda, c)\big\}$. We deduce immediately that $C^*(\Lambda, c)_j \subseteq \overline{\operatorname{span}}\{s_\mu s_\nu^* : \delta(\mu) - \delta(\nu) = i\}$, and the reverse implication follows as well because $a \mapsto (a + (-1)^j \alpha_\delta(a))/2$ is linear and continuous, and $C^*(\Lambda, c) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda\}$. ∎

NOTATION 6.2. Consider nonnegative integers k, l, and let $P := \mathbb{N}^k \times \mathbb{Z}_2^l$, regarded as a finitely generated abelian monoid. We denote by $E_P := \{e_i : 1 \leqslant i \leqslant k + l\}$ the canonical set of generators of P. We call elements of Λ with degree in E_P *edges*. Let $\pi : P \to \mathbb{Z}_2$ be the unique homomorphism such that $\pi(e_i) = 1$ for all $1 \leqslant i \leqslant k + l$. Given a P-graph Λ, there is a functor $\delta_\Lambda : \Lambda \to \mathbb{Z}_2$ such that

$$\delta_\Lambda(\lambda) = \pi(d(\lambda)) \quad \text{for all } \lambda \in \Lambda. \tag{6.2}$$

The following examples illustrate the connection between gradings of P-graph C^*-algebras and twisted P-graph C^*-algebras. Specifically, the graded tensor product of graded P-graph C^*-algebras can frequently be realised as a twisted C^*-algebra of the cartesian-product graph. We return to this in Theorem 7.1.

EXAMPLES 6.3. (i) For $k \geqslant 1$ recall from Examples 5.1(iv) that T_k denotes the k-graph $\mathbb{N}^k$ with degree functor given by the identity functor, and C^*-algebra isomorphic to $C(\mathbb{T}^k)$. Endow $C^*(T_1)$ with the grading automorphism $\alpha_{\delta_{T_1}}$ induced by δ_{T_1} (see Notation 6.2); so the unitary generator is homogeneous of odd degree. (Under the canonical isomorphism $C^*(T_1) \cong C(\mathbb{T})$ such that $s_1 \mapsto z$, the grading automorphism $\alpha_{\delta_{T_1}}$ corresponds to the automorphism induced by the homeomorphism $z \mapsto -z$ of $\mathbb{T}$.) Then the graded tensor product $C^*(T_1) \mathbin{\widehat{\otimes}} C^*(T_1)$ is not abelian. To see this, let s_1 denote the unitary generator of $C^*(T_1)$. Then

$$(s_1 \mathbin{\widehat{\otimes}} 1)(1 \mathbin{\widehat{\otimes}} s_1) = (s_1 \mathbin{\widehat{\otimes}} s_1) \neq -(s_1 \mathbin{\widehat{\otimes}} s_1) = (1 \mathbin{\widehat{\otimes}} s_1)(s_1 \mathbin{\widehat{\otimes}} 1).$$

In particular, the graded tensor product $C^*(T_1) \mathbin{\widehat{\otimes}} C^*(T_1)$ is not isomorphic to $C^*(T_2)$.

Instead, we claim that $C^*(T_1) \mathbin{\widehat{\otimes}} C^*(T_1) \cong C^*(T_2, c)$ with grading automorphism $\alpha_{\delta_{T_2}}$ and twisting 2-cocycle $c : (T_2)^{*2} \to \mathbb{T}$ with values in $\{\pm 1\}$ given by

$$c(m, n) = (-1)^{m_2 n_1} \quad \text{for } (m, n) \in \mathbb{N}^2 \times \mathbb{N}^2 = (T_2)^{*2}.$$

To see this, for $n \in \mathbb{N}$ let $s_n \in C^*(T_1)$ denote the corresponding generator. Define elements $\{t_{(m,n)} : (m, n) \in \mathbb{N}^2\} \subseteq C^*(T_1) \mathbin{\widehat{\otimes}} C^*(T_1)$ by $t_{(m,n)} := s_m \mathbin{\widehat{\otimes}} s_n$. Routine calculations using the definition of multiplication and involution in the graded tensor product show that the $t_{(m,n)}$ are a Cuntz–Krieger (T_2, c)-family; for example, we can check (CK2) as follows:

$$t_{(m,n)} t_{(p,q)} = (s_m \mathbin{\widehat{\otimes}} s_n)(s_p \mathbin{\widehat{\otimes}} s_q) = (-1)^{np}(s_m s_p \mathbin{\widehat{\otimes}} s_n s_q) = c((m, n), (p, q)) t_{m+p, n+q}.$$

The universal property of $C^*(T_2, c)$ gives a homomorphism $\psi : C^*(T_2, c) \to C^*(T_1) \mathbin{\widehat{\otimes}} C^*(T_1)$. An application of the gauge-invariant uniqueness theorem (Theorem 5.10) shows that ψ is an isomorphism. Finally, one checks on generators that ψ intertwines the grading automorphisms.

(ii) Recall that Z_2 denotes $\mathbb{Z}_2$ considered as a $\mathbb{Z}_2$-graph as in Examples 5.1(ii). Let $\delta := \delta_{Z_2}$ be the identity map $Z_2 \to \mathbb{Z}_2$; so the associated grading automorphism α_δ of $C^*(Z_2)$ makes the generator u homogeneous of degree one. Then there is an isomorphism $(C^*(Z_2), \alpha_\delta) \cong \mathbb{C}\mathrm{liff}_1$ that takes s_1 to $(1, -1)$ and s_0 to $(1, 1)$.

(iii) Consider the graded tensor product $(C^*(Z_2), \alpha_\delta) \mathbin{\widehat{\otimes}} (C^*(Z_2), \alpha_\delta)$. As above, this is, in general, a nonabelian C^*-algebra. Indeed, let $c \in Z^2(Z_2 \times Z_2, \mathbb{T})$ be the cocycle given by $c(m, n) = (-1)^{m_2 n_1}$. Write $\delta := \delta_{Z_2}$ and recall that $\delta_{Z_2 \times Z_2} : Z_2 \times Z_2 \to \mathbb{Z}_2$ satisfies $\delta(i, j) = i + j$. Then

$$(C^*(Z_2), \alpha_\delta) \mathbin{\widehat{\otimes}} (C^*(Z_2), \alpha_\delta) \cong (C^*(Z_2 \times Z_2, c), \alpha_{\delta_{Z_2 \times Z_2}}).$$

Since $(C^*(Z_2), \alpha_\delta) \cong \mathbb{C}\mathrm{liff}_1$ as graded algebras, it follows that there is a graded isomorphism $C^*(Z_2 \times Z_2, c) \cong \mathbb{C}\mathrm{liff}_2$. Indeed, we will see in Corollary 7.5 that for any $l \geqslant 1$, if c is the 2-cocycle on the $\mathbb{Z}_2^l$-graph Z_2^l described at (5.1), then $\mathbb{C}\mathrm{liff}_n \cong C^*(Z_2^n, c)$ as graded C^*-algebras, where $C^*(Z_2^n, c)$ carries the grading induced by $\delta_{Z_2^n}$ as above.

(iv) Expanding on (i), let A be a C^*-algebra with grading automorphism α. Then, as at (4.10), there is a grading β^1 of $A \rtimes_\alpha \mathbb{Z}$ given by $\beta^1(i_A(a)i_{\mathbb{Z}}(n)) = (-1)^n i_A(\alpha(a))i_{\mathbb{Z}}(n)$; that is, the copy of A retains its given grading, and the generating unitary in the copy of $C^*(\mathbb{Z})$ is odd. Let T_1 denote $\mathbb{N}$ regarded as a 1-graph. The universal property of $A \rtimes_\alpha \mathbb{Z}$ and straightforward computations show that the map $i_A(a)i_{\mathbb{Z}}(n) \mapsto a \mathbin{\widehat{\otimes}} s_n$ defines a graded isomorphism $A \rtimes_\alpha \mathbb{Z} \cong A \mathbin{\widehat{\otimes}} C^*(T_1)$. We return to this in the context of graph C^*-algebras in Examples 8.9(i).

In Section 7, motivated by Examples 6.3(i) and (iii), we will investigate graded tensor products of graded P-graph C^*-algebras, and show that these often coincide with twisted C^*-algebras of cartesian-product graphs. To do this, we first need an alternative description of the graded C^*-algebras of appropriate P-graphs as universal graded C^*-algebras.

Let Λ be a P-graph with $P = \mathbb{N}^k \times \mathbb{Z}_2^l$, and let E_P be the standard generators $\{e_i : 1 \leqslant i \leqslant k + l\}$ of P as in Notation 6.2.

THEOREM 6.4. *Let $P = \mathbb{N}^k \times \mathbb{Z}_2^l$ and let Λ be a P-graph. Let c_Λ be the 2-cocycle of Proposition 5.4. Then for each $e \in E_P$ such that $2e = 0$ and each $\lambda \in \Lambda^e$, there is a unique $\lambda^* \in s(\lambda)\Lambda^e$. This λ^* satisfies $s(\lambda^*) = r(\lambda)$, $\lambda\lambda^* = r(\lambda)$, and $\lambda^*\lambda = s(\lambda)$. Moreover there exists a C^*-algebra D such that:*

(i) *D is generated by partial isometries $\{t_\lambda : d(\lambda) \in E_P\}$ and mutually orthogonal projections $\{p_v : v \in \Lambda^0\}$ such that:*

(a) *$t_\lambda t_\mu = -t_{\mu'} t_{\lambda'}$ for all $\lambda, \lambda' \in \Lambda^e$, $\mu, \mu' \in \Lambda^{e'}$ with $\lambda\mu = \mu'\lambda'$, $e, e' \in E_P$ and $e \neq e'$;*

(b) *if $d(\lambda) = e \in E_P$ and $2e = 0$ then $t_\lambda^* = t_{\lambda^*}$;*

(c) *$t_\lambda^* t_\lambda = p_{s(\lambda)}$ for all $\lambda \in \Lambda^e$, $e \in E_P$;*

(d) *for all $v \in \Lambda^0$ and $e \in E_P$ we have*

$$p_v = \sum_{\lambda \in v\Lambda^e} t_\lambda t_\lambda^*.$$

(ii) *D is universal in the sense that for any other C^*-algebra D' generated by elements t'_λ satisfying (a)–(d), there is a homomorphism $D \to D'$ satisfying $t_\lambda \mapsto t'_\lambda$.*

The C^*-algebra D carries a grading automorphism α satisfying $\alpha(t_\lambda) = -t_\lambda$ whenever $d(\lambda) \in E_P$, and $\alpha(p_v) = p_v$ for all $v \in \Lambda^0$. Moreover, if α_{δ_Λ} is the grading of $C^*(\Lambda, c_\Lambda)$ obtained from Lemma 6.1 applied to the functor (6.2), then there is a graded isomorphism $\pi : C^*(\Lambda, c_\Lambda) \to D$ such that

$$\pi(s_v) = p_v \quad \text{for all } v \in \Lambda^0, \quad \text{and} \quad \pi(s_\lambda) = t_\lambda \quad \text{whenever } d(\lambda) \in E_P.$$

Proof. For the first statement, suppose that $d(\lambda) = e$ with $2e = 0$. We have $d(s(\lambda)) = 0 = e + e$ and so the factorisation property shows that there exist $\lambda^*, \nu \in \Lambda^e$ such that $s(\lambda) = \lambda^*\nu$. Since $r(\lambda^*) = s(\lambda)$, the pair (λ, λ^*) is composable and since $\lambda\lambda^* \in \Lambda^{2e} = \Lambda^0$ we have $s(\lambda^*) = \lambda\lambda^* = r(\lambda)$. Hence, the pair

(λ^*, λ) is composable and $\lambda^*\lambda = r(\lambda)$. The factorisation property ensures that this λ^* is unique.

Let $\{s_\lambda : \lambda \in \Lambda\}$ be the generating partial isometries in $C^*(\Lambda, c_\Lambda)$. Let D be the universal C^*-algebra generated by a family of t_λ's and p_v's satisfying (a)–(d). The universal property guarantees that D carries a grading automorphism α as described.

Define $T_\lambda = s_\lambda$ and $P_v = s_v$. Then the family $\{P, T\}$ satisfies conditions (c) and (d) above. Suppose that $e \in E_P$ satisfies $2e = 0$ and that $\lambda \in \Lambda^e$. Using (CK2)–(CK4) and the first paragraph, we have

$$s_{\lambda^*} = (s_\lambda^* s_\lambda) s_{\lambda^*} = s_\lambda^* c_\Lambda(\lambda, \lambda^*) s_{\lambda\lambda^*} = s_\lambda^* s_{r(\lambda)} = s_\lambda^*(s_\lambda s_\lambda^*) = s_\lambda^*$$

and hence $T_{\lambda^*} = T_\lambda^*$.

It remains to check property (a): if $i \neq j$ then $c_\Lambda(e_i, e_j) = -1$ if $j < i$ and $c_\Lambda(e_i, e_j) = 1$ otherwise. Let $\lambda, \lambda' \in \Lambda^{e_i}$, $\mu, \mu' \in \Lambda^{e_j}$ with $\lambda\mu = \mu'\lambda'$. Suppose that $j < i$. Then

$$\begin{aligned} T_\lambda T_\mu &= s_\lambda s_\mu = c_\Lambda(\lambda, \mu) s_{\lambda\mu} = (-1)^1 s_{\lambda\mu} = -s_{\lambda\mu}, \quad \text{and} \\ T_{\mu'} T_{\lambda'} &= s_{\mu'} s_{\lambda'} = c_\Lambda(\mu', \lambda') s_{\mu'\lambda'} = (-1)^0 s_{\lambda\mu} = s_{\lambda\mu}. \end{aligned}$$

Hence $T_\lambda T_\mu = -T_{\mu'} T_{\lambda'}$. If $i < j$, the same argument applies (switching the λ's and μ's). Hence by the universal property of D there is a map $\phi : D \to C^*(\Lambda, c_\Lambda)$ such that $\phi(p_v) = s_v$ for all $v \in \Lambda^0$ and $\phi(t_\lambda) = s_\lambda$ for all edges λ.

To show that ϕ has an inverse, for each $v \in \Lambda^0$ set $s_v = p_v$. For $\lambda \in \Lambda$ with $d(\lambda) = \sum_{i=1}^{k+l} m_i e_i$, use the factorisation property to write

$$\lambda = \lambda_1^1 \cdots \lambda_1^{m_1} \lambda_2^1 \cdots \lambda_2^{m_2} \cdots \lambda_{k+l}^1 \cdots \lambda_{k+l}^{m_{k+l}} \tag{6.3}$$

where $d(\lambda_i^j) = e_i$ for $j = 1, \ldots, m_i$ and $i = 1, \ldots, k + l$, and define

$$S_\lambda := t_{\lambda_1^1} \cdots t_{\lambda_1^{m_1}} t_{\lambda_2^1} \cdots t_{\lambda_2^{m_2}} \cdots t_{\lambda_{k+l}^1} \cdots t_{\lambda_{k+l}^{m_{k+l}}}.$$

Direct calculation shows that $\{S_\lambda : \lambda \in \Lambda\}$ is a Cuntz–Krieger (Λ, c_Λ)-family in D. For example, to verify (CK2), observe that an induction will prove the general case if we can establish (CK2) whenever $d(\mu) \in E_P$. So fix $\nu = \nu_1 \cdots \nu_h$ with each $d(\nu_i) \in E_P$, and fix $\mu \in \Lambda r(\nu)$ with $d(\mu) = e_j \in E_P$. Factorise $\nu = \nu_1 \cdots \nu_h$ such that for $a < b$ if $d(\nu_a) = e_p$ and $d(\nu_b) = e_q$, then $p \leqslant q$ (so ν is factorised as in (6.3)). Define $c \in \{0, \ldots, h\}$ to be the unique value such that for $a \leqslant c$ we have $d(\nu_a) = e_i$ for some $i < j$ and for $a > c$ we have $d(\nu_a) = e_i$ for some $i \geqslant j$. Let $\mu_0 := \mu$ and for $l \leqslant c$, define $\mu_l \in \Lambda^{e_j}$ and $\nu_l' \in \Lambda^{d(\nu_l)}$ to be the unique elements such that $\mu_{a-1}\nu_a = \nu_a' \mu_a$. Then a applications of (a) show that $S_\mu S_\nu = (-1)^a t_{\nu_1'} \cdots t_{\nu_a'} t_{\mu_a} t_{\nu_{a+1}} \cdots t_{\nu_h}$. By definition of c_Λ (in particular, see the definition of κ in (5.1)), the right-hand side of this expression is precisely $c_\Lambda(\mu, \nu) S_{\mu\nu}$.

By the universal property of $C^*(\Lambda, c_\Lambda)$ there is a map $\psi : C^*(\Lambda, c_\Lambda) \to D$ such that $\psi(s_\lambda) = S_\lambda$. By construction the maps ψ and ϕ are mutually inverse and so $D \cong C^*(\Lambda, c_\Lambda)$.

The final assertion follows by the universality of D. ∎

7. GRADED TENSOR PRODUCTS OF TWISTED P-GRAPH C^*-ALGEBRAS

Let $P = \mathbb{N}^k \times \mathbb{Z}_2^a$ and let $Q = \mathbb{N}^l \times \mathbb{Z}_2^b$. Let Λ be a P-graph and Γ a Q-graph. Then $\Lambda \times \Gamma$ is a $P \times Q$ graph. The functor $\delta_{\Lambda\times\Gamma}$ and the 2-cocycle $c_{\Lambda\times\Gamma}$ defined in Proposition 5.4 via (5.1), still make sense as we are using the map $\pi : P \times Q \to \mathbb{Z}_2^{(k+a)+(l+b)}$ to define $c_{\Lambda\times\Gamma}$.

THEOREM 7.1. *Let $P = \mathbb{N}^k \times \mathbb{Z}_2^a$ and let $Q = \mathbb{N}^l \times \mathbb{Z}_2^b$. Let Λ be a P-graph and Γ a Q-graph. Then there is an isomorphism of graded C^*-algebras*

$$C^*(\Lambda \times \Gamma, c_{\Lambda\times\Gamma}) \cong C^*(\Lambda, c_\Lambda) \mathbin{\widehat{\otimes}} C^*(\Gamma, c_\Gamma)$$

with respect to the gradings $\alpha_{\delta_{\Lambda\times\Gamma}}$ and $\alpha_{\delta_\Lambda} \mathbin{\widehat{\otimes}} \alpha_{\delta_\Gamma}$.

Proof. By Theorem 6.4, the graded C^*-algebra $C^*(\Lambda \times \Gamma, c_{\Lambda\times\Gamma})$ is universal for families $\{p_{(v,w)} : v, w \in \Lambda^0 \times \Gamma^0\}$, $\{t_{(\lambda,w)} : \lambda \in \Lambda, d(\lambda) \in E_P \text{ and } w \in \Gamma^0\}$ and $\{t_{(v,\mu)} : v \in \Lambda^0, \mu \in \Gamma \text{ and } d(\mu) \in E_Q\}$ satisfying (a)–(d) of Theorem 6.4. Define elements of $C^*(\Lambda, c_\Lambda) \mathbin{\widehat{\otimes}} C^*(\Gamma, c_\Gamma)$ as follows:

$$\begin{aligned} T_{(\lambda,w)} &= s_\lambda \mathbin{\widehat{\otimes}} s_w \quad \text{for } \lambda \in \Lambda^{e_i}, w \in \Gamma^0, \\ T_{(v,\mu)} &= s_v \mathbin{\widehat{\otimes}} s_\mu \quad \text{for } \mu \in \Gamma^{e_j}, v \in \Lambda^0, \quad \text{and} \\ P_{(v,w)} &= s_v \mathbin{\widehat{\otimes}} s_w \quad \text{for } v \in \Lambda^0, w \in \Gamma^0. \end{aligned}$$

Then (since s_v is zero graded in $C^*(\Lambda, c_\Lambda)$ for each $v \in \Lambda^0$ and s_w is zero graded in $C^*(\Gamma, c_\Gamma)_0$ for each $w \in \Gamma^0$), the set $\{P_{(v,w)} : (v, w) \in \Lambda^0 \times \Gamma^0\}$ is a family of mutually orthogonal projections. We must check that the $P_{(v,w)}$ and the $T_{(\lambda,w)}$ and $T_{(v,\mu)}$ satisfy relations (a)–(d) for the $(P \times Q)$-graph $\Lambda \times \Gamma$ and the cocycle $c_{\Lambda\times\Gamma}$. Condition (a) is the most difficult, and we present it here; (b)–(d) are routine.

Let $\overline{\lambda}, \overline{\lambda}', \overline{\mu}, \overline{\mu}'$ be edges in $\Lambda \times \Gamma$ such that $d(\overline{\lambda}) = d(\overline{\lambda}')$, $d(\overline{\mu}) = d(\overline{\mu}')$, $d(\overline{\lambda}) \neq d(\overline{\mu})$, $s(\overline{\lambda}) = r(\overline{\mu})$, $s(\overline{\mu}') = r(\overline{\lambda}')$, and $\overline{\lambda}\overline{\mu} = \overline{\mu}'\overline{\lambda}'$. There are four combinations to check according to whether

$$\begin{aligned} &d(\overline{\lambda}) = (e_i, 0) \quad 1 \leqslant i \leqslant k, \quad \text{or} \quad d(\overline{\lambda}) = (0, e_i) \quad 1 \leqslant i \leqslant l; \quad \text{and} \\ &d(\overline{\mu}) = (e_j, 0) \quad 1 \leqslant j \leqslant k, \quad \text{or} \quad d(\overline{\mu}) = (0, e_j) \quad 1 \leqslant j \leqslant l. \end{aligned}$$

First suppose that $d(\overline{\lambda}) = (e_i, 0)$ and $d(\overline{\mu}) = (e_j, 0)$ where $i \neq j$. Then $\overline{\lambda} = (\lambda, v)$, $\overline{\mu} = (\mu, v)$, $\overline{\lambda}' = (\lambda', v)$ and $\overline{\mu}' = (\mu', v)$ for some $v \in \Gamma^0$ and some $\lambda, \mu' \in \Lambda^{e_i}$, $\mu, \lambda' \in \Lambda^{e_j}$ with $\lambda\mu = \mu'\lambda'$. We then have

$$T_{\overline{\lambda}} T_{\overline{\mu}} = T_{(\lambda,v)} T_{(\mu,v)} = (s_\lambda \mathbin{\widehat{\otimes}} s_v)(s_\mu \mathbin{\widehat{\otimes}} s_v) = (-1)^{\partial s_v \cdot \partial s_\mu} (s_\lambda s_\mu \mathbin{\widehat{\otimes}} s_v) = (s_{\lambda\mu} \mathbin{\widehat{\otimes}} s_v)$$

since $\partial s_v = 0$. On the other hand,

$$\begin{aligned} T_{\overline{\mu}'} T_{\overline{\lambda}'} &= T_{(\mu',v)} T_{(\lambda',v)} = (s_{\mu'} \mathbin{\widehat{\otimes}} s_v)(s_{\lambda'} \mathbin{\widehat{\otimes}} s_v) \\ &= (-1)^{\partial s_v \cdot \partial s_{\lambda'}} (s_{\mu'} s_{\lambda'} \mathbin{\widehat{\otimes}} s_v) = s_{\mu'} s_{\lambda'} \mathbin{\widehat{\otimes}} s_v = -(s_{\lambda\mu} \mathbin{\widehat{\otimes}} s_v). \end{aligned}$$

Now suppose that $d(\overline{\lambda}) = (e_i, 0)$ and $d(\overline{\mu}) = (0, e_j)$. Then $\overline{\lambda} = (\lambda, r(\mu))$, $\overline{\mu} = (s(\lambda), \mu)$, $\overline{\mu}' = (r(\lambda), \mu)$ and $\overline{\lambda}' = (\lambda, s(\mu))$ for some $\lambda \in \Lambda^{e_i}$ and $\mu \in \Gamma^{e_j}$. So

$$\begin{aligned} T_{\overline{\lambda}} T_{\overline{\mu}} = T_{(\lambda, r(\mu))} T_{(s(\lambda), \mu)} &= (s_\lambda \mathbin{\widehat{\otimes}} s_{r(\mu)})(s_{s(\lambda)} \mathbin{\widehat{\otimes}} s_\mu) \\ &= (-1)^{\partial s_{r(\mu)} \cdot \partial s_{r(\lambda)}} (s_{s(\lambda)} s_\lambda \mathbin{\widehat{\otimes}} s_\mu s_{s(\mu)}) = (s_\lambda \mathbin{\widehat{\otimes}} s_\mu), \end{aligned}$$

whereas, using that $\partial s_{\lambda'} = \partial s_{\mu'} = 1$,

$$\begin{aligned} T_{\overline{\mu}'} T_{\overline{\lambda}'} = T_{(r(\lambda'), \mu')} T_{(\lambda', s(\mu'))} &= (s_{r(\lambda')} \mathbin{\widehat{\otimes}} s_{\mu'})(s_{\lambda'} \mathbin{\widehat{\otimes}} s_{s(\mu')}) \\ &= (-1)^{\partial s_{\mu'} \cdot \partial s_{\lambda'}} (s_{r(\lambda')} s_{\lambda'} \mathbin{\widehat{\otimes}} s_{\mu'} s_{s(\mu')}) = -(s_\lambda \mathbin{\widehat{\otimes}} s_\mu). \end{aligned}$$

The remaining two cases are similar.

By the universal property of $C^*(\Lambda \times \Gamma, c_{\Lambda\times\Gamma})$ there is a map $\pi : C^*(\Lambda \times \Gamma, c_{\Lambda\times\Gamma}) \to C^*(\Lambda, c_\Lambda) \mathbin{\widehat{\otimes}} C^*(\Gamma, c_\Gamma)$ such that $\pi(t_{(\lambda,w)}) = T_{(\lambda,w)}$, $\pi(t_{(v,\mu)}) = T_{(v,\mu)}$ and $\pi(p_{(v,w)}) = P_{(v,w)}$. This π is surjective because $C^*(\Lambda, c_\Lambda) \mathbin{\widehat{\otimes}} C^*(\Gamma, c_\Gamma)$ is generated by the elements $s_\lambda \mathbin{\widehat{\otimes}} s_\mu = T_{(\lambda, r(\mu))} T_{(s(\lambda),\mu)}$. We aim to apply the gauge invariant uniqueness theorem for twisted P-graph C^*-algebras given in Proposition 5.10 to show that π is injective. For this, observe that the projections $P_{(v,w)} = s_v \mathbin{\widehat{\otimes}} s_w$ are nonzero, so it suffices to show that, identifying $\widehat{G}_{P\times Q}$ with $\widehat{G}_P \times \widehat{G}_Q$ in the canonical way, π is equivariant for the gauge-action $\gamma^{\Lambda\times\Gamma}$ on $C^*(\Lambda \times \Gamma, c_{\Lambda\times\Gamma})$ and the action $\gamma^\Lambda \mathbin{\widehat{\otimes}} \gamma^\Gamma$ on $C^*(\Lambda, c_\Lambda) \mathbin{\widehat{\otimes}} C^*(\Gamma, c_\Gamma)$ such that

$$(\gamma^\Lambda_\chi \mathbin{\widehat{\otimes}} \gamma^\Gamma_{\chi'})(s_\lambda \mathbin{\widehat{\otimes}} s_\mu) = \chi(d(\lambda))\chi'(d(\mu))(s_\lambda \mathbin{\widehat{\otimes}} s_\mu)$$

for all $(\chi, \chi') \in \widehat{G}_P \times \widehat{G}_Q$, $\lambda \in \Lambda$ and $\mu \in \Gamma$.

Since $\gamma^{\Lambda\times\Gamma}_{(\chi,\chi')} s_{(\lambda,\mu)} = \chi(d(\lambda))\chi'(d(\mu)) s_{(\lambda,\mu)}$ we see that π is equivariant on the generators $p_{(v,w)}$, $t_{(\lambda,w)}$, $t_{(v,\mu)}$, and therefore on $C^*(\Lambda \times \Gamma, c_{\Lambda\times\Gamma})$. ∎

An interesting special case of Theorem 7.1 occurs when Γ is the $\mathbb{Z}_2$-graph Z_2, so that $C^*(\Gamma) \cong \mathbb{C}\mathrm{liff}_1$ as graded C^*-algebras.

COROLLARY 7.2. *Let $P = \mathbb{N}^k \times \mathbb{Z}_2^a$ and let Λ be a P-graph. Then*

$$C^*(\Lambda \times Z_2, c_{\Lambda\times Z_2}) \cong C^*(\Lambda, c_\Lambda) \mathbin{\widehat{\otimes}} C^*(Z_2) \cong C^*(\Lambda, c_\Lambda) \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1,$$

with respect to the gradings $\alpha_{\delta_{\Lambda\times Z_2}}$ and $\alpha_{\delta_\Lambda} \mathbin{\widehat{\otimes}} \alpha_{\delta_{Z_2}}$.

Proof. The $\mathbb{Z}_2$-graph Z_2 has trivial second cohomology, so Lemma 5.8 yields an isomorphism $C^*(Z_2, c_{Z_2}) \cong C^*(Z_2)$ that clearly preserves gradings. The first statement therefore follows from Theorem 7.1. The second statement follows from Examples 6.3(ii). ∎

REMARK 7.3. Since the graded tensor product with $\mathbb{C}\mathrm{liff}_1$ is like a graded suspension operation, Corollary 7.2 has implications for graded K-theory. Let $P = \mathbb{N}^k \times \mathbb{Z}_2^l$ for some k, l, and let Λ be a P-graph. Then $K_i^{\mathrm{gr}}(C^*(\Lambda \times Z_2, c_{\Lambda\times Z_2})) \cong K_{i+1}^{\mathrm{gr}}(C^*(\Lambda, c_\Lambda))$, and then inductively

$$K_i^{\mathrm{gr}}(C^*(\Lambda \times Z_2^n, c_{\Lambda\times Z_2})) \cong K_{i+n}^{\mathrm{gr}}(C^*(\Lambda, c_\Lambda)).$$

COROLLARY 7.4. *Let Λ be the $\mathbb{Z}_2^l$-graph $\prod_{i=1}^{l} Z_2$. Then $C^*(\Lambda, c_\Lambda) \cong \mathbb{C}\mathrm{liff}_l$, the l-th complex Clifford algebra. This isomorphism is a graded isomorphism with respect to the grading δ_Λ of $C^*(\Lambda, c_\Lambda)$.*

Proof. We have $C^*(Z_2) \cong \mathbb{C}\mathrm{liff}_1$ as graded C^*-algebras as discussed in Example 6.3(ii). So the result follows from an induction argument using Corollary 7.2 and the definition $\mathbb{C}\mathrm{liff}_{l+1} = \mathbb{C}\mathrm{liff}_l \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1$. ∎

So far we have discussed gradings arising from functors from k-graphs into $\mathbb{Z}_2$, but there are other possible gradings including those arising from order two automorphisms of k-graphs. Let θ be an order two automorphism of a row-finite k-graph Λ with no sources. Then θ induces a grading β_θ of $C^*(\Lambda)$ satisfying $\beta_\theta(s_\lambda) = s_{\theta(\lambda)}$. With respect to this grading,

$$\begin{aligned} C^*(\Lambda)_0 &= \overline{\operatorname{span}}\{s_\lambda s_\mu^* + s_{\theta(\lambda)} s_{\theta(\mu)}^* : s(\lambda) = s(\mu)\}, \quad \text{and} \\ C^*(\Lambda)_1 &= \overline{\operatorname{span}}\{s_\lambda s_\mu^* - s_{\theta(\lambda)} s_{\theta(\mu)}^* : s(\lambda) = s(\mu)\}. \end{aligned}$$

PROPOSITION 7.5. *With notation as above there is a graded isomorphism*

$$\rho : C^*(\Lambda \times_\theta \mathbb{Z}_2) \to C^*(\Lambda) \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1$$

such that $\rho(s_{(\lambda,i)}) = s_\lambda \mathbin{\widehat{\otimes}} u^i$, where $\Lambda \times_\theta \mathbb{Z}_2$ is the crossed-product $(\mathbb{N}^k \times \mathbb{Z}_2)$-graph, and $C^*(\Lambda \times_\theta \mathbb{Z}_2)$ has grading automorphism $\widetilde{\beta}_\theta$ given by $\widetilde{\beta}_\theta(s_{(\lambda,i)}) = (-1)^i s_{(\theta(\lambda),i)}$.

Proof. Direct calculation shows that the elements $t_{(\lambda,i)} := s_\lambda \mathbin{\widehat{\otimes}} u^i$ constitute a Cuntz–Krieger $(\Lambda \times_\theta \mathbb{Z}_2)$-family in $C^*(\Lambda) \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1$. So the universal property of $C^*(\Lambda \times_\theta \mathbb{Z}_2)$ gives a homomorphism $\rho : C^*(\Lambda \times_\theta \mathbb{Z}_2) \to C^*(\Lambda) \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1$ taking $s_{(\lambda,i)}$ to $t_{(\lambda,i)} = s_\lambda \mathbin{\widehat{\otimes}} u^i$. An application of the gauge-invariant uniqueness theorem (Proposition 5.10) shows that ρ is injective; it is surjective because its image contains the generators of $C^*(\Lambda) \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}_1$. Therefore ρ is an isomorphism. Let α be the grading automorphism of $\mathbb{C}\mathrm{liff}_1$. Then

$$\begin{aligned} \rho(\widetilde{\beta}_\theta(s_{(\lambda,i)})) = \rho((-1)^i s_{(\theta(\lambda),i)}) &= (-1)^i s_{\theta(\lambda)} \mathbin{\widehat{\otimes}} u^i \\ &= (\beta_\theta \mathbin{\widehat{\otimes}} \alpha)(s_\lambda \mathbin{\widehat{\otimes}} u^i) = (\beta_\theta \mathbin{\widehat{\otimes}} \alpha)\rho(s_{(\lambda,i)}); \end{aligned}$$

hence ρ intertwines the grading automorphisms of the two algebras. ∎

8. GRADED K-THEORY OF GRAPH C^*-ALGEBRAS

In this section we apply the sequence (4.9) to a graph C^*-algebra $C^*(E)$ graded by an automorphism α_δ determined by a function $\delta : E^1 \to \mathbb{Z}_2$; see Corollary 8.3.

REMARK 8.1. Following [11] (see also Section 8 of [33]), given a 1-graph E, we can realise $C^*(E)$ as the Cuntz–Pimsner algebra of the module $X(E)$ defined as the Hilbert-bimodule completion of $C_c(E^1)$, regarded as a $C_0(E^0)$-$C_0(E^0)$-bimodule under the left and right actions given by $(a \cdot x \cdot b)(e) = a(r(e))x(e)b(s(e))$, under the inner-product $\langle x, y\rangle_{C_0(E^0)}(v) = \sum\limits_{s(e)=v} \overline{x(e)}y(e)$. By Proposition 4.4 of [11] this left action is by compacts if E is row-finite. The left action is injective if E has no sources. Observe that $C_0(E^0)$, is separable and nuclear. It carries the trivial grading $\alpha_{C_0(E^0)} = \mathrm{id}$.

Fix a function $\delta : E^1 \to \mathbb{Z}_2$. This δ extends uniquely to a functor $\delta : E^* \to \mathbb{Z}_2$. There is a grading $\alpha_{X(E)}$ on $X(E)$ determined by $\alpha_{X(E)}(1_e) = (-1)^{\delta(e)}1_e$. It is straightforward to check that (X, α_X) is a graded $C_0(E^0)$-$C_0(E^0)$-correspondence. By Proposition 12 of [10] (see also Example 8.13 of [33]) we have $C^*(X(E)) \cong C^*(E)$. Hence (4.9) becomes

$$0 \to K_1^{\mathrm{gr}}(C^*(E), \alpha) \hookrightarrow K_0^{\mathrm{gr}}(C_0(E^0)) \xrightarrow{1-[X(E)]} K_0^{\mathrm{gr}}(C_0(E^0)) \twoheadrightarrow K_0^{\mathrm{gr}}(C^*(E), \alpha) \to 0 \tag{8.1}$$

since $K_1^{\mathrm{gr}}(A, \alpha_A) = \bigoplus\limits_{v \in E^0} K_1(\mathbb{C}) = 0$.

To apply (8.1) to compute the graded K-theory of the C^*-algebra associated to a 1-graph E we need to examine the central terms more closely. We describe $K_0^{\mathrm{gr}}(C_0(E^0))$ in a way which allows us to compute the map $\widehat{\otimes}_{C_0(E^0)}(1 - [X(E)])$.

Let E be a row-finite 1-graph with no sources. We have $K_0^{\mathrm{gr}}(C_0(E^0)) = KK(\mathbb{C}, \mathbb{C}^{E^0})$. Let $\mathbb{C}\delta_v$ be a copy $\{z\delta_v : z \in \mathbb{C}\}$ of $\mathbb{C}$ as a vector space with inner-product given by $\langle z\delta_v, z'\delta_v\rangle_{\mathbb{C}^{E^0}}(u) = \delta_{v,u}\bar{z}z'$ and right action $z\delta_v \cdot a = a(v)z\delta_v$. It carries a left action φ_v of $\mathbb{C}$ by multiplication. The tuple $(\mathbb{C}\delta_v, \varphi_v, 0, \mathrm{id})$ is a Kasparov $\mathbb{C}$-$\mathbb{C}^{E^0}$-module. The group $KK(\mathbb{C}, \mathbb{C}^{E^0})$ is generated by the Kasparov $\mathbb{C}$-$\mathbb{C}^{E^0}$ modules $[\mathbb{C}\delta_v] := [\mathbb{C}\delta_v, \varphi_v, 0, \mathrm{id}]$ for $v \in E^0$, and there is an isomorphism $\theta : \mathbb{Z}E^0 \to KK_0(\mathbb{C}, C_0(E^0))$ such that $\theta(1_v) = [\mathbb{C}\delta_v]$, where 1_v is the generator of $\mathbb{Z}E^0$ corresponding to v.

Now we describe the map $\widehat{\otimes}_{C_0(E^0)}[X(E)]$ on $\mathbb{Z}E^0$ induced by the isomorphism θ. Let A_E^δ be the $E^0 \times E^0$ matrix defined by

$$A_E^\delta(v, w) = \sum_{e \in vE^1w} (-1)^{\delta(e)} \tag{8.2}$$

(the empty sum is 0 by convention). If E_j denotes the subgraph $(E^0, \delta^{-1}(j), r, s)$ of E for $j = 0, 1$, then A_E^δ is just $A_{E_0} - A_{E_1}$.

LEMMA 8.2. *Let E be a row-finite 1-graph with no sources. Then with notation as above, the following diagram commutes.*

$$\begin{tikzcd} \mathbb{Z}E^0 \arrow[r, "\theta"] \arrow[d, "(A_E^\delta)^{\mathrm{t}}"'] & KK_0(\mathbb{C}, \mathbb{C}^{E^0}) \arrow[d, "\widehat{\otimes}_{C_0(E^0)}[X(E)]"] \\ \mathbb{Z}E^0 \arrow[r, "\theta"] & KK_0(\mathbb{C}, \mathbb{C}^{E^0}). \end{tikzcd}$$

Proof. It suffices to check that the diagram commutes on generators 1_v. Fix $v \in E^0$. Using Lemma 4.1 at the second equality we calculate:

$$\begin{aligned}\theta(1_v)\,\widehat{\otimes}_{C_0(E^0)}\,[X(E)] &= [\mathbb{C}\delta_v]\,\widehat{\otimes}_{C_0(E^0)}\,[X(E)] = [\mathbb{C}\delta_v]\,\widehat{\otimes}_{C_0(E^0)}\,([X(E)_0] - [X(E)_1]) \\ &= \sum_{e \in vE^1w,\, \delta(e)=0} [\mathbb{C}\delta_w] - \sum_{f \in vE^1w,\, \delta(f)=1} [\mathbb{C}\delta_w] = \sum_{g \in vE^1w} (-1)^{\delta(g)} [\mathbb{C}\delta_w].\end{aligned}$$

This is precisely $\theta((A_E^\delta)^{\mathrm{t}} 1_v)$. ∎

We now use Corollary 4.5 and Lemma 8.2 to compute the graded K-theory of graph C^*-algebras for suitable gradings.

COROLLARY 8.3. *Let E be a row-finite 1-graph with no sources, and let $\delta : E \to \mathbb{Z}_2$ be the functor determined by the function $\delta : E^1 \to \mathbb{Z}_2$. Let α_δ be the associated grading $\alpha_\delta(s_e) = (-1)^{\delta(e)} s_e$ of $C^*(E)$. Then, with A_E^δ as in* (8.2),

$$K_0^{gr}(C^*(E), \alpha_\delta) \cong \operatorname{coker}(1 - (A_E^\delta)^{\mathrm{t}} : \mathbb{Z}E^0 \to \mathbb{Z}E^0) \quad \textit{and}$$
$$K_1^{\mathrm{gr}}(C^*(E), \alpha_\delta) \cong \ker(1 - (A_E^\delta)^{\mathrm{t}} : \mathbb{Z}E^0 \to \mathbb{Z}E^0).$$

Note that if E^0 is finite, then $K_1^{\mathrm{gr}}(C^*(E), \alpha_\delta)$ is a free abelian group with the same rank as $K_0^{\mathrm{gr}}(C^*(E), \alpha_\delta)$.

REMARK 8.4. Corollary 8.3 is a direct parallel to Corollary 4.2.5 of [30] (see also Example 7.2 of [33]): given $\delta : E^1 \to \mathbb{Z}_2$, the graded K_0-group $K_0^{\mathrm{gr}}(C^*(E), \alpha_\delta)$ is generated as an abelian group by the classes of the vertex projections $\{p_v : v \in E^0\}$ subject only to the relations

$$[p_v] = \Big[\sum_{e \in vE^1} s_e s_e^*\Big] = \sum_{e \in vE^1} (-1)^{\delta(e)} [s_e^* s_e] = \sum_{w \in E^0} (A_E^\delta)^{\mathrm{t}}(v, w)[p_w]$$

coming from Example 3.5. This motivates, in part, our conjecture in Section 9 below.

In particular, taking $\delta \equiv 0$, we recover the well-known formula for the (ungraded) K-theory of a 1-graph C^*-algebra ([30], Theorem 4.2.4).

EXAMPLES 8.5. (i) As in Examples 5.1, for $1 \leqslant n < \infty$ let B_n be the 1-graph with one vertex and n edges. Fix $\delta : B_n^1 \to \mathbb{Z}_2$, and let $p := |\delta^{-1}(1)|$ and $q = |\delta^{-1}(0)|$, so that $p + q = n$. Then $(A_{B_n}^\delta)^{\mathrm{t}}$ is the 1×1 matrix $(q - p)$. Since $C^*(B_n) \cong$

$\mathcal{O}_n$ we recover the formula for $K_*^{\mathrm{gr}}(\mathcal{O}_n)$ obtained by Haag in Proposition 4.11 of [15]:

$$K_*^{\mathrm{gr}}(\mathcal{O}_n, \alpha_\delta) \cong \begin{cases} (\mathbb{Z}_{|1+p-q|}, 0) & \text{if } 1+p-q \neq 0, \\ (\mathbb{Z}, \mathbb{Z}) & \text{otherwise.} \end{cases}$$

(ii) Let K_2 be the 1-graph associated to the complete directed graph on two vertices. Endow K_2 with the map $\delta' : K_2^1 \to \mathbb{Z}_2$ for which $A_{K_2}^{\delta'} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$. Then $K_0^{\mathrm{gr}}(C^*(K_2), \alpha_{\delta'}) \cong \mathbb{Z}_5$. In particular, this and (i) above show that although $C^*(K_2) \cong \mathcal{O}_2 \cong C^*(B_2)$, there is no graded isomorphism from $(C^*(K_2), \alpha_{\delta'})$ to $(C^*(B_2), \alpha_\delta)$ for any $\delta : B_2^1 \to \{0, 1\}$.

(iii) More generally, let Λ be a row-finite k-graph with no sources and fix $p \in \mathbb{N}^k$. Recall that the *dual graph* $p\Lambda := \{\lambda \in \Lambda : d(\lambda) \geqslant p\}$ is a k-graph as follows: $d_p(\lambda) = d(\lambda) - p$, and if we use the factorisation property in Λ to write each $\lambda \in \Lambda$ as $\lambda = \overline{\lambda} t(\lambda) = h(\lambda)\underline{\lambda}$ with $d(t(\lambda)) = d(h(\lambda)) = p$, then the range and source maps on $p\Lambda$ are h and t respectively, and composition in $p\Lambda$ is given by $\lambda \circ_p \mu = \overline{\lambda}\mu = \lambda\underline{\mu}$ whenever $t(\lambda) = h(\mu)$ (cf. Proposition 3.2 of [1]). By Theorem 3.5 of [1] there is an isomorphism $\theta : C^*(p\Lambda) \to C^*(\Lambda)$ such that $s_\lambda^{p\Lambda} \mapsto s_\lambda^\Lambda (s_{t(\lambda)}^\Lambda)^*$. So any functor $\delta_p : p\Lambda \to \mathbb{Z}_2$ induces a grading α_p of $C^*(p\Lambda)$ and hence a grading α of $C^*(\Lambda)$. As seen in the preceding example, this grading typically does not arise from a functor from Λ to $\mathbb{Z}_2$, but for $k = 1$, we can still apply Corollary 8.3 (to $p\Lambda$) to compute $K_*^{\mathrm{gr}}(C^*(\Lambda), \alpha)$.

(iv) Let F be the 1-graph with vertices $\{v_n : n \in \mathbb{N}\}$ and edges $\{e_n, f_n : n \in \mathbb{N}\}$ where $r(e_n) = r(f_n) = v_n$ and $s(e_n) = s(f_n) = v_{n+1}$. Then $C^*(F)$ is Morita equivalent to the UHF-algebra M_{2^∞}, and so $K_*(C^*(F)) = (\mathbb{Z}[\frac{1}{2}], 0)$. Define $\delta : F^1 \to \mathbb{Z}_2$ by $\delta(e_n) = 0$ and $\delta(f_n) = 1$ for all n. Then the matrix A_F^δ is the zero matrix. Hence $1 - A_F^\delta$ is the identity map from $\mathbb{Z}F^0$ to $\mathbb{Z}F^0$, and we obtain $K_*^{\mathrm{gr}}(C^*(F), \alpha_\delta) = (0, 0)$ by Corollary 8.3. (We can also recover this result by taking a direct-limit decomposition as in Example 8.7 below.)

REMARK 8.6. Suppose that Λ is a bipartite P-graph. That is, $\Lambda^0 = L \sqcup R$ and for every edge $\lambda \in \Lambda$ either $s(\lambda) \in L$ and $r(\lambda) \in R$, or vice versa. Then the gradings α_{δ_Λ} of $C^*(\Lambda)$ and $C^*(\Lambda, c_\Lambda)$ induced by the functors δ_Λ of (6.2) are inner because the grading automorphism is implemented by the self-adjoint multiplier unitary $U = P_L - P_R$. Hence 14.5.2 of [2] gives $K_*^{\mathrm{gr}}(C^*(\Lambda), \alpha_\Lambda) \cong K_*(C^*(\Lambda))$.

To see why this observation is useful, observe that the skew-product of a k-graph Λ by the degree functor $\Lambda \times_d \mathbb{Z}^k$ is bipartite with $L = \Lambda^0 \times \Big\{n \in \mathbb{Z}^k : \sum_i n_i \text{ is even}\Big\}$ and $R = \Lambda^0 \times \Big\{n \in \mathbb{Z}^k : \sum_i n_i \text{ is odd}\Big\}$. If Λ is the 1-graph B_2 from (i), then $B_2 \times_d \mathbb{Z} \cong F$ where F as in (iv) above. Hence, as graded algebras $C^*(B_2 \times_d \mathbb{Z}) \cong C^*(F)$.

Also, let Λ be a k-graph and let $\delta = \delta_\Lambda : \Lambda \to \mathbb{Z}_2$ be as in (6.2). Then the skew product graph $\Lambda \times_\delta \mathbb{Z}_2$ is bipartite (with $L = \Lambda^0 \times \{0\}$ and $R = \Lambda^0 \times \{1\}$), and so the grading on $C^*(\Lambda \times_\delta \mathbb{Z}_2)$ induced by $\delta_{\Lambda \times_\delta \mathbb{Z}_2}$ is inner.

EXAMPLE 8.7. Consider again the graph and functor of Examples 8.5(iv). We have $C^*(F) = \overline{\bigcup C^*(F_n)}$ where F_n is the subgraph of F with

$$F_n^0 = \{v_1, \dots, v_n\} \quad \text{and} \quad F_n^1 = \{e_1, f_1, \dots, e_{n-1}, f_{n-1}\}.$$

Fix $n \in \mathbb{N}$. Let $\{\theta_{\mu,\nu} : \mu, \nu \in Fv_n\}$ denote the canonical matrix units for M_{Fv_n}. We have $C^*(F_n) \cong M_{Fv_n}$ via $s_\mu s_\nu^* \mapsto \theta_{\mu,\nu}$. Extend δ to F by setting $\delta(\mu) = \sum_{i=1}^{|\mu|} \delta(\mu_i)$, and define

$$U = \sum_{\mu \in Fv_n} (-1)^{\delta(\mu)} s_\mu s_\mu^*.$$

Then U is a self-adjoint unitary in $C^*(F_n)$ that implements the grading by conjugation. So the grading on each $C^*(F_n)$ is inner, and therefore $K_*^{\mathrm{gr}}(C^*(F_n), \alpha_\delta) = K_*(C^*(F_n)) = K_*(M_{Fv_n}) = (\mathbb{Z}, 0)$, with generator $[s_{v_n}]$.

The inclusion map $\iota_n : C^*(F_n) \hookrightarrow C^*(F_{n+1})$ is given by $s_\mu s_\nu^* \mapsto s_{\mu e_n} s_{\nu e_n}^* + s_{\mu f_n} s_{\nu f_n}^*$. In particular $\iota_n(s_{v_n}) = s_{e_n} s_{e_n}^* + s_{f_n} s_{f_n}^*$. The partial isometry $V := s_{e_n} s_{f_n}^*$ is odd and satisfies $VV^* = s_{e_n} s_{e_n}^*$ and $V^*V = s_{f_n} s_{f_n}^*$. So $\iota_n(s_{v_n}) = V^*V + VV^*$. By Example 3.5, we have $[V^*V] = -[VV^*]$ in $K_0^{\mathrm{gr}}(C^*(F_{n+1}), \alpha_\delta)$, and it follows that $\iota_* : K_0^{\mathrm{gr}}(C^*(F_n), \alpha_\delta) \to K_0^{\mathrm{gr}}(C^*(F_{n+1}), \alpha_\delta)$ sends $[s_{v_n}]$ to zero and hence is the zero map. Hence

$$K_*^{\mathrm{gr}}(C^*(F), \alpha_\delta) \cong (\varinjlim(K_0^{\mathrm{gr}}(C^*(F_n), \alpha_\delta), \iota_*), 0) \cong (\varinjlim(\mathbb{Z}, 0), 0) = (0, 0) \text{ as before.}$$

We turn next to some applications of Corollary 4.7 to the crossed products of the C^*-algebra of a 1-graph E. To do this, we first need to describe the map in graded K-theory induced by an automorphism determined by a function from E^1 to $\mathbb{Z}_2$.

LEMMA 8.8. *Let E be a row-finite 1-graph with no sources and $\delta : E^1 \to \mathbb{Z}_2$ a function. Let $\delta : E^* \to \mathbb{Z}_2$ be the induced functor. The map α_* on $K_*^{\mathrm{gr}}(C^*(E), \alpha_\delta)$ induced by the automorphism α_δ is the identity map.*

Proof. Let $X(E)$ denote the graph module described in Remark 8.1. For $v \in E^0$, and $e \in E^1$, the grading operator α_δ on $X(E)$ satisfies

$$\alpha_\delta(1_v \cdot 1_e) = \delta_{v,r(e)} \alpha_\delta(1_e) = \delta_{v,r(e)} (-1)^{\delta(e)} 1_e = 1_v \cdot \alpha_\delta(1_e)$$

and similarly $\alpha_\delta(1_e \cdot 1_v) = \alpha_\delta(1_e) \cdot 1_v$. So, by linearity and continuity, $\alpha_\delta : X(E) \to X(E)$ is a bimodule map. Moreover for $e, f \in E^1$ we have

$$\langle \alpha_\delta(1_e), \alpha_\delta(1_f) \rangle_{C_0(E^0)} = (-1)^{\delta(e)+\delta(f)} \langle 1_e, 1_f \rangle_{C_0(E^0)} = \begin{cases} 1_{s(e)} & \text{if } e = f, \\ 0 & \text{otherwise,} \end{cases}$$

which is precisely $\langle 1_e, 1_f \rangle_{C_0(E^0)}$. So α_δ is a graded automorphism of $X(E)$.

Thus the final statement of Theorem 4.4 implies that α_δ induces an automorphism of the exact sequence

$$0 \to K_1^{\mathrm{gr}}(C^*(E), \alpha_\delta) \hookrightarrow K_0^{\mathrm{gr}}(C_0(E^0)) \xrightarrow{1-[X(E)]} K_0^{\mathrm{gr}}(C_0(E^0)) \twoheadrightarrow K_0^{\mathrm{gr}}(C^*(E), \alpha_\delta) \to 0.$$

Since this automorphism is the identity map on the two middle terms in the sequence, we deduce that it is the identity map on $K_*^{\mathrm{gr}}(C^*(E), \alpha_\delta)$ as claimed. ∎

EXAMPLES 8.9. (i) Recall Example 6.3(iv). Let E be a row-finite 1-graph with no sources. Give $C^*(E)$ the grading α induced by the functor $\delta(\lambda) = |\lambda| \pmod 2$. Consider the crossed product $C^*(E) \rtimes_\alpha \mathbb{Z}$ under the grading $\widetilde{\alpha}$ satisfying $\widetilde{\alpha}(i_A(a) i_{\mathbb{Z}}(n)) = (-1)^n i_A(\alpha(a)) i_{\mathbb{Z}}(n)$. By applying Corollary 4.7 with $k = 1$ (so that $\widetilde{\alpha} = \beta^1$), and Lemma 8.8 we obtain the following exact sequence:

$$\begin{array}{ccccc} K_0^{\mathrm{gr}}(C^*(E), \alpha_\delta) & \xrightarrow{\times 2} & K_0^{\mathrm{gr}}(C^*(E), \alpha_\delta) & \xrightarrow{i_*} & K_0^{\mathrm{gr}}(C^*(E) \rtimes_{\alpha_\delta} \mathbb{Z}, \widetilde{\alpha}) \\ \uparrow & & & & \downarrow \\ K_1^{\mathrm{gr}}(C^*(E) \rtimes_{\alpha_\delta} \mathbb{Z}, \widetilde{\alpha}) & \xleftarrow{i_*} & K_1^{\mathrm{gr}}(C^*(E), \alpha_\delta) & \xleftarrow{\times 2} & K_1^{\mathrm{gr}}(C^*(E), \alpha_\delta). \end{array} \tag{8.3}$$

By Example 6.3(iv), we have a graded isomorphism

$$(C^*(E) \mathbin{\widehat{\otimes}} C^*(T_1), \alpha_{\delta_E} \mathbin{\widehat{\otimes}} \alpha_{\delta T_1}) \cong (C^*(E) \rtimes_\alpha \mathbb{Z}, \widetilde{\alpha}).$$

We use this to compute $K_*^{\mathrm{gr}}(C^*(E) \mathbin{\widehat{\otimes}} C^*(T_1))$.

Since $K_1^{\mathrm{gr}}(C^*(E), \alpha_\delta)$ has no torsion, multiplication by 2 is injective on that group, so exactness implies that the right-hand boundary map is zero. Therefore $K_0^{\mathrm{gr}}(C^*(E) \mathbin{\widehat{\otimes}} C^*(T_1))$ is isomorphic to the cokernel of the times-two map on $K_0^{\mathrm{gr}}(C^*(E), \alpha_\delta)$; that is

$$K_0^{\mathrm{gr}}(C^*(E) \mathbin{\widehat{\otimes}} C^*(T_1)) \cong K_0^{\mathrm{gr}}(C^*(E), \alpha_\delta) / 2K_0^{\mathrm{gr}}(C^*(E), \alpha_\delta).$$

Exactness of the bottom row gives

$$i_*(K_1^{\mathrm{gr}}(C^*(E), \alpha_\delta)) \cong K_1^{\mathrm{gr}}(C^*(E), \alpha_\delta) / 2K_1^{\mathrm{gr}}(C^*(E), \alpha_\delta),$$

so $K_1^{\mathrm{gr}}(C^*(E) \mathbin{\widehat{\otimes}} C^*(T_1))$ is an extension of the 2-torsion subgroup

$$\{a \in K_0^{\mathrm{gr}}(C^*(E), \alpha_\delta) : 2a = 0\}$$

by $K_1^{\mathrm{gr}}(C^*(E), \alpha_\delta) / 2K_1^{\mathrm{gr}}(C^*(E), \alpha_\delta)$.

In particular, if $K_0^{\mathrm{gr}}(C^*(E), \alpha_\delta)$ has no 2-torsion, then we obtain

$$K_1^{\mathrm{gr}}(C^*(E) \mathbin{\widehat{\otimes}} C^*(T_1)) \cong K_1^{\mathrm{gr}}(C^*(E), \alpha_\delta) / 2K_1^{\mathrm{gr}}(C^*(E), \alpha_\delta);$$

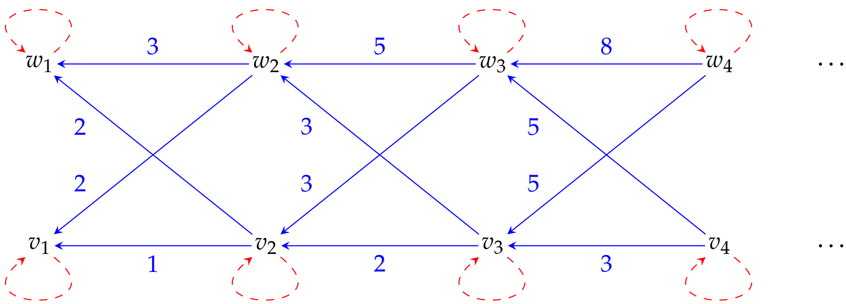
but even if $K_0^{\mathrm{gr}}(C^*(E), \alpha_\delta)$ does contain 2-torsion, we can deduce, for example, that the order of every element of $K_1^{\mathrm{gr}}(C^*(E) \mathbin{\widehat{\otimes}} C^*(T_1))$ divides 4.

(ii) Recall from Examples 6.3(i) that $C^*(T_1) \mathbin{\widehat{\otimes}} C^*(T_1) \cong C^*(T_2, c)$ where c is the 2-cocycle $c(m,n) = (-1)^{m_2 n_1}$ for $(m,n) \in \mathbb{N}^2$. We can compute the graded K-theory $K_*^{\mathrm{gr}}(C^*(T_2, c), \delta_{T_2})$, by taking $E = T_1$ in (i) above. Since $T_1 = B_1$ we have $K_0^{\mathrm{gr}}(C^*(T_1), \delta_{T_1}) = \mathbb{Z}_2$ and $K_1^{\mathrm{gr}}(C^*(T_1), \delta_{T_1}) = \{0\}$ by Examples 8.5(i). Then by (8.3), the times-two map on $K_0^{\mathrm{gr}}(C^*(T_1), \delta_{T_1})$ is the zero map, and so the exact sequence above for $K_*^{\mathrm{gr}}((C^*(T_2), c), \delta_{T_2})$ collapses to give $K_*^{\mathrm{gr}}(C^*(T_2, c), \delta_{T_2}) \cong (\mathbb{Z}_2, \mathbb{Z}_2)$. Observe that $C^*(T_2, c)$ is the rational rotation algebra $A_{1/2}$, so its ungraded K-theory is $(\mathbb{Z}^2, \mathbb{Z}^2)$ (see [7]).

REMARK 8.10. More generally, by Theorem 7.1, if E is any row-finite 1-graph with no sources endowed with the grading induced by the functor $\delta(e) = 1$ for all $e \in E^1$, then $C^*(E) \mathbin{\widehat{\otimes}} C^*(T_1) \cong C^*(E \times T_1, c_{E \times T_1})$ with the grading induced by $\delta_{E \times T_1}$. Thus Example 8.9(i) computes the graded K-theory of this twisted 2-graph C^*-algebra.

We finish with an example describing a 2-graph C^*-algebra $C^*(\Lambda)$ that is Morita equivalent to an irrational-rotation algebra, and a grading α on $C^*(\Lambda)$ such that $K_*^{\mathrm{gr}}(C^*(\Lambda), \alpha) = (\mathbb{Z}_2 \times \mathbb{Z}_2, 0)$.

EXAMPLE 8.11. Consider the following 2-coloured graph (see Example 6.5 of [31]):

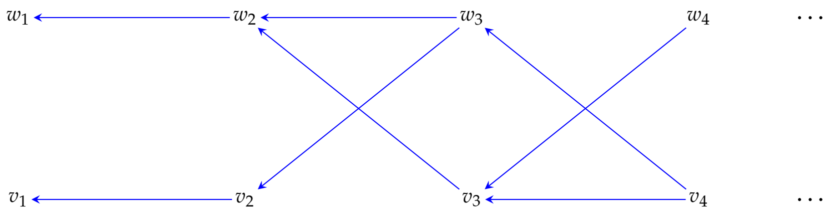


where the label on a blue (solid) edge indicates the number of parallel blue edges. The pattern is that the numbers of edges are Fibonacci numbers. The red edges are drawn dashed. Let $E = E_{\mathrm{blue}}$ be the subgraph consisting of blue edges. For $v, w \in E^0$ such that $wE^1_{\mathrm{blue}}v \neq \emptyset$, fix a partition $vE^1_{\mathrm{blue}}w = S^0(v,w) \sqcup S^1(v,w)$ of $vE^1_{\mathrm{blue}}w$ such that $|S^0(v,w)| = |S^1(v,w)|$ if $|vE^1_{\mathrm{blue}}w|$ is even, and $|S^0(v,w)| = |S^1(v,w)| + 1$ if $|vE^1_{\mathrm{blue}}w|$ is odd.

Choose a permutation ρ of E^1_{blue} that preserves ranges and sources, and cyclicly permutes the elements of each $S^j(v,w)$, for $j = 0, 1$. For each $v \in E^0$, let f_v be the dashed loop based at v. Let Λ be the 2-graph with the above skeleton, and with factorisation rules given by $f_{r(e)}e = \rho(e) f_{s(e)}$ for all $e \in E^1_{\mathrm{blue}}$.

Since the numbers of parallel edges grow exponentially fast, Λ has large-permutation factorisations in the sense of Definition 5.6 in [31]. It is also cofinal, and so $C^*(\Lambda)$ is simple with real-rank zero. Elliott's classification theorem [8] combined with Theorem 4.3 of [31] implies that $C^*(\Lambda)$ is Morita equivalent to the irrational-rotation algebra A_θ where $\theta = \frac{1+\sqrt{5}}{2}$ (see Example 6.5 of [31]). Define

$\delta : E^1_{\text{blue}} \to \mathbb{Z}_2$ by $\delta(e) = k$ whenever $e \in S^k(v, w)$ for some v, w. This induces a functor $\delta : E^*_{\text{blue}} \to \mathbb{Z}_2$. The matrix $A^{\delta}_{E_{\text{blue}}}$ defined in Corollary 8.3 has entries in $\{0, 1\}$, and corresponds to the 1-graph F with skeleton



where the pattern of connecting edges repeats every three levels (note: there are no parallel edges). By telescoping these three levels, and arguing as in Example 8.7, we see that $K_0^{\text{gr}}(C^*(E), \alpha_\delta) \cong \varinjlim(\mathbb{Z}^2, \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right))$. This matrix has determinant 1, and so $K_*^{\text{gr}}(C^*(E), \alpha_\delta) \cong (\mathbb{Z}^2, 0)$.

As in [9], the permutation ρ of E^1 defining the factorisation rules induces an automorphism $\widetilde{\rho}$ of $C^*(E)$, and $C^*(\Lambda) \cong C^*(E) \rtimes_{\widetilde{\rho}} \mathbb{Z}$ by Theorem 3.4 of [9]. There are two natural extensions of δ to a $\mathbb{Z}_2$-valued functor on Λ: namely δ^0, determined by $\delta^0(f) = 0$ for all $f \in \Lambda^{e_2}$, and δ^1, determined by $\delta^1(f) = 1$ for all $f \in \Lambda^{e_2}$. The grading automorphisms α_{δ^k} correspond under the identification $C^*(\Lambda) \cong C^*(E) \rtimes_{\widetilde{\rho}} \mathbb{Z}$ with the automorphisms β^k of Corollary 4.7, for $k = 0, 1$. So we can compute the graded K-theory of $C^*(\Lambda, \alpha_{\delta^k})$ by applying that result. The automorphism $\widetilde{\rho}$ of $C^*(E)$ permutes equivalent projections in approximating finite-dimensional subalgebras of $C^*(E)$. So the automorphism $\widetilde{\rho}_*$ of $K_*^{\text{gr}}(C^*(E))$ induced by $\alpha_\delta \circ \widetilde{\rho}$ is the identity. By Lemma 8.8 we therefore have $(-(\alpha_\delta)_*)^k \widetilde{\rho}_* = (-1)^k \,\text{id}$ for $k = 0, 1$. Thus $\text{id} - (-\alpha_*)^0 \widetilde{\rho}_* = 0$ and Corollary 4.7 gives $K_*^{\text{gr}}(C^*(\Lambda), \alpha_{\delta^0}) \cong (\mathbb{Z}^2, \mathbb{Z}^2)$ (which is isomorphic to $K_*(C^*(\Lambda))$ as a pair of abelian groups). And $\text{id} - (-(\alpha_\delta)_*)^1 \widetilde{\rho}_* = 2 \cdot \text{id}$, so Corollary 4.7 gives $K_*^{\text{gr}}(C^*(\Lambda), \alpha_{\delta^1}) = (\mathbb{Z}_2 \times \mathbb{Z}_2, 0)$.

9. CONJECTURE

In this section we deal exclusively with a unital (and in particular σ-unital) graded C^*-algebra (A, α). Our results and examples, particularly Example 3.5, lead us to ask whether $K_0^{\text{gr}}(A)$ consists of equivalence classes of homogeneous projections over A subject to the relation

$$[v^* v] = (-1)^{\partial v} [v v^*] \quad \text{for every homogeneous partial isometry } v.$$

To make this concrete, let (A, α) be a unital graded C^*-algebra. Let $P_0(A)$ denote the collection of homogeneous projections in $\mathcal{K}(\widehat{\mathcal{H}}_A)$. Let $p, q \in P_0(A)$; we write $p \sim q$ if there is an *even* partial isometry v such that $p = v^* v$ and $q = v v^*$. If $p \perp q$, then $p + q$ is a projection and we write $[p] + [q] = [p + q]$. Define $V_0(A) :=$

$P_0(A)/\sim$, which is an abelian monoid under the binary operation induced by orthogonal addition. Given a homotopy $t \mapsto p_t$ in $P_0(A)$, we have $[p_0] = [p_1]$ (see 2.2.7 of [39] or Section 4 of [2]).

Note that $V_0(A)$ may be identified with the set of isomorphism classes of graded, projective modules over A. By a projective module over A we mean a right-Hilbert module of the form $p\mathcal{H}_A$ where $p \in \mathcal{K}(\mathcal{H}_A)$. Since $\mathcal{H}_A = \ell^2(A)$ is countably generated, projective modules are countably generated — in fact, by Corollary 3.10 of [12], finitely generated. For any $p \in P_0(A)$ we may form the graded projective module $p\widehat{\mathcal{H}}_A$ (with grading inherited from $\widehat{\mathcal{H}}_A$). Given $p, q \in P_0(A)$. We have $p \sim q$ if and only if $p\widehat{\mathcal{H}}_A \cong q\widehat{\mathcal{H}}_A$. Moreover, if $p \perp q$, then $(p+q)\widehat{\mathcal{H}}_A \cong p\widehat{\mathcal{H}}_A \oplus q\widehat{\mathcal{H}}_A$. By the stabilisation theorem (see Theorem 14.6.1 of [2]) every graded projective module is isomorphic to a summand of $\widehat{\mathcal{H}}_A$ and therefore is isomorphic to $p\widehat{\mathcal{H}}_A$ for some $p \in P_0(A)$. Thus we may and do regard $V_0(A)$ as the semigroup of isomorphism classes of graded projective modules over A with the binary operation given by direct sum, that is, $[X] + [Y] = [X \oplus Y]$ where X and Y are graded projective modules.

A graded projective module Z is said to be *degenerate* if there is a graded projective module X such that $Z \cong X \oplus X^{\mathrm{op}}$. For $p \in P_0(A)$, the graded projective module $p\widehat{\mathcal{H}}_A$ is degenerate if and only if there is an odd partial isometry v such that $p = v^*v + vv^*$. Let $D_0(A)$ denote the collection of isomorphism classes of degenerate graded projective modules in $P_0(A)$. Observe that $D_0(A)$ forms a submonoid of $V_0(A)$.

Let X, Y be graded projective modules; we write $X \approx Y$ if there are degenerate graded projective modules Z, W such that $X \oplus Z \cong Y \oplus W$. Then $\approx$ forms an equivalence relation on graded projective modules (coarser than isomorphism) and we let $L(A, \alpha)$ denote the collection of equivalence classes. We write $[X]_L$ for the equivalence class of the graded projective module X. It is routine to show that the direct sum of graded projective modules yields a well-defined binary operation on $L(A, \alpha)$ which makes it an abelian semigroup. Recall that $\ell : \mathbb{C} \to \mathcal{L}(X)$ denotes the left action of $\mathbb{C}$ by scalar multiplication on X.

PROPOSITION 9.1. *Let (A, α) be a unital graded C^*-algebra. The semigroup $L(A, \alpha)$ is an abelian group with inverse given by $-[X]_L = [X^{\mathrm{op}}]_L$ for X a graded projective module and zero element given by the class of the trivial module (or any degenerate module). There is a group homomorphism $\varpi : L(A, \alpha) \to K_0^{\mathrm{gr}}(A, \alpha)$ such that*

$$\varpi([X]_L) = [\ell, X, 0, \alpha_X] \in KK(\mathbb{C}, A) = K_0^{\mathrm{gr}}(A, \alpha)$$

for every graded projective module X.

Proof. Let X be a graded projective module. Then $[X]_L + [X^{\mathrm{op}}]_L = [X \oplus X^{\mathrm{op}}]_L = [0]_L$. The map ϖ is well defined since Example 3.5 shows that the Kasparov element associated to a degenerate graded projective module maps to zero and ϖ is clearly additive. ∎

CONJECTURE 9.2. *The homomorphism ϖ of Proposition* 9.1 *is an isomorphism.*

It should not be difficult to show that our conjecture holds when A is trivially graded. In this case $L(A, \mathrm{id}) \cong K_0(A)$ since graded projective modules over A are all of the form $X \cong Y \oplus Z^{\mathrm{op}}$ where Y and Z are trivially graded projective modules over A and $\alpha_X \cong (\mathrm{id}, -\mathrm{id})$. Moreover, $[Y_1 \oplus Z_1^{\mathrm{op}}]_L = [Y_2 \oplus Z_2^{\mathrm{op}}]_L$ if and only if $([Y_1], [Z_1]) \sim ([Y_2], [Z_2])$ in the Grothendieck group $K_0(A)$. This is closely related to the argument that $KK_0(\mathbb{C}, A) \cong K_0(A)$ for ungraded A; see Proposition 17.5.5 of [2].

Appendix A. TRANSITIVITY OF HOMOTOPY EQUIVALENCE IN KASPAROV THEORY

In this appendix, we provide a proof that homotopy equivalence is an equivalence relation among Kasparov A-B-modules. This is implicit in both [20] and [2], but — as the anonymous referee points out — it is not explicitly proved in either place. So it seems worthwhile to record a detailed proof here.

PROPOSITION A.1. *Let A and B be σ-unital graded C^*-algebras. The relation $\sim_{\mathrm{h}}$ is an equivalence relation on Kasparov A-B-modules.*

To prove the proposition, we need the following standard fact about the structure of $\mathcal{K}(X)$ if X is a right-Hilbert $C([0,1], B)$-module.

LEMMA A.2. *Suppose that B is a σ-unital C^*-algebra, that S is a compact Hausdorff space, and that X is a countably generated right-Hilbert $C(S, B)$-module. For $s \in S$ let $\varepsilon_s : C(S, B) \to B$ be given by evaluation at s. Then*

$$\mathcal{K}(X) = \{T \in \mathcal{L}(X) : \widetilde{\varepsilon}_s(T) \in \mathcal{K}(X \otimes_{\varepsilon_s} B) \text{ for all } s \in S\}.$$

Proof. First suppose that $T \in \mathcal{K}(X)$. Fix $s \in S$ and $\delta > 0$. Fix finitely many $\xi_i, \eta_i \in X$ and $b_i, c_i \in C(S, B)$ such that $\Big\| T - \sum_i \theta_{\xi_i \cdot b_i, \eta_i \cdot c_i} \Big\| < \delta$. For each i we have $\varepsilon_s(\theta_{\xi_i \cdot b_i, \eta_i \cdot c_i}) = \theta_{\xi_i \otimes_{\varepsilon_s} b_i, \eta_i \otimes_{\varepsilon_s} c_i}$, and since $\widetilde{\varepsilon}_s$ is norm decreasing we deduce that $\Big\| \widetilde{\varepsilon}_s(T) - \sum_i \theta_{\xi_i \otimes_{\varepsilon_s} b_i, \eta_i \otimes_{\varepsilon_s} c_i} \Big\| < \delta$. Therefore $\widetilde{\varepsilon}_s(T) \in \mathcal{K}(X \otimes_{\varepsilon_s} B)$. This proves $\subseteq$.

For $\supseteq$, we need a little preliminary work. It is routine to verify that there is a unital homomorphism $\iota : C(S) \to \mathcal{Z}\mathcal{L}(X)$ such that $\iota(f)(\xi \cdot b) = \xi \cdot (bf)$ for all $\xi \in X$ and $b \in C(S, B)$. Therefore $\mathcal{L}(X)$ is a $C(S)$-algebra. For $s \in S$, we write J_s for the ideal of $\mathcal{L}(X)$ generated by $\iota(\{f \in C(S) : f(s) = 0\})$. It is straightforward to check that $I_s = \{T \in \mathcal{L}(X) : \widetilde{\varepsilon}_s(T) = 0\}$, and therefore that $T + J_s \mapsto \widetilde{\varepsilon}_s(T)$ is an isomorphism of $\mathcal{L}(X)/J_s$ onto $\widetilde{\varepsilon}(L(X))$. In particular, $\|T + J_s\| = \|\widetilde{\varepsilon}_s(T)\|$ for all s. By Proposition C.23 and Theorem C.26 of [46] we then have $\|T\| = \sup_s \|\widetilde{\varepsilon}_s(T)\|$ for $T \in \mathcal{L}(X)$.

Now suppose that $T \in \mathcal{L}(X)$ and that $\widetilde{\varepsilon}_s(T) \in \mathcal{K}(X \otimes_{\varepsilon_s} B)$ for all $s \in S$. Fix an approximate identity $(E_i)_{i \in \mathbb{N}}$ for $\mathcal{K}(X)$. For each s, the sequence $(\widetilde{\varepsilon}_s(E_i))_{i \in \mathbb{N}}$

is an approximate identity for $\mathcal{K}(X \otimes_{\varepsilon_s} B)$. If $s_n \to s$ in S and if $T \in \mathcal{L}(X)$ and $\xi_i \in X$ and $b_i \in B$ for $1 \leqslant i \leqslant n$, then

$$\Big\|\widetilde{\varepsilon}_{s_n}(T)\Big(\sum_i \xi_i \otimes b_i\Big)\Big\|^2 = \Big\|\sum_{i,j} b_i^* \varepsilon_{s_n}(\langle T\xi_i, T\xi_j\rangle_{C([0,1],B)})b_j\Big\|$$
$$\to \Big\|\sum_{i,j} b_i^* \varepsilon_s(\langle T\xi_i, T\xi_j\rangle_{C([0,1],B)})b_j\Big\| = \Big\|\widetilde{\varepsilon}_s(T)\Big(\sum_i \xi_i \otimes b_i\Big)\Big\|^2.$$

Since finite linear combinations $\sum_i \xi_i \otimes b_i$ are dense in the unit ball of each $X \otimes_{\varepsilon_s} B$, we deduce that

$$\|\widetilde{\varepsilon}_s(T)\| = \sup_{\|x\|=1} \|\widetilde{\varepsilon}_s(T)(x)\|^2 = \sup_{\|x\|=1} \lim_n \|\widetilde{\varepsilon}_{s_n}(T)(x)\|^2$$
$$\leqslant \lim_n \sup_{\|x\|=1} \|\widetilde{\varepsilon}_{s_n}(T)(x)\|^2 = \lim_n \|\widetilde{\varepsilon}_{s_n}(T)\|,$$

and so $s \mapsto \|\widetilde{\varepsilon_s}(T)\|$ is lower semicontinuous. Using this, the compactness of S, and that $\varepsilon_s(E_iT) \to \varepsilon_s(T)$ pointwise with respect to s, we see that $\varepsilon_s(E_iT) \to \varepsilon_s(T)$ uniformly with respect to s. Therefore $\sup_s \|\varepsilon_s(E_iT) - \varepsilon_s(T)\| \to 0$, and by the preceding paragraph we obtain $E_iT \to T$. Hence $T \in \mathcal{K}(X)$. This proves $\supseteq$. ∎

Proof of Proposition A.1. Reflexivity is clear: if $(X_0, \phi_0, F_0, \alpha_{X_0})$ is a Kasparov A-B module, the external tensor product $(X_0 \otimes C([0,1]), \phi_0 \otimes \mathrm{id}, F_0 \otimes \mathrm{id}, \alpha_{X_0} \otimes \mathrm{id})$ is a homotopy from $(X_0, \phi_0, F_0, \alpha_{X_0})$ to itself. Symmetry is also clear: given a homotopy (X, ϕ, F, α_X) from $(X_0, \phi_0, F_0, \alpha_{X_0})$ to $(X_1, \phi_1, F_1, \alpha_{X_1})$, if we denote by $\mathcal{F} : C([0,1]) \to C([0,1])$ the flip map $\mathcal{F}(f)(t) = f(1-t)$, we see that

$$(X \mathbin{\widehat{\otimes}}_{\mathcal{F}\ C([0,1])} C([0,1])_{C([0,1])}, \widetilde{\mathcal{F}} \circ \phi, \widetilde{\mathcal{F}} \circ F, \alpha_X \mathbin{\widehat{\otimes}} \mathrm{id})$$

is a homotopy from $(X_1, \phi_1, F_1, \alpha_{X_1})$ to $(X_0, \phi_0, F_0, \alpha_{X_0})$. So we just have to establish transitivity.

Let (W, ϕ, F, α) be a homotopy from $(X_0, \phi_0, F_0, \alpha_{X_0})$ to $(X_1, \phi_1, F_1, \alpha_{X_1})$, and let (Y, ψ, G, β) be a homotopy from $(X_1, \phi_1, F_1, \alpha_{X_1})$ to $(X_2, \phi_2, F_2, \alpha_{X_2})$. By definition of homotopy, there are zero-graded unitaries $U_t \in \mathcal{L}(W \otimes_{\epsilon_t\ B} B_B, X_t)$ for $i = 0, 1$ and $V_t \in \mathcal{L}(Y \otimes_{\epsilon_{t-1}\ B} B_B, X_t)$ for $t = 1, 2$ that implement unitary equivalences

$$(W \otimes_{\epsilon_t\ B} B_B, \widetilde{\epsilon}_t \circ \phi, \widetilde{\epsilon}_t(F), \alpha \mathbin{\widehat{\otimes}} \alpha_B) \sim_{U_t} (X_t, \phi_t, F_t, \alpha_{X_t}) \quad \text{and}$$
$$(Y \otimes_{\epsilon_{t-1}\ B} B_B, \widetilde{\epsilon}_{t-1} \circ \psi, \widetilde{\epsilon}_{t-1}(G), \beta \mathbin{\widehat{\otimes}} \alpha_B) \sim_{V_t} (X_t, \phi_t, F_t, \alpha_{X_t}).$$

Consider the direct-sum module $W \oplus Y$ as an A-$(C([0,1], B) \oplus C([0,1], B))$-module. Observe that $W \otimes_{\epsilon_t\ B} B_B$ can be identified with the quotient module $W/\{w \in W : \langle w, w\rangle_{B \otimes C([0,1])} \in B \otimes C_0([0,1] \setminus \{t\})\}$ via the map that sends $w \otimes_{\epsilon_t} b$ to the coset of $w \cdot (b \otimes 1)$. For $w \in W$, we write w_t for the image of w in $W \otimes_{\epsilon_t\ B} B_B$ obtained from this isomorphism; and similarly for $y \in Y$ we write y_t for the image of y in $Y \otimes_{\epsilon_t\ B} B_B$.

Let $Z \subseteq W \oplus Y$ be the subspace

$$Z := \{(w, y) \in X \oplus Y : U_1 w_1 = V_1 y_0\}.$$

If $(w, y), (w', y') \in Z$, then, since U_1 and V_1 are unitary, we have

$$\begin{aligned}\varepsilon_1(\langle w, w' \rangle_{B \otimes C([0,1])}) &= \langle w_1, w_1' \rangle_B = \langle U_1 w_1, U_1 w_1' \rangle_B \\ &= \langle V_1 y_0, V_1 y_0' \rangle_B = \langle y_0, y_0' \rangle_B = \varepsilon_0(\langle y, y' \rangle_{B \otimes C([0,1])}).\end{aligned}$$

So, identifying $B \otimes C([0,1])$ with $C([0,1], B)$ as usual, we have

$$\begin{aligned}\langle Z, Z \rangle \subseteq C &:= \{(f, g) \in C([0,1], B) \oplus C([0,1], B) : f(1) = g(0)\} \\ &= \{f \in C([0,1] \times \{0,1\}, B) : f(1,0) = f(0,1)\}.\end{aligned}$$

Since adjointable operators on a Hilbert module are linear in the right action on the module, for $(f, g) \in C$ and $(w, y) \in Z$, we have $U_1((w \cdot f)_1) = U_1 w_1 \cdot f(1) = U_1 w_1 \cdot g(0) = V_1 y_0 \cdot g(0) = V_1((y \cdot g)_0)$ and so Z is invariant for the right action of C, and it is clearly norm-closed so it is a right-Hilbert C-module under the inner-product and action inherited from $W \oplus Y$.

By definition of unitary equivalence, U_1 intertwines the left A-actions on $W \otimes_{\epsilon_1} {}_B B_B$ and on X_1 and likewise V_1 intertwines the A-actions on X_1 and $Y \otimes_{\epsilon_0} {}_B B_B$. So for $a \in A$ and $(w, y) \in Z$, we have

$$\begin{aligned}U_1(\phi(a)w)_1 &= U_1(\widetilde{\epsilon}_1 \circ \phi(a))w_1 = \phi_1(a) U_1 w_1 = \phi_1(a) V_1 y_0 = V_1(\widetilde{\epsilon}_0 \circ \psi(a)) y_0 \\ &= V_1(\psi(a) y)_0,\end{aligned}$$

so Z is invariant for the left action of A on $W \oplus Y$, so is a Hilbert A-C-bimodule under this action. We write $\rho : A \to \mathcal{L}(Z)$ for the homomorphism $a \mapsto (\phi(a) \oplus \psi(a))|_Z$ that implements the left action.

By definition of unitary equivalence, we have $U_1(\alpha \mathbin{\widehat{\otimes}} \alpha_B) = \alpha_{X_1} U_1$ and $V_1(\beta \mathbin{\widehat{\otimes}} \alpha_B) = \alpha_{X_1} V_1$. So for $(w, y) \in Z$ we have

$$U_1 \alpha(w)_1 = U_1(\alpha \mathbin{\widehat{\otimes}} \alpha_B)(w_1) = \alpha_{X_1} U_1 w_1 = \alpha_{X_1} V_1 y_0 = V_1(\beta \mathbin{\widehat{\otimes}} \alpha_B)(y_0) = V_1(\beta y)_0.$$

So $\alpha \oplus \beta$ restricts to an operator $\gamma \in \mathcal{L}(Z)$, and we have $\gamma^2 = \alpha^2 \oplus \beta^2 = \mathrm{id} \oplus \mathrm{id}$. Moreover, this γ is a grading operator on Z because $\alpha \oplus \beta$ is a grading operator on $W \oplus Y$.

Similarly, we have $U_1 \widetilde{\epsilon}_1(F) = F_1 U_1$ and $V_1 \widetilde{\epsilon}_0(G) = F_1 V_1$. So for $(w, y) \in Z$ we have

$$U_1(Fw)_1 = U_1 \widetilde{\epsilon}_1(F) w_1 = F_1 U_1 w_1 = F_1 V_1 y_0 = V_1 \widetilde{\epsilon}_0(G) y_0 = V_1(Gy)_0.$$

So $F \oplus G$ restricts to an operator $H \in \mathcal{L}(Z)$. For $a \in A$, the operators $(H^2 - 1)\rho(a)$ and $(H - H^*)\rho(a)$ are the restrictions of the compact operators $(F^2 - 1)\phi(a) \oplus (G^2 - 1)\psi(a)$ and $(F - F^*)\phi(a) \oplus (G - G^*)\psi(a)$ to Z. Since $\gamma = (\alpha \oplus \beta)|_Z$, the induced grading of $\mathcal{L}(Z)$ is the restriction of the induced grading of $\mathcal{L}(W) \oplus \mathcal{L}(Y)$ as at (2.3), and so we have $[H, \rho(a)]^{\mathrm{gr}} = ([F, \phi(a)]^{\mathrm{gr}} \oplus [G, \psi(a)]^{\mathrm{gr}})|_Z$ for all $a \in A$. Again, since $[F, \phi(a)]^{\mathrm{gr}}$ and $[G, \psi(a)]^{\mathrm{gr}}$ are compact, we deduce that for every $a \in A$, the operator $[H, \rho(a)]^{\mathrm{gr}}$ is the restriction to Z of a compact operator on $W \oplus Z$.

We regard $W \oplus Y$ as a right-Hilbert $C([0,1] \times \{0,1\}, B)$-module and let $\widetilde{H} := F \oplus G$. Lemma A.2 applied to $W \oplus Y$ now shows that if T is of the form $(\widetilde{H}^2 - 1)\rho(a)$, $(\widetilde{H} - \widetilde{H}^*)\rho(a)$ or $[\widetilde{H}, \rho(a)]^{\mathrm{gr}}$ for some a, then $\widetilde{\varepsilon}_{(t,i)}(T)$ belongs to $\mathcal{K}((W \oplus Y) \otimes_{\varepsilon_{(t,i)}} B)$ for all $(t,i) \in [0,1] \times \{0,1\}$; similarly Lemma A.2 applied to Z regarded as a right-Hilbert $C([0,1], B)$-module under the canonical identification (see below)

$$\{f \in C([0,1] \times \{0,1\}, B) : f(1,0) = f(0,1)\} = C([0,1], B)$$

shows that $T|_Z \in \mathcal{K}(Z)$. Hence, $(H^2 - 1)\rho(a)$, $(H - H^*)\rho(a)$, $[H\rho(a)]^{\mathrm{gr}} \in \mathcal{K}(Z)$ for all a.

We have $H \circ \gamma = (F \circ \alpha \oplus G \circ \beta)|_Z = ((-\alpha \circ F) \oplus (-\beta \circ G))|_Z = -\gamma \circ H$.

We have now established that (Z, ρ, H, γ) is a Kasparov A-C-module. We can identify C with $C([0,1], B)$ via the isomorphism

$$(f,g) \mapsto \left(t \mapsto \begin{cases} f(2t) & \text{if } t \in [0, \frac{1}{2}], \\ g(2t-1) & \text{if } t \in [\frac{1}{2}, 1], \end{cases}\right),$$

and then with $B \otimes C([0,1])$ in the usual way. This identification makes(Z, ρ, H, γ) into a Kasparov A-$(B \otimes C([0,1]))$-module with

$$Z_0 = W \otimes_{\epsilon_0\ B} B_B \sim_{U_0} X_0 \quad \text{and} \quad Z_1 = Y \otimes_{\epsilon_1\ B} B_B \sim_{V_2} X_2.$$

That is, (Z, ρ, H, γ) is a homotopy from $(X_0, \phi_0, F_0, \alpha_{X_0})$ to $(X_2, \phi_2, F_2, \alpha_{X_2})$. ∎

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