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EQUIVALENCE OF FELL BUNDLES OVER GROUPS

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ABSTRACT. We give a notion of equivalence for Fell bundles over groups, not necessarily saturated nor separable. The equivalence between two Fell bundles is implemented by a bundle of Hilbert bimodules with some extra structure. Suitable cross-sectional spaces of such a bundle turn out to be imprimitivity bimodules for the cross-sectional C^*-algebras of the involved Fell bundles. We show that amenability is preserved under this equivalence.

KEYWORDS: *Fell bundles, Morita equivalence.*

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1. INTRODUCTION

Towards the end of the sixth decade of the past century, J.M.G. Fell introduced the notion of Banach $*$-algebraic bundle over a group, to study and extend the Mackey normal subgroup analysis (see [10] and the references therein). In particular he defined C^*-algebraic bundles, today known as Fell bundles. A Fell bundle can be thought of as the abstraction of a grading of a C^*-algebra over a locally compact and Hausdorff (LCH) group, and its cross-sectional C^*-algebras generalize crossed products by actions, even by twisted actions. The Fell bundles that have received more attention are those called *saturated*, in which every fiber B_t (over a group element t) is naturally a B_e-B_e-imprimitivity bimodule (e being the unit of the group). The main reason for this is probably that such bundles were enough to study most of the examples. That situation changed with the introduction of partial crossed products in the nineties (see [8], [9] and [12]), since partial crossed products give rise to non-saturated Fell bundles.

In the present work we introduce and study a notion of equivalence between arbitrary Fell bundles over groups, these understood as C^*-algebraic bundles as in [10]. This notion is already present, implicitly, in [1] and [3] (see the Examples 2.2.1 and 2.2.2 below), where it was used to prove Morita–Rieffel equivalences between several crossed products by partial actions or their enveloping actions (in this paper we use the expression Morita–Rieffel equivalence to mean strong Morita equivalence, as defined by Rieffel). In those works, in general at least one of the involved Fell bundles is not saturated, because it is the Fell bundle associated to a partial action, which is saturated if and only if the action is global. Still, both the reduced and the universal cross-sectional algebras of these Fell bundles are shown to be Morita–Rieffel equivalent, due to a kind of Morita equivalence between the bundles themselves. It is precisely this notion of equivalence between Fell bundles that we study in this article.

Many authors have studied equivalence of Fell bundles over *groupoids* (see [13], [15] and the references therein). In this setting the bundles considered are such that each fiber B_γ over any element γ of the groupoid is an imprimitivity bimodule that establishes a Morita–Rieffel equivalence between the fibers over $s(\gamma)$ and $r(\gamma)$, the source and range of γ respectively. In case the groupoid is actually a group, this means that the bundle is saturated, which is equivalent to the fact that, for all elements r, s in the group, the linear span of B_rB_s is dense in B_{rs}. Therefore this theory does not apply to non-saturated Fell bundles over groups, for instance those associated to partial actions.

An equivalence between groupoid based Fell bundles is possible even when the base groupoids, say G and H, are not isomorphic groupoids, but there exists a $(G\text{-}H)$-equivalence ([14], Definition 2.1) between them. In case G and H are groups, the existence of a $(G\text{-}H)$-equivalence implies that G and H are isomorphic as topological groups. Thus, when dealing with equivalence of Fell bundles over groups, we may suppose that the bundles have the same group as a base space.

Another feature of Fell bundles over groupoids (in the sense of [15]) is that the norm is supposed to be only upper semicontinuous instead of continuous, as must be the case for C^*-algebraic bundles [10]. However, when the base space is a group, an upper semicontinuous norm on the bundle is automatically continuous (see Lemma 3.30 of [6]).

On the other hand the notion of Fell bundle over a groupoid includes some separability conditions. The result is that, when the base space is actually a group, the usual notion of Fell bundle over a groupoid amounts to a separable and saturated C^*-algebraic bundle over a second countable group. Our aim is to remove all these restrictions, especially that of saturation, and even so to develop a useful notion of equivalence between Fell bundles.

The organization of our exposition is the following.

The next section is devoted to the introduction of equivalence bundles between Fell bundles, which can be thought of as the abstraction of a grading of an imprimitivity bimodule over a group. At the beginning we fix notations and review some aspects of the theory of Fell bundles, as well as some functors associated to them, especially the functors C^* and C^*_{r}, which to every Fell bundle associate its full (also called universal) and reduced cross-sectional C^*-algebras respectively. Then we present a couple of examples from [1] which have guided us, not only to the definition of Hilbert $\mathcal{B}$-bundles and equivalence bundles, but also to the proof that equivalent Fell bundles have Morita–Rieffel equivalent cross-sectional C^*-algebras. After the proof of an important technical result, Lemma 2.8, we define morphisms between equivalence bundles, thus obtaining a category, and we prove that there exist two special functors from this category into the category of Fell bundles.

The aim of the third section is to pave the way to the proof that equivalent Fell bundles give rise to Morita–Rieffel equivalent cross-sectional C^*-algebras, to be accomplished in Section 4. Namely, we proceed to the construction of the "linking Fell bundle" associated to an equivalence bundle. In fact, as it is shown later in the same section, it is possible to start just with a right Hilbert bundle $\mathcal{X}$ over the Fell bundle $\mathcal{B}$, because it is automatically a $\mathbb{K}(\mathcal{X})$-$\mathcal{B}$-equivalence bundle, where $\mathbb{K}(\mathcal{X})$ is a graded version of the usual C^*-algebra of generalized compact operators of a Hilbert module. The assignment of a linking Fell bundle to each equivalence bundle is itself a functor, which later will be shown, in Section 4, to commute with the functor C^*, suitably extended from the category of Fell bundles to the category of equivalence bundles. Essentially, what happens is the following. If $\mathcal{X}$ is an equivalence bundle between the Fell bundles $\mathcal{A}$ and $\mathcal{B}$, then there exists a $C^*(\mathcal{A})$-$C^*(\mathcal{B})$-imprimitivity bimodule $C^*(\mathcal{X})$, which is a certain completion of $C_c(\mathcal{X})$, such that if $\mathbb{L}(\mathcal{X})$ is the linking Fell bundle of $\mathcal{X}$, then $C^*(\mathbb{L}(\mathcal{X})) = \mathbb{L}(C^*(\mathcal{X}))$ (see Theorem 4.5 and Corollary 4.10).

A more general situation can be considered, and this is done at the end of Section 4, in which the functor C^* is replaced by other type of functors from the category of Fell bundles to the category of C^*-algebras, for instance the functor C^*_r. The functors we consider are a generalization of the crossed product functors considered in [5].

The last section of the article is quite technical. Its main objective is to prove that the equivalence of Fell bundles is transitive, thus an equivalence relation. To this end internal tensor products of Hilbert bundles are defined, and it is shown that their cross-sectional algebras are isomorphic to the internal tensor product of the corresponding cross-sectional algebras of the Hilbert bundles.

2. EQUIVALENCE BUNDLES

2.1. SOME PRELIMINARIES AND NOTATIONS. In this paper we will be dealing with Fell bundles over groups. Throughout the work G will always denote by a fixed locally compact and Hausdorff group with unit element e. We also fix a left invariant Haar measure, $\mathrm{d}t$, of G and denote Δ the modular function of G. We understand by a Fell bundle a C^*-algebraic bundle in the sense of Fell [10]. If $\mathcal{A}$ and $\mathcal{B}$ are Fell bundles over G, a *morphism* $\pi : \mathcal{A} \to \mathcal{B}$ is a continuous, multiplicative and $*$-preserving map, such that for each $t \in G$, $\pi(A_t) \subset B_t$ and $\pi|_{A_t}$ is linear. Fell bundles with these morphisms form a category $\mathscr{F}$. Note that every fiber of a Fell bundle is a C^*-ternary ring (in the sense of [18]) with the product $(a,b,c)\mapsto ab^*c$, and the restriction of a morphism of Fell bundles to a fiber is a homomorphism of C^*-ternary rings (see [2], [18]). See below for some additional information about C^*-ternary rings and their homomorphisms.

There are several functors of interest to us defined on the category $\mathscr{F}$. It is the purpose of this section to review some of them, and to introduce some notation along the way. The reader is referred to [1] for more details.

Recall that if $\mathcal{B}$ is a Fell bundle, then $C_c(\mathcal{B})$ is a $*$-algebra. Besides, given a morphism $\phi:\mathcal{A}\to\mathcal{B}$ of Fell bundles, we have a homomorphism of $*$-algebras: $\phi_c: C_c(\mathcal{A})\to C_c(\mathcal{B})$, $\phi_c(f)=\phi\circ f$. This functor ($\phi\mapsto\phi_c$) extends to a functor from Fell bundles into Banach $*$-algebras: $(\mathcal{A}\stackrel{\phi}{\to}\mathcal{B})\longmapsto(L^1(\mathcal{A})\stackrel{\phi_1}{\to}L^1(\mathcal{B}))$, where $L^1(\mathcal{A})$ and $L^1(\mathcal{B})$ are the Banach $*$-algebras obtained by completing respectively the $*$-algebras $C_c(\mathcal{A})$ and $C_c(\mathcal{B})$ with respect to the norm $\|f\|_1=\int_G\|f(t)\|\mathrm{d}t$, and ϕ_1 is the continuous extension of ϕ_c. Composing the latter functor with the functor from the category of Banach $*$-algebras into the category of C^*-algebras which consists of taking the enveloping C^*-algebra, we obtain another functor $C^*:\mathscr{F}\to\mathscr{C}$: $(\mathcal{A}\stackrel{\phi}{\to}\mathcal{B})\longmapsto(C^*(\mathcal{A})\stackrel{\phi^*}{\to}C^*(\mathcal{B}))$, from the category of Fell bundles into the category $\mathscr{C}$ of C^*-algebras (with their usual homomorphisms). We call $C^*(\mathcal{A})$ the *universal or full cross-sectional algebra* of $\mathcal{A}$.

The universal C^*-algebra of a Fell bundle has the property that its nondegenerate representations are in a bijective correspondence with the nondegenerate representations of the bundle ([10], VIII 12.8).

Given a Fell bundle $\mathcal{A}$, let $L^2(\mathcal{A})$ be the right Hilbert A_e-module obtained by completing $C_c(\mathcal{A})$ with respect to the inner product $\langle f,g\rangle:=\int_G f(t)^*g(t)\mathrm{d}t$. If $\mathcal{A}\stackrel{\phi}{\to}\mathcal{B}$ is a morphism of Fell bundles, we have a map $\phi_2: L^2(\mathcal{A})\to L^2(\mathcal{B})$, which is the continuous extension of ϕ_c (note $\langle\phi_c(f),\phi_c(g)\rangle_{L^2(\mathcal{B})}=\phi(\langle f,g\rangle_{L^2(\mathcal{A})})$, $\forall f,g\in C_c(\mathcal{A})$). The map ϕ_2 is a morphism of C^*-ternary rings (see [1], [2]). Thus $(\mathcal{A}\stackrel{\phi}{\to}\mathcal{B})\longmapsto(L^2(\mathcal{A})\xrightarrow{\phi_2}L^2(\mathcal{B}))$ is a functor from the category of Fell bundles into the category of C^*-ternary rings.

Among the representations of a Fell bundle $\mathcal{A}=(A_t)_{t\in G}$ there is the so called (left) regular representation, which we briefly recall now. If $a_t\in A_t$ and $\xi\in C_c(\mathcal{A})$, we define $\Lambda_{a_t}\xi\in C_c(\mathcal{B})$ by $\Lambda_{a_t}\xi(s):=a_t\xi(t^{-1}s)$. Then Λ_{a_t} extends to an adjointable map $\Lambda_{a_t}\in\mathbb{B}(L^2(\mathcal{A}))$, where $\mathbb{B}(L^2(\mathcal{A}))$ denotes the C^*-algebra of adjointable maps of $L^2(\mathcal{A})$. Besides, the map $\Lambda:\mathcal{A}\to\mathbb{B}(L^2(\mathcal{A}))$ is a representation of the bundle $\mathcal{A}$, called the regular representation of $\mathcal{A}$. The integrated form of Λ, that is, its associated representation $\Lambda^{\mathcal{A}}:C^*(\mathcal{A})\to\mathbb{B}(L^2(\mathcal{A}))$ is also called regular representation, and its image $C^*_r(\mathcal{A})$ is called the *reduced cross-sectional algebra* of $\mathcal{A}$. We say that $\mathcal{A}$ is amenable when $\Lambda^{\mathcal{A}}:C^*(\mathcal{A})\to C^*_r(\mathcal{A})$ is an isomorphism. Since $\Lambda^{\mathcal{A}}$ is injective on $C_c(\mathcal{A})$, we will consider $C^*_r(\mathcal{A})$ as a completion of $C_c(\mathcal{A})$.

If $\phi : \mathcal{A} \to \mathcal{B}$ is a morphism, it is easily checked that, for all $x \in C^*(\mathcal{A})$, ϕ_2 intertwines $\Lambda^{\mathcal{B}}_{\phi^*(x)}$ and $\Lambda^{\mathcal{A}}_x$: $\Lambda^{\mathcal{B}}_{\phi^*(x)}\phi_2 = \phi_2\Lambda^{\mathcal{A}}_x$. In case ϕ is surjective, both ϕ_2 and ϕ^* are also surjective, because they have closed and dense ranges, since $\phi_c(C_c(\mathcal{A}))$ is uniformly dense in $C_c(\mathcal{B})$ by 14.1 of [10]. So in this case $\phi^*(\ker \Lambda^{\mathcal{A}}) \subseteq \ker \Lambda^{\mathcal{B}}$, and ϕ^* induces a unique homomorphism $\phi_r : C^*_r(\mathcal{A}) \to C^*_r(\mathcal{B})$ such that the following diagram commutes:

$$\begin{tikzcd} C^*(\mathcal{A}) \arrow[r, "\Lambda_{\mathcal{A}}"] \arrow[d, "\phi^*"'] & C^*_r(\mathcal{A}) \arrow[d, "\phi_r"] \\ C^*(\mathcal{B}) \arrow[r, "\Lambda_{\mathcal{B}}"'] & C^*_r(\mathcal{B}). \end{tikzcd} \tag{2.1}$$

In case ϕ is not surjective, let $\mathcal{C}$ be its image. Then $\mathcal{C}$ is a Fell subbundle of $\mathcal{B}$, and $\psi : \mathcal{A} \to \mathcal{C}$ such that $\psi(a) := \phi(a)$, $\forall a \in \mathcal{A}$, is a surjective morphism, and $\phi = \iota\psi$, where $\iota : \mathcal{C} \to \mathcal{B}$ is the natural inclusion. By Proposition 3.2 of [1], $C^*_r(\mathcal{C})$ is isomorphic to the closure of $C_c(\mathcal{C})$ in $C^*_r(\mathcal{B})$, so we have an injective homomorphism $\iota_r : C^*_r(\mathcal{C}) \to C^*_r(\mathcal{B})$ such that $\iota_r\Lambda_{\mathcal{C}} = \Lambda_{\mathcal{B}}\iota^*$. Then the map $\phi_r := \iota_r\psi_r : C^*_r(\mathcal{A}) \to C^*_r(\mathcal{B})$ also makes commutative the diagram above, so again $\phi^*(\ker \Lambda^{\mathcal{A}}) \subseteq \ker \Lambda^{\mathcal{B}}$. Therefore we have another functor $C^*_r : \mathscr{F} \to \mathscr{C}$, such that $(\mathcal{A} \xrightarrow{\phi} \mathcal{B}) \longmapsto (C^*_r(\mathcal{A}) \xrightarrow{\phi_r} C^*_r(\mathcal{B}))$, and $\Lambda : C^* \to C^*_r$ is a natural transformation.

A particular type of Fell bundle, which is a guiding example for us, is that associated to a partial action $\alpha = (\{D_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ of a group G on a C^*-algebra A. This bundle, denoted $\mathcal{B}_\alpha$, has total space $\{(t,a) : a \in D_t\} \subseteq G \times A$ with the topology inherited from the product topology on $G \times A$. We set $\|(t,a)\| := \|a\|$. The operations on $\mathcal{B}_\alpha$ are defined as follows: $(t,a)^* := (t^{-1}, \alpha_{t^{-1}}(a^*))$, and $(r,a)(s,b) := (rs, \alpha_r(\alpha_r^{-1}(a)b))$. If X is an A-B-imprimitivity bimodule (see [17]) between the C^*algebras A and B, we will say that A and B are Morita–Rieffel equivalent (rather than strong Morita equivalent, the original terminology used by Rieffel), and X will be referred just as an equivalence bimodule between A and B. When convenient, Hilbert modules will be regarded as ternary C^*-rings (C^*-trings for short) and often we will work with homomorphisms of C^*-trings as in [1], [2], where the reader is referred to for more information. Here we just recall the basic definitions and properties. A C^*-tring is a Banach space X with a ternary product $X \times X \times X \to X$ that is linear in the odd variables and conjugate linear in the second variable, satisfies a certain associativity property, and moroever $\|(x,y,z)\| \leqslant \|x\|\,\|y\|\,\|z\|$ and $\|(x,x,x)\| = \|x\|^3$, $\forall x,y,z \in X$. For instance a right Hilbert B-module X can be seen as a C^*-tring with the product $(x,y,z) := x\langle y,z\rangle$. As shown in [18], this is almost the general case: for every C^*-tring X there exist a C^*-algebra X^r and a map $\langle\cdot,\cdot\rangle_r : X \times X \to X^r$ such that X is a full right Hilbert X^r-module (except that $\langle x,x\rangle_r$ does not need to be a positive element of X^r) that satisfies: $(x,y,z) = x\langle y,z\rangle_r$, $\forall x,y,z \in X$. Besides the pair $(\langle\cdot,\cdot\rangle_r, X^r)$ is unique up to canonical isomorphisms. There also exists an essentially unique pair $(\langle\cdot,\cdot\rangle_{\mathrm{l}}, X^{\mathrm{l}})$ with similar properties, but now X is a left X^{l}-module, and $(x,y,z) = \langle x,y\rangle_{\mathrm{l}} z$, $\forall x,y,z \in X$. A homomorphism $\pi : X \to Y$ of C^*-trings is a linear map that preserves the ternary products. Such a homomorphism π induces homomorphisms of C^*-algebras $\pi^{\mathrm{l}} : X^{\mathrm{l}} \to Y^{\mathrm{l}}$ and $\pi^{\mathrm{r}} : X^{\mathrm{r}} \to Y^{\mathrm{r}}$ determined by the properties $\pi^{\mathrm{l}}(\langle x,y\rangle_{\mathrm{l}})\pi(z) = (\pi(x),\pi(y),\pi(z)) = \pi(y)\pi^{\mathrm{r}}(\langle y,z\rangle_{\mathrm{l}})$, $\forall x,y,z \in X$. In this way we have two functors from the category of C^*-trings $\mathscr{T}$ into the category of C^*-algebras $\mathscr{C}$: the left functor $(X \stackrel{\pi}{\to} Y) \longmapsto (X^{\mathrm{l}} \stackrel{\pi^{\mathrm{l}}}{\to} Y^{\mathrm{l}})$, and the right functor: $(X \stackrel{\pi}{\to} Y) \longmapsto (X^{\mathrm{r}} \stackrel{\pi^{\mathrm{r}}}{\to} Y^{\mathrm{r}})$. We especially want to call the attention of the reader to the following facts: every homomorphism of C^*-trings is contractive; for such maps, being injective is equivalent to being an isometry and, finally, the image of a C^*-tring homomorphism is closed, and so a C^*-tring itself.

Finally we fix some more notation. Letters $\mathcal{H}$ and $\mathcal{K}$ will be used to denote Hilbert spaces, while (right) Hilbert modules [17] will be denoted X, or X_A if it is necessary to indicate the involved algebra A. On the set of adjointable operators from X_A to Y_A, $\mathbb{B}(X_A, Y_A)$, we consider the operator norm and we regard this set as a ternary C^*-ring [18] with the operation $(R,S,T) \mapsto RS^*T$. We denote by $\mathbb{K}(X_A, Y_A)$ the set of generalized compact operators, which is a Banach subspace of $\mathbb{B}(X_A, Y_A)$ (in fact an ideal of $\mathbb{B}(X_A, Y_A)$ in the sense of [1], [2]).

The letters $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ will be used to represent Fell bundles over G and $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ to denote Banach bundles over G (in the sense of [10]). The fiber over t of $\mathcal{A}, \mathcal{B}, \ldots, \mathcal{X}, \ldots$ will be denoted $A_t, B_t, \ldots, X_t, \ldots$ respectively. The space of compactly supported continuous sections of the Banach bundle $\mathcal{X}$ will be denoted $C_c(\mathcal{X})$.

2.2. MOTIVATING EXAMPLES. In [1] (see also [3]) it was shown that Morita equivalent partial actions give rise to Morita–Rieffel equivalent crossed products, and also that the crossed product of a partial action is Morita–Rieffel equivalent to the crossed product by its enveloping action (and even to the crossed product by its Morita enveloping action). These results were obtained as particular cases of more general results about cross-sectional algebras of Fell bundles. The involved Fell bundles are related by a kind of equivalence bundles, whose properties inspired our Definition 2.2, and which we briefly review below.

2.2.1. ENVELOPING ACTIONS. Let β be a continuous action by automorphisms of G on the C^*-algebra B and A an ideal of B, with $B = \overline{\operatorname{span}}\{\beta_t(A) : t \in G\}$. Now let α be the partial action obtained as the restriction of β to A [1], so that β is an enveloping action for α.

We think of $\mathcal{B}_\alpha$ and $\mathcal{X} := A \times G$ as Banach subbundles of $\mathcal{B}_\beta$. Since $\mathcal{B}_\alpha\mathcal{X} \subset \mathcal{X}$, $\mathcal{X}\mathcal{B}_\beta \subset \mathcal{X}$ and $\mathcal{X}\mathcal{X}^* \subset \mathcal{B}_\alpha$, we have the operations

$$\mathcal{B}_\alpha \times \mathcal{X} \to \mathcal{X}\ (a,x) \mapsto ax, \quad {}_{\mathcal{B}_\alpha}\langle\cdot,\cdot\rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{B}_\alpha\ (x,y) \mapsto xy^*, \tag{2.2}$$

$$\mathcal{X} \times \mathcal{B}_\beta \to \mathcal{X}\ (x,b) \mapsto xb, \quad \text{and} \quad \langle\cdot,\cdot\rangle_{\mathcal{B}_\beta} : \mathcal{X} \times \mathcal{X} \to \mathcal{B}_\beta\ (x,y) \mapsto x^*y. \tag{2.3}$$

Note that, if $X_r := \{(a,r) : a \in A\}$, and A_r, B_r are respectively the fibers of $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$ at r, then we have that ${}_{\mathcal{B}_\alpha}\langle X_r, X_s\rangle \subseteq A_{rs^{-1}}$, $\langle X_r, X_s\rangle_{\mathcal{B}_\beta} \subseteq B_{r^{-1}s}$, $A_r X_s \subseteq X_{rs}$, $X_r B_s \subseteq X_{rs}$, $A_t = \overline{\operatorname{span}}\{{}_{\mathcal{B}_\alpha}\langle X_{ts}, X_s\rangle : s \in G\}$, and $B_t = \overline{\operatorname{span}}\{\langle X_r, X_{rt}\rangle_{\mathcal{B}_\beta} : s \in G\}$, $\forall t \in G$. These are precisely the properties that inspire the definition of equivalence between Fell bundles in (2.2).

It is shown in [1] that $C^*_{\mathrm{r}}(\mathcal{B}_\alpha)$ is a full hereditary C^*-subalgebra of $C^*_{\mathrm{r}}(\mathcal{B}_\beta)$. Moreover, the canonical equivalence bimodule is the closure of $C_{\mathrm{c}}(\mathcal{X})$ within $C^*_{\mathrm{r}}(\mathcal{B}_\beta)$. In terms of the operations described in (2.2) and (2.3), the bimodule structure is given by

$$uf(t) = \int_G u(tr)f(r^{-1})\,\mathrm{d}r \tag{2.4}$$

$$fv(t) = \int_G f(r)v(r^{-1}t)\,\mathrm{d}r \tag{2.5}$$

$${}_{C_{\mathrm{c}}(\mathcal{B}_\alpha)}\langle f, g\rangle(t) = \int_G {}_{\mathcal{B}_\alpha}\langle f(tr), g(r)\rangle \Delta(r)\,\mathrm{d}r \tag{2.6}$$

$$\langle f, g\rangle_{C_{\mathrm{c}}(\mathcal{B}_\beta)}(t) = \int_G \langle f(r), g(rt)\rangle_{\mathcal{B}_\beta}\,\mathrm{d}r \tag{2.7}$$

for $u \in C_{\mathrm{c}}(\mathcal{B}_\alpha)$, $f, g \in C_{\mathrm{c}}(\mathcal{X})$ and $v \in C_{\mathrm{c}}(\mathcal{B}_\beta)$. Note that $\mathcal{B}_\beta$ is a saturated Fell bundle (i.e.: $\operatorname{span} B_r B_s$ is dense in B_{rs}, $\forall r, s \in G$), while $\mathcal{B}_\alpha$ is saturated only if α is a global action, that is: $A = B$ and $\alpha = \beta$.

2.2.2. MORITA EQUIVALENCE OF PARTIAL ACTIONS. Assume now that α and β are Morita equivalent partial actions of G on the C^*-algebras A and B, respectively. Thus (using the notation of [1]) there exists a partial action $\gamma = (\{X_t\}_{t\in G}, \{\gamma_t\}_{t\in G})$ of G on the A-B-equivalence bimodule X with $\alpha = \gamma^{\mathrm{l}}$ and $\beta = \gamma^{\mathrm{r}}$. If $\mathbb{L}(\gamma)$ is the linking partial action of γ, then $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$ are Fell subbundles of $\mathcal{B}_{\mathbb{L}(\gamma)}$ and we may think of $\mathcal{B}_{\mathbb{L}(\gamma)}$ as a Banach subbundle of $G \times \mathbb{L}(X)$. Recall that in Exel's notation the element $(t, S) \in G \times \mathbb{L}(X)$ is represented by $S\delta_t$. Consider

$$\mathcal{X}_\gamma := \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \delta_t : x \in X_t,\ t \in G \right\}. \tag{2.8}$$

Then $\mathcal{X}_\gamma$ is a Banach subbundle of $\mathcal{B}_{\mathbb{L}(\gamma)}$.

In this situation it is possible to define the following operations:

$$\mathcal{B}_\alpha \times \mathcal{X}_\gamma \to \mathcal{X}_\gamma : (a, x) \mapsto ax, \quad {}_{\mathcal{B}_\alpha}\langle \cdot, \cdot\rangle : \mathcal{X}_\gamma \times \mathcal{X}_\gamma \to \mathcal{B}_\alpha : (x, y) \mapsto xy^*, \tag{2.9}$$

$$\mathcal{X}_\gamma \times \mathcal{B}_\beta \to \mathcal{X}_\gamma : (x, b) \mapsto xb, \quad \text{and} \quad \langle \cdot, \cdot\rangle_{\mathcal{B}_\beta} : \mathcal{X}_\gamma \times \mathcal{X}_\gamma \to \mathcal{B}_\beta : (x, y) \mapsto x^*y. \tag{2.10}$$

Using the product and involution of $C_{\mathrm{c}}(\mathcal{B}_{\mathbb{L}(\gamma)})$ we can make $C_{\mathrm{c}}(\mathcal{X}_\gamma)$ into a pre $C_{\mathrm{c}}(\mathcal{B}_\alpha)$-$C_{\mathrm{c}}(\mathcal{B}_\beta)$ Hilbert bimodule. More precisely, the operations are given by

$$uf := u * f, \quad fv := f * v, \ {}_{C_{\mathrm{c}}(\mathcal{B}_\alpha)}\langle f, g\rangle := f * g^* \quad \text{and} \quad \langle f, g\rangle_{C_{\mathrm{c}}(\mathcal{B}_\alpha)} := f^* * g,$$

for $u \in C_c(\mathcal{B}_\alpha)$, $v \in C_c(\mathcal{B}_\beta)$ and $f, g \in C_c(\mathcal{X}_\gamma)$. It is easy to check that the operations above correspond to the expressions (2.4)–(2.7).

The techniques of Section 3 in [1] can be used to show that the completion of $C_c(\mathcal{X}_\gamma)$ in $C^*(\mathcal{B}_{\mathbb{L}(\gamma)})$, $C^*(\mathcal{X}_\gamma)$, is a $C^*(\mathcal{B}_\alpha)$-$C^*(\mathcal{B}_\beta)$ Hilbert module. Moreover, $C^*(\mathcal{B}_{\mathbb{L}(\gamma)})$ is isomorphic to the linking algebra of $C^*(\mathcal{X}_\gamma)$ (see Theorem 1.1 of [3] and Corollary 5.3 of [2] for details). This will turn out to be a particular case of a general situation (Theorem 4.5).

2.3. EQUIVALENCE BUNDLES. A Fell bundle is an abstraction of a grading of a C^*-algebra over a group. In the same way, a Hilbert $\mathcal{B}$-bundle over a group, as we define below, can be thought of as an abstraction of a graded Hilbert module.

Let $\mathcal{A}$ and $\mathcal{B}$ be Fell bundles over G. Considering our motivating examples and the discussion in the Introduction we state the following definition.

DEFINITION 2.1. A *right Hilbert $\mathcal{B}$-bundle* is a complex Banach bundle over G, $\mathcal{X} := \{X_t\}_{t\in G}$, with continuous maps

$$\mathcal{X} \times \mathcal{B} \to \mathcal{X}, \quad (x, b) \mapsto xb, \quad \text{and} \quad \langle \cdot, \cdot \rangle_{\mathcal{B}} : \mathcal{X} \times \mathcal{X} \to \mathcal{B}, \quad (x, y) \mapsto \langle x, y \rangle_{\mathcal{B}}$$

such that:

(1R) $X_r B_s \subset X_{rs}$ and $\langle X_r, X_s \rangle_{\mathcal{B}} \subset B_{r^{-1}s}$, for all $r, s \in G$.

(2R) $X_r \times B_s \to X_{rs}$, $(x, b) \mapsto xb$, is bilinear for all $r, s \in G$.

(3R) $X_s \to B_{r^{-1}s}$, $y \mapsto \langle x, y \rangle_{\mathcal{B}}$, is linear for all $x \in X_r$ and $s \in G$.

(4R) $\langle x, yb \rangle_{\mathcal{B}} = \langle x, y \rangle_{\mathcal{B}} b$ and $\langle x, y \rangle_{\mathcal{B}}^* = \langle y, x \rangle_{\mathcal{B}}$ for all $x, y \in \mathcal{X}$ and $b \in \mathcal{B}$.

(5R) $\langle x, x \rangle_{\mathcal{B}} \geqslant 0$, for all $x \in \mathcal{X}$, and $\langle x, x \rangle_{\mathcal{B}} = 0$ implies $x = 0$. Besides, each fiber X_t is complete with respect to the norm $x \mapsto \|\langle x, x \rangle_{\mathcal{B}}\|^{1/2}$.

(6R) For all $x \in \mathcal{X}$, $\|x\|^2 = \|\langle x, x \rangle_{\mathcal{B}}\|$.

(7R) $\overline{\operatorname{span}}\,\{\langle X_s, X_s \rangle_{\mathcal{B}} : s \in G\} = B_e$.

Condition (6R) expresses the compatibility of the norm of the fibers of $\mathcal{X}$ with the norm given by considering each one of the fibers as a right pre Hilbert B_e-module (with the action and inner product specified in the definition).

A word must be said about condition (7R). As mentioned before, we may think of a Hilbert bundle as a grading of a Hilbert module. In this sense, conditions (1R)–(6R) would be enough to define a Hilbert $\mathcal{B}$-bundle. However, along this work we are mainly interested in obtaining equivalence bimodules as completions of cross-sectional spaces of Hilbert bundles, so we consider just Hilbert bundles analogous to *full* Hilbert modules, that is, Hilbert bundles satisfying (7R). On the other hand, it is this crucial property that will allow to have, for instance, an equivalence between a saturated Fell bundle with a non-saturated one. Note that we are not requiring, unlike the usual notion in the Fell bundles over groupoids context, that $\overline{\operatorname{span}}\,\langle X_s, X_s \rangle_{\mathcal{B}} = B_e$, $\forall s \in G$, but only that the sum of *all* these spaces is dense in B_e. Note that this also implies $B_t = \overline{\operatorname{span}}\,\{\langle X_r, X_{rt} \rangle_{\mathcal{B}} : r \in G\}$, $\forall t \in G$ (a proof of this fact will be provided in (vi) of Lemma 2.7).

Left Hilbert bundles are defined similarly: properties (1R)–(7R) are changed to:

(1L) $A_r X_s \subset X_{rs}$ and ${}_{\mathcal{A}}\langle X_r, X_s\rangle \subset A_{rs^{-1}}$, for all $r,s \in G$.
(2L) $A_r \times X_s \to X_{rs}$, $(a,x) \mapsto ax$, is bilinear for all $r,s \in G$.
(3L) $X_s \to A_{sr^{-1}}$, $y \mapsto {}_{\mathcal{A}}\langle y,x\rangle$, is linear for all $x \in X_r$ and $s \in G$.
(4L) ${}_{\mathcal{A}}\langle ax,y\rangle = a\,{}_{\mathcal{A}}\langle x,y\rangle$ and ${}_{\mathcal{A}}\langle x,y\rangle^* = {}_{\mathcal{A}}\langle y,x\rangle$ for all $x,y \in \mathcal{X}$ and $a \in \mathcal{A}$.
(5L) ${}_{\mathcal{A}}\langle x,x\rangle \geqslant 0$, for all $x \in \mathcal{X}$, and ${}_{\mathcal{A}}\langle x,x\rangle = 0$ implies $x = 0$. Besides, each fiber X_t is complete with respect to the norm $x \mapsto \|{}_{\mathcal{A}}\langle x,x\rangle\|^{1/2}$.
(6L) For all $x \in \mathcal{X}$, $\|x\|^2 = \|{}_{\mathcal{A}}\langle x,x\rangle\|$.
(7L) $\overline{\operatorname{span}}\,\{{}_{\mathcal{A}}\langle X_s, X_s\rangle : s \in G\} = A_e$.

DEFINITION 2.2. We say that $\mathcal{X}$ is an *$\mathcal{A}$-$\mathcal{B}$-equivalence bundle* if it is both a left Hilbert $\mathcal{A}$-bundle, a right Hilbert $\mathcal{B}$-bundle and ${}_{\mathcal{A}}\langle x,y\rangle z = x\langle y,z\rangle_{\mathcal{B}}$, for all $x,y,z \in \mathcal{X}$. Besides, we say that $\mathcal{A}$ is equivalent to $\mathcal{B}$ if there exists an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle.

EXAMPLE 2.3. In the context of Section 2.2.1, $\mathcal{X}$ is a $\mathcal{B}_\alpha$-$\mathcal{B}_\beta$-equivalence bundle.

EXAMPLE 2.4. Suppose that γ is a partial action on the A-B Hilbert bimodule X, as in Section 2.2.2. Then the bundle $\mathcal{X}_\gamma$ is a $\mathcal{B}_{\gamma^{\mathrm{l}}}$-$\mathcal{B}_{\gamma^{\mathrm{r}}}$-equivalence bundle.

EXAMPLE 2.5. If $\pi : \mathcal{A} \to \mathcal{B}$ is an isomorphism of Fell bundles, then the bundle $\mathcal{X} = \mathcal{B}$ is an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle with the operations

$$\mathcal{A} \times \mathcal{X} \to \mathcal{X} : (a,x) \mapsto \pi(a)x, \quad {}_{\mathcal{A}}\langle\cdot,\cdot\rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A} : (x,y) \mapsto \pi^{-1}(xy^*),$$
$$\mathcal{X} \times \mathcal{B} \to \mathcal{X} : (x,b) \mapsto xb, \quad \text{and} \quad \langle\cdot,\cdot\rangle_{\mathcal{B}} : \mathcal{X} \times \mathcal{X} \to \mathcal{B} : (x,y) \mapsto x^*y.$$

Then isomorphic Fell bundles are equivalent and every Fell bundle is equivalent to itself.

There are two questions that can be immediately answered using our motivating examples: Do equivalent Fell bundles have Morita–Rieffel equivalent unit fibers? Can a non-saturated Fell bundle be equivalent to a saturated Fell bundle?

To answer both questions at once consider the action β of $G = \mathbb{R}$ on $B = C_0(\mathbb{R})$ given by $\beta_t(f(\cdot)) = f(\cdot + t)$ and let A be the C^*-ideal of B corresponding to the open set $(0,1) \cup (1,2)$. Since β is the enveloping action of α, $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$ are equivalent by Section 2.2.1. Now the unit fibers of $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$ are $C_0((0,1) \cup (1,2))$ and $C_0(\mathbb{R})$, respectively, which are not Morita–Rieffel equivalent because they are commutative and not isomorphic. Also note that $\mathcal{B}_\beta$ is saturated but $\mathcal{B}_\alpha$ is not.

An interesting fact to be proved later in Corollary 5.15, which will follow from the main result of [1] and the transitivity of equivalence of Fell bundles, is that every Fell bundle associated to a partial action is equivalent to the Fell bundle associated to an action.

As a preparation for future sections we prove some basic facts about right Hilbert bundles. Of course left Hilbert bundles will have similar properties. A way of translating results from left to right and vice versa is to consider adjoint bundles, which we introduce next.

Assume that $\mathcal{X}$ is a left Hilbert $\mathcal{A}$-bundle over G, and let $\widetilde{\mathcal{X}}$ be the retraction of $\mathcal{X}$ by the inversion map of G (according to II 13.3 of [10]), except that the product by scalars is given by $\lambda\widetilde{x} = \widetilde{\overline{\lambda}x}$, where $\lambda \in \mathbb{C}$ and $\widetilde{x}$ is the element $x \in \mathcal{X}$ seen as an element of $\widetilde{X}$. Thus the fiber of $\widetilde{\mathcal{X}}$ over $t \in G$ is $\widetilde{X}_{t^{-1}}$, where $\widetilde{X}_t$ is the complex-conjugate Banach space of X_t. Then the *adjoint* of $\mathcal{X}$ is the right Hilbert $\mathcal{A}$-bundle $\widetilde{\mathcal{X}}$ where the action $\widetilde{\mathcal{X}} \times \mathcal{A} \to \widetilde{\mathcal{X}}$ is given by $(\widetilde{x}, a) \mapsto \widetilde{a^*x}$, and the inner product $\widetilde{\mathcal{X}} \times \widetilde{\mathcal{X}} \to \mathcal{A}$ by $(\widetilde{x}, \widetilde{y}) \mapsto {}_{\mathcal{A}}\langle x, y\rangle$.

REMARK 2.6. A similar construction can be performed on a right Hilbert bundle to obtain a left Hilbert bundle. In case $\mathcal{X}$ is an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle, $\widetilde{\mathcal{X}}$ is a $\mathcal{B}$-$\mathcal{A}$-equivalence bundle. Also note that $\widetilde{\widetilde{\mathcal{X}}} = \mathcal{X}$.

LEMMA 2.7. *Given a right Hilbert $\mathcal{B}$-bundle $\mathcal{X}$, for all $x, y \in \mathcal{X}$ and $b, c \in \mathcal{B}$, the following relations hold:*

(i) $\langle xb, y\rangle_{\mathcal{B}} = b^*\langle x, y\rangle_{\mathcal{B}}$;
(ii) $\|xb\| \leqslant \|x\|\|b\|$;
(iii) $\|\langle x, y\rangle_{\mathcal{B}}\| \leqslant \|x\|\|y\|$;
(iv) $(xb)c = x(bc)$;
(v) $\|b\| = \sup\{\|zb\| : z \in \mathcal{X}, \|z\| \leqslant 1\}$;
(vi) $B_t = \overline{\operatorname{span}}\,\{\langle X_r, X_{rt}\rangle_{\mathcal{B}} : r \in G\}$.

In case $\mathcal{X}$ is an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle, the following equalities hold for all $a \in \mathcal{A}$, $x, y \in \mathcal{X}$, and $b \in \mathcal{B}$:

(vii) $(ax)b = a(xb)$;
(viii) ${}_{\mathcal{A}}\langle xb, y\rangle = {}_{\mathcal{A}}\langle x, yb^*\rangle$;
(ix) $\langle ax, y\rangle_{\mathcal{B}} = \langle x, a^*y\rangle_{\mathcal{B}}$.

Proof. The proof of (i) is left to the reader. Meanwhile (ii) holds because

$$\|xb\|^2 = \|\langle xb, xb\rangle_{\mathcal{B}}\| = \|b^*\langle x, x\rangle_{\mathcal{B}} b\| \leqslant \|x\|^2\|b\|^2. \tag{2.11}$$

Note that $x\langle x, y\rangle_{\mathcal{B}}$ and y belong to the same fiber of $\mathcal{X}$, say X_t. Regarding X_t as a right Hilbert B_e-module we have:

$$\|\langle x, y\rangle_{\mathcal{B}}\|^2 = \|\langle x\langle x, y\rangle_{\mathcal{B}}, y\rangle_{\mathcal{B}}\| \leqslant \|x\langle x, y\rangle_{\mathcal{B}}\|\|y\| \leqslant \|\langle x, y\rangle_{\mathcal{B}}\|\|x\|\|y\|,$$

so (iii) follows. To prove (iv) note that, since $(xb)c$ and $x(bc)$ belong to the same fiber, say X_t, the identity $(xb)c = x(bc)$ holds if and only if $\langle z, (xb)c\rangle_{\mathcal{B}} = \langle z, x(bc)\rangle_{\mathcal{B}}$, for all $z \in X_t$, and the latter equality holds, because

$$\langle z, (xb)c\rangle_{\mathcal{B}} = \langle z, xb\rangle_{\mathcal{B}} c = \langle z, x\rangle_{\mathcal{B}} bc = \langle z, x(bc)\rangle_{\mathcal{B}}, \quad \forall z \in X_t.$$

To prove (v) set $\tau(b) := \sup\{\|xb\| : x \in \mathcal{X}, \|x\| \leqslant 1\}$. From (ii) follows that $\tau(b) \leqslant \|b\|$. Consider each fiber X_t as a left $\mathbb{K}(X_t)$ Hilbert module, and let B_e^{op} be the opposite C^*-algebra of B_e. Then we have a representation $\phi_t : B_e^{\mathrm{op}} \to \mathbb{B}_{\mathbb{K}(X_t)}(X_t)$, $\phi_t(c)u = uc$. Note that

$$\|\phi_t(c)\|^2 = \sup\{\|{}_{\mathbb{K}(X_t)}\langle xc, xc\rangle\| : \|x\| \leqslant 1\} = \sup\{\|\langle xc, xc\rangle_{\mathcal{B}}\| : \|x\| \leqslant 1\}. \tag{2.12}$$

Since condition (7R) together with $\phi_t(c) = 0$ for all $t \in G$ implies $B_e c = 0$, thus $c = 0$, the direct sum $\phi = \bigoplus_{t\in G} \phi_t$ is injective, and therefore isometric. This implies $\|b\| = \tau(b)$ because by (2.12) we have

$$\|b\| = \sup_{t\in G} \|\phi_t(b)\| = \sup_{t\in G} \sup\{\|xb\| : x \in X_t,\ \|x\| \leqslant 1\} = \tau(b).$$

Finally, if b is any element of $\mathcal{B}$, we have $\|b\|^2 = \tau(bb^*) \leqslant \tau(b)\|b\|$, which shows that $\|b\| \leqslant \tau(b)$.

As for (vi), from VIII 16.3 of [10] and (7R) it follows that

$$B_t = \overline{\operatorname{span}}\,\{\langle X_r, X_r\rangle_{\mathcal{B}} B_t : r \in G\} = \overline{\operatorname{span}}\,\{\langle X_r, X_{rt}\rangle_{\mathcal{B}} : r \in G\} \subset B_t.$$

For the rest of the proof we assume that $\mathcal{X}$ is an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle. Using the conclusion of the last paragraph together with continuity arguments, we see that it suffices to show (vii)–(ix) for elements a and b of the form $a = {}_{\mathcal{A}}\langle u, v\rangle$ and $b = \langle z, w\rangle_{\mathcal{B}}$. The proof finishes after we observe that

$$a(xb) = {}_{\mathcal{A}}\langle u, v\rangle(x\langle z, w\rangle_{\mathcal{B}}) = {}_{\mathcal{A}}\langle {}_{\mathcal{A}}\langle u, v\rangle x, z\rangle w = ({}_{\mathcal{A}}\langle u, v\rangle x)\langle z, w\rangle_{\mathcal{B}} = (ax)b;$$
$${}_{\mathcal{A}}\langle xb, y\rangle = {}_{\mathcal{A}}\langle x\langle z, w\rangle_{\mathcal{B}}, y\rangle = {}_{\mathcal{A}}\langle x, z\rangle_{\mathcal{A}}\langle w, y\rangle = {}_{\mathcal{A}}\langle x, y\langle w, \rangle_{\mathcal{B}}\rangle = {}_{\mathcal{A}}\langle x, yb^*\rangle;$$
$$\langle ax, y\rangle_{\mathcal{B}} = \langle u\langle v, x\rangle_{\mathcal{B}}, y\rangle_{\mathcal{B}} = \langle x, v\rangle_{\mathcal{B}}\langle u, y\rangle_{\mathcal{B}} = \langle x, {}_{\mathcal{A}}\langle v, u\rangle y\rangle_{\mathcal{B}} = \langle x, a^* y\rangle_{\mathcal{B}}. \quad ∎$$

Approximate units of Fell bundles are a powerful tool. In what follows we construct a special kind of approximate units, which will prove to be extremely useful.

The expression $\mathbb{M}_n(X)$ stands for the $n \times n$ matrices with entries in the set X. The (i, j) entry of $M \in \mathbb{M}_n(X)$ will be denoted $M_{i,j}$.

LEMMA 2.8 (cf. Lemma 5.1 of [2]). *Let $\mathcal{X}$ be an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle. Then*

(i) *$\mathcal{A}$ and $\mathcal{B}$ have approximate units, $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$, such that for all $i \in I$ and $j \in J$ there exist $x_1, \ldots, x_{n_i}, y_1, \ldots, y_{n_j} \in \mathcal{X}$ such that*

$$a_i = \sum_{k=1}^{n_i} {}_{\mathcal{A}}\langle x_k, x_k\rangle \quad \textit{and} \quad b_j = \sum_{k=1}^{n_j} \langle y_k, y_k\rangle_{\mathcal{B}}.$$

(ii) *For all $\mathbf{t} = (t_1, \ldots, t_n) \in G^n$, the set*

$$\mathbb{M}_{\mathbf{t}}(\mathcal{B}) := \{M \in \mathbb{M}_n(\mathcal{B}) : M_{i,j} \in B_{t_i^{-1}t_j}\ \forall\, i, j = 1, \ldots, n\}$$

is a C^*-algebra with entrywise vector space operations, matrix multiplication as product and $*$-transpose (${M^*}_{i,j} = {M_{j,i}}^*$) as involution. Moreover, its C^*-norm is equivalent to the supremum norm $\|M\|_\infty := \max_{i,j} \|M_{i,j}\|$.

(iii) *For all $t, r_i \in G$, $x_i \in X_{r_i}$ and $y_i \in X_{r_i t}$ ($i = 1, \ldots, n$):*

$$\Big\|\sum_{i=1}^{n}\langle x_i, y_i\rangle_{\mathcal{B}}\Big\|^2 \leqslant \Big\|\sum_{i=1}^{n}\langle x_i, x_i\rangle_{\mathcal{B}}\Big\|\Big\|\sum_{i=1}^{n}\langle y_i, y_i\rangle_{\mathcal{B}}\Big\|. \tag{2.13}$$

Proof. Let Λ be the set

$$\Big\{b \in B_e : \|b\| < 1,\ \exists\, y_1, \ldots, y_n \in \mathcal{X} \text{ such that } b = \sum_{j=1}^{n} \langle y_k, y_k\rangle_{\mathcal{B}}\Big\}.$$

To show that Λ is a directed set take $b_1, b_2 \in \Lambda$, with $b_j = \sum_{k=1}^{n} \langle x_k^j, x_k^j\rangle_{\mathcal{B}}$. Each fiber X_t is a Hilbert B_e-module, so we may think of X_t as a Hilbert module over the unitization of B_e. Set

$$c' := b_1(1-b_1)^{-1} + b_2(1-b_2)^{-1} = \sum_{j=1}^{2}\sum_{k=1}^{n} \langle x_k^j(1-b_j)^{-1/2}, x_k^j(1-b_j)^{-1/2}\rangle_{\mathcal{B}}.$$

Then $c' \geqslant 0$ and, using functional calculus, we see that $c := c'(1+c')^{-1}$ equals

$$\sum_{j=1}^{2}\sum_{k=1}^{n} \langle x_k^j(1-b_j)^{-1/2}(1+c')^{-1/2}, x_k^j(1-b_j)^{-1/2}(1+c')^{-1/2}\rangle_{\mathcal{B}},$$

and belongs to Λ. Moreover, it can be shown that $b_1, b_2 \leqslant c$ (see page 78 of [16]).

To show that $\{\lambda\}_{\lambda\in\Lambda}$ is an approximate unit of B_e (and so also of $\mathcal{B}$) it suffices to show that $\|b - b\lambda\| \to 0$, for all $b \in B_e^+$ with $\|b\| < 1$. From the proof of Theorem 3.1.1 of [16] we know $\lambda \mapsto \|b - b\lambda\|$ is decreasing, so it suffices to show that, given $\varepsilon > 0$, there is $\lambda \in \Lambda$ such that $\|b - b\lambda\| < \varepsilon$. To this end, fix $\varepsilon > 0$ and consider the right Hilbert B_e-module obtained as the direct sum of all the right B_e-Hilbert modules X_t, $M := \bigoplus_{t\in G} X_t$, which is full by (7R). From Lemma 7.2 of [11] we know there exist $\xi_1, \ldots, \xi_n \in M$ such that $\Big\|b - b\sum_{k=1}^{n} \langle \xi_k, \xi_k\rangle\Big\| < \varepsilon$ and $\Big\|\sum_{k=1}^{n} \langle \xi_k, \xi_k\rangle\Big\| < 1$. Since $\sum_{k=1}^{n} \langle \xi_k, \xi_k\rangle$ lies in the closure of Λ, there exists $\lambda \in \Lambda$ such that $\|b - b\lambda\| < \varepsilon$.

As for the $*$-algebra structure of $\mathbb{M}_{\mathbf{t}}(\mathcal{B})$, note that the product and involution are defined, because given $M, N \in \mathbb{M}_{\mathbf{t}}(\mathcal{B})$ we have $M_{i,k}N_{k,j} \in B_{t_i t_j^{-1}}$ and $M_{i,j}{}^* \in B_{t_i t_j^{-1}}{}^* = B_{t_j t_i^{-1}}$. The routine algebraic verifications needed to see that $\mathbb{M}_{\mathbf{t}}(\mathcal{B})$ is a $*$-algebra are left to the reader.

In order to define a C^*-norm on $\mathbb{M}_{\mathbf{t}}(\mathcal{B})$, take a representation $T : \mathcal{B} \to \mathbb{B}(\mathcal{H})$ such that $T|_{B_e}$ is faithful (and so an isometry). Then the restriction of T to each fiber is an isometry and we have a $*$-representation $T^{\mathbf{t}} : \mathbb{M}_{\mathbf{t}}(\mathcal{B}) \to M_n(\mathbb{B}(\mathcal{H})) \cong \mathbb{B}(\mathcal{H}^n)$, given by $T^{\mathbf{t}}(M)_{i,j} = T_{M_{i,j}}$. Observe that $\mathbb{M}_{\mathbf{t}}(\mathcal{B})$ is $\|\cdot\|_\infty$-complete and that $T^{\mathbf{t}}$ is an isometry when we consider on its domain and range the supremum norm. Thus $T^{\mathbf{t}}(\mathbb{M}_{\mathbf{t}}(\mathcal{B}))$ is a C^*-subalgebra of $M_n(\mathbb{B}(\mathcal{H}))$ and its C^*-norm is equivalent to $\|\cdot\|_\infty$. Hence $\mathbb{M}_{\mathbf{t}}(\mathcal{B})$ is a C^*-algebra and its C^*-norm is equivalent to the supremum norm.

To prove claim (iii) we start by noticing that

$$\Big\| \sum_{i=1}^{n} \langle x_i, y_i \rangle_{\mathcal{B}} \Big\|^2 = \Big\| \sum_{i,j=1}^{n} \langle x_i \langle x_j, y_j \rangle_{\mathcal{B}}, y_i \rangle_{\mathcal{B}} \Big\| = \Big\| \sum_{i,j=1}^{n} \langle {}_{\mathcal{A}}\langle x_i, x_j \rangle y_j, y_i \rangle_{\mathcal{B}} \Big\|$$

and that the last term looks like the norm of a matrix multiplication. Given $\mathbf{r} := (r_1, \ldots, r_n) \in G^n$, consider the sum of right Hilbert B_e-modules $\mathbb{X}_{\mathbf{r}} := X_{r_1} \oplus \cdots \oplus X_{r_n}$. Writing the elements of $\mathbb{X}_{\mathbf{r}}$ as column matrices, matrix multiplication gives an action of $\mathbb{M}_{\mathbf{r}^{-1}}(\mathcal{A})$ on $\mathbb{X}_{\mathbf{r}}$ by adjointable operators. Moreover, the formula ${}_{\mathbb{M}_{\mathbf{r}^{-1}}(\mathcal{A})}\langle \eta, \zeta \rangle := ({}_{\mathcal{A}}\langle \eta_i, \zeta_j \rangle)_{i,j=1}^{n}$ defines a $\mathbb{M}_{\mathbf{r}^{-1}}(\mathcal{A})$-valued inner product making $\mathbb{X}_{\mathbf{r}}$ a $\mathbb{M}_{\mathbf{r}^{-1}}(\mathcal{A})$-$B_e$ Hilbert bimodule (not full in general). We should warn the reader that we gave no justification of the positivity of the left inner product of $\mathbb{X}_{\mathbf{r}}$. This actually follows from the positivity of the right inner product because, according to [2], this implies $\mathbb{X}_{\mathbf{r}}$ is a positive C^*-tring, so the left inner product must also be positive. Despite the previous comment, we give next a direct proof of this fact, for the convenience of the reader. Note that $I := \operatorname{span}\{{}_{\mathbb{M}_{\mathbf{r}^{-1}}(\mathcal{A})}\langle \eta, \zeta \rangle : \eta, \zeta \in \mathbb{X}_{\mathbf{r}^{-1}}\}$ is a $*$-ideal of $\mathbb{M}_{\mathbf{r}^{-1}}(\mathcal{A})$ and that VI 19.11 of [10] implies it has a unique C^*-norm (see also Corollary 3.8 of [2] for a complete argumentation). If $\phi : \mathbb{M}_{\mathbf{r}^{-1}}(\mathcal{A}) \to \mathbb{B}(\mathbb{X}_{\mathbf{r}})$ is the homomorphism corresponding to the action of $\mathbb{M}_{\mathbf{r}^{-1}}(\mathcal{A})$ on $\mathbb{X}_{\mathbf{r}}$ by adjointable operators, its restriction $\phi|_I : I \to \mathbb{K}(\mathbb{X}_{\mathbf{r}})$ is injective, because for every $a \in I$ the condition $\phi(a)\xi = 0, \forall\, \xi \in \mathbb{X}_{\mathbf{r}}$, implies $aa^* = 0$. Then the closure of I in $\mathbb{M}_{\mathbf{r}^{-1}}(\mathcal{A})$ is isomorphic, as a C^*-algebra, to the closure of $\phi(I)$ in $\mathbb{K}(\mathbb{X}_{\mathbf{r}})$. Finally note that $\phi({}_{\mathbb{M}_{\mathbf{r}^{-1}}(\mathcal{A})}\langle \eta, \eta \rangle)$ is the generalized compact operator $\zeta \mapsto \eta\langle \eta, \zeta \rangle_{B_e}$, which is positive. So ${}_{\mathbb{M}_{\mathbf{r}^{-1}}(\mathcal{A})}\langle \eta, \eta \rangle$ is positive in $\mathbb{M}_{\mathbf{r}^{-1}}(\mathcal{A})$.

The discussion above implies that

$$\|({}_{\mathcal{A}}\langle x_i, x_j \rangle)_{i,j=1}^{n}\| = \|(x_1, \ldots, x_n)^t\|^2 = \Big\| \sum_{i=1}^{n} \langle x_i, x_i \rangle_{\mathcal{B}} \Big\|. \tag{2.14}$$

On the other hand, if $\mathbf{r}t := (r_1 t, \ldots, r_n t)$, so $\mathbb{X}_{\mathbf{r}t} = X_{r_1 t} \oplus \cdots \oplus X_{r_n t}$, matrix multiplication gives a representation $\varphi : \mathbb{M}_{\mathbf{r}^{-1}}(\mathcal{A}) \to \mathbb{B}(\mathbb{X}_{\mathbf{r}t})$ and, if $x = (x_1, \ldots, x_n)^t \in \mathbb{X}_{\mathbf{r}}$ and $y = (y_1, \ldots, y_n)^t \in \mathbb{X}_{\mathbf{r}t}$:

$$\Big\| \sum_{i=1}^{n} \langle x_i, y_i \rangle_{\mathcal{B}} \Big\|^2 = \|\langle \varphi({}_{\mathbb{M}_{\mathbf{r}^{-1}}(\mathcal{A})}\langle x, x \rangle) y, y \rangle_{B_e}\| \leqslant \|y\|^2 \|({}_{\mathcal{A}}\langle x_i, x_j \rangle)_{i,j=1}^{n}\|. \tag{2.15}$$

Finally, since $\|y\|^2 = \Big\| \sum\limits_{i=1} \langle y_i, y_i \rangle_{\mathcal{B}} \Big\|$, (2.13) follows from (2.14) and (2.15). ∎

REMARK 2.9. Given a Fell bundle $\mathcal{B}$ over G, consider the complex vector space $\mathbf{k}_{\mathrm{c}}(\mathcal{B})$ formed by all the functions $k : G \times G \to \mathcal{B}$ with compact support and such that $k(r,s) \in B_{rs^{-1}}, \forall r, s \in G$. The product and involution we consider on $\mathbf{k}_{\mathrm{c}}(\mathcal{B})$ are $k_1 * k_2(r,s) := \int_G k_1(r,t) k_2(t,s) \mathrm{d}t$ and $k^*(r,s) := k(s,r)^*$. As shown in [1], $\mathbf{k}_{\mathrm{c}}(\mathcal{B})$ is a $*$-algebra, and has a C^*-completion $\mathbf{k}(\mathcal{B})$ (which in fact is equal to $C^*_{\mathrm{r}}(\mathcal{B}) \rtimes_\delta G$, where δ is the dual coaction of G on $C^*_{\mathrm{r}}(\mathcal{B})$). Note that, given

$\mathbf{t} = (t_1, \ldots, t_n) \in G^n$, we can think of every element of $M_{\mathbf{t}}$ as a function $M : G \times G \to \mathcal{B}$ supported in $\{t_1, \ldots, t_n\}^2$ and such that $M(r^{-1}s) \in B_{r^{-1}s}$, $\forall r, s \in G$. Then we have a natural inclusion of $*$-algebras $M_{\mathbf{t}} \hookrightarrow \mathbf{k}_c(\mathcal{B})$, given by $M \mapsto k_M$, where $k_M(r,s) := M(r^{-1}, s^{-1})$, $\forall r, s \in G$. This is an alternative way of proving that $M_{\mathbf{t}}$ has a C^*-algebra structure. Besides it follows that, when G is discrete, we have $C^*_{\mathrm{r}}(\mathcal{B}) \rtimes_\delta G = \varinjlim_{\mathbf{t}} M_{\mathbf{t}}$.

2.4. MORPHISMS OF EQUIVALENCE BUNDLES. In order to define a map between the equivalence bundles $\mathcal{X}$ and $\mathcal{Y}$ that takes into account the equivalence bundle structure, it is convenient to think of $\mathcal{X}$ and $\mathcal{Y}$ as bundles of C^*-trings. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are an $\mathcal{A}$-$\mathcal{B}$ and a $\mathcal{C}$-$\mathcal{D}$-equivalence bundle, respectively (all of them bundles over G).

DEFINITION 2.10. We say that $\rho : \mathcal{X} \to \mathcal{Y}$ is a *morphism of equivalence bundles* if it is a continuous map such that:

(i) $\rho(X_t) \subset Y_t$, and the restriction of ρ to X_t is linear, for all $t \in G$;

(ii) $\rho(x\langle y, z\rangle_{\mathcal{B}}) = \rho(x)\langle \rho(y), \rho(z)\rangle_{\mathcal{D}}$, for all $x, y, z \in \mathcal{X}$.

It is easy to check that equivalence bundles and their morphisms form a category. We denote this category by $\mathscr{E}$.

We have a "bundle version" of Proposition 3.1 in [1].

THEOREM 2.11. *Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{A}$-$\mathcal{B}$ and $\mathcal{C}$-$\mathcal{D}$-equivalence bundles, respectively. Assume $\rho : \mathcal{X} \to \mathcal{Y}$ is a morphism of equivalence bundles. Then*

(i) *$\|\rho(x)\| \leqslant \|x\|$, for all $x \in \mathcal{X}$.*

(ii) *There are unique morphisms of Fell bundles over G, denoted $\rho^{\mathrm{l}} : \mathcal{A} \to \mathcal{C}$ and $\rho^{\mathrm{r}} : \mathcal{B} \to \mathcal{D}$, such that for all $x, y \in \mathcal{X}$ we have:*

$$\rho^{\mathrm{l}}({}_{\mathcal{A}}\langle x, y\rangle) = {}_{\mathcal{C}}\langle \rho(x), \rho(y)\rangle \quad \text{and} \quad \rho^{\mathrm{r}}(\langle x, y\rangle_{\mathcal{B}}) = \langle \rho(x), \rho(y)\rangle_{\mathcal{D}}.$$

(iii) *$\rho^{\mathrm{l}}(a)\rho(x) = \rho(ax)$ and $\rho(xb) = \rho(x)\rho^{\mathrm{r}}(b)$, for all $a \in \mathcal{A}$, $x \in \mathcal{X}$, and $b \in \mathcal{B}$.*

(iv) *In case ρ is bijective, it is an isomorphism of equivalence bundles over G, ρ^{r} and ρ^{l} are isomorphisms, $(\rho^{\mathrm{r}})^{-1} = (\rho^{-1})^{\mathrm{r}}$ and $(\rho^{\mathrm{l}})^{-1} = (\rho^{-1})^{\mathrm{l}}$.*

Proof. Take $t \in G$ and consider X_t as a C^*-tring with the ternary operation $(x, y, z) = x\langle y, z\rangle_{\mathcal{B}}$. Then $\mu_t : X_t \to Y_t$ such that $x \mapsto \rho(x)$, is a homomorphism of C^*-trings and Proposition 3.1 of [1] implies $\|\rho(x)\| \leqslant \|x\|$. Moreover, if $I_t := \overline{\mathrm{span}}\,\langle X_t, X_t\rangle_{\mathcal{B}}$ and $J_t := \overline{\mathrm{span}}\,\langle X_t, X_t\rangle_{\mathcal{D}}$, the above cited proposition implies there exists a unique $*$-homomorphism $\mu_t^{\mathrm{r}} : I_t \to J_t$ sending $\langle x, y\rangle_{\mathcal{B}}$ to $\langle \rho(x), \rho(y)\rangle_{\mathcal{D}}$.

Set $B_t^0 := \mathrm{span}\{\langle X_r, X_{rt}\rangle_{\mathcal{B}} : r \in G\}$. We claim that there exists a unique linear contraction $\nu_t : B_t^0 \to D_t$ such that $\nu_t(\langle x, y\rangle_{\mathcal{B}}) = \langle \rho(x), \rho(y)\rangle_{\mathcal{D}}$. Take $b = \sum_{j=1}^{n} \langle x_j, y_j\rangle_{\mathcal{B}} \in B_t^0$ and set $d := \sum_{j=1}^{n} \langle \rho(x_j), \rho(y_j)\rangle_{\mathcal{D}}$. It suffices to show that $\|d\| \leqslant$

$\|b\|$, which follows from

$$\|d\|^2 = \|\langle d, d\rangle_{\mathcal{D}}\| = \Big\| \sum_{j,k=1}^{n} \langle \rho(x_k \langle x_j, y_j\rangle_{\mathcal{B}}), \rho(y_k)\rangle_{\mathcal{D}} \Big\| = \|\mu_t^{\mathrm{r}}(b^*b)\| \leqslant \|b\|^2.$$

Then ν_t has a unique extension to a linear contraction $\rho_t^{\mathrm{r}} : B_t \to D_t$. Let $\rho : \mathcal{B} \to \mathcal{D}$ be such that $\rho|_{B_t} = \rho_t^{\mathrm{r}}$. It is readily checked that, given $b \in B_s^0$ and $c \in B_t^0$, we have $\rho^{\mathrm{r}}(bc) = \nu_{st}(bc) = \nu_s(b)\nu_t(c) = \rho^{\mathrm{r}}(b)\rho^{\mathrm{r}}(c)$ and $\rho^{\mathrm{r}}(b^*) = \nu_{s^{-1}}(b^*) = \nu_s(b)^* = \rho^{\mathrm{r}}(b)^*$, from where it follows that ρ^{r} is multiplicative and $*$-preserving.

We use II 13.16 of [10] to show that ρ^{r} is continuous. Given $f, g \in C_c(\mathcal{X})$ and $t \in G$, let $[f, g, t] : G \to \mathcal{B}$ be defined as $[f, g, t](r) = \langle f(t), g(tr)\rangle_{\mathcal{B}}$. Then $\Gamma_{\mathcal{B}}^{\mathcal{X}} := \{[f, g, t] : f, g \in C_c(\mathcal{X}),\ t \in G\}$ satisfies:

(i) $B_s = \overline{\operatorname{span}}\,\{u(s) : u \in \Gamma_{\mathcal{B}}\}$ for all $s \in G$ (by Lemma 2.7);

(ii) $\rho^{\mathrm{r}} \circ f \in \Gamma_{\mathcal{D}}^{\mathcal{Y}}$, for all $f \in \Gamma_{\mathcal{B}}^{\mathcal{X}}$.

Hence ρ^{r} is continuous.

From the last two paragraphs it follows that ρ^{r} is a morphism of Fell bundles over G. In order to prove the existence of ρ^{l} define $\widetilde{\rho} : \widetilde{\mathcal{X}} \to \widetilde{\mathcal{Y}}$ as $\widetilde{\rho} = \rho$ (recall that, as topological spaces, $\widetilde{\mathcal{X}} = \mathcal{X}$). Then ρ^{l} is nothing but $\widetilde{\rho}^{\mathrm{r}}$. We leave to the reader the routine verification of the claims in (iii).

Regarding (iv), if ρ is injective then each $\rho|_{X_t}$ is an injective homomorphism of C^*-trings and thus an isometry. Hence, by II 13.17 of [10], ρ^{-1} is continuous. The rest of the proof, which follows from the remark below, is left to the reader. ∎

REMARK 2.12. As a result of the second part of Theorem 2.11 we get two functors $\mathscr{E} \to \mathscr{F}$: the left functor, given by $(\mathcal{X} \xrightarrow{\rho} \mathcal{Y}) \longmapsto (\mathcal{A} \xrightarrow{\rho^{\mathrm{l}}} \mathcal{C})$, and the right functor, given by $(\mathcal{X} \xrightarrow{\rho} \mathcal{Y}) \longmapsto (\mathcal{B} \xrightarrow{\rho^{\mathrm{r}}} \mathcal{D})$.

3. FELL BUNDLES ASSOCIATED TO HILBERT BUNDLES

3.1. THE LINKING BUNDLE. Each of the equivalence bundles presented in our motivating examples was constructed inside an ambient Fell bundle. Although this will turn out to be the general situation, a priori we do not have a Fell bundle that contains a given equivalence bundle. The first purpose of this section is precisely to show that any equivalence bundle can be included in a certain Fell bundle, the so called linking Fell bundle, provided by Theorem 3.2 below. To this end we follow the idea used in Proposition 4.5 of [1] to define the linking partial action of two Morita equivalent partial actions (see Example 2.2.2).

The next result will help us to prove the continuity of the operations to be defined along the construction of the linking bundle.

PROPOSITION 3.1. *Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be (real or complex) Banach bundles over the* LCH *spaces X, Y and Z, respectively. Assume $\Phi : \mathcal{U} \times \mathcal{V} \to \mathcal{W}$ is a function for which* *there exist a continuous map $f : X \times Y \to Z$, a constant $k \geqslant 0$ and sets of sections $\Gamma_Q \subset C(Q)$ for $Q \in \{\mathcal{U}, \mathcal{V}, \mathcal{W}\}$ such that:*

(i) *$\Phi(U_x \times V_y) \subset W_{f(x,y)}$ and $U_x \times V_y \to W_{f(x,y)}$, $(u,v) \mapsto \Phi(u,v)$, is $\mathbb{R}$-bilinear, for all $(x,y) \in X \times Y$;*

(ii) *$\|\Phi(u,v)\| \leqslant k\|u\|\|v\|$, for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$;*

(iii) *for all $x \in X$, $\{\xi(x) : \xi \in \Gamma_{\mathcal{U}}\}$ is dense in U_x and analogous conditions hold for $\mathcal{V}$ and $\mathcal{W}$;*

(iv) *for all $\xi \in \Gamma_{\mathcal{U}}$, $\eta \in \Gamma_{\mathcal{V}}$ and $\zeta \in \Gamma_{\mathcal{W}}$ the function*

$$X \times Y \to \mathbb{R}, \quad (x,y) \mapsto \|\Phi(\xi(x), \eta(y)) - \zeta(f(x,y))\|,$$

is continuous.
Then Φ is continuous.

Proof. We start by observing that the inequality

$$\|\Phi(u_1, v_1) - \Phi(u_2, v_2)\| \leqslant k(\|u_1\| + \|v_2\|)(\|u_1 - u_2\| + \|v_1 - v_2\|) \tag{3.1}$$

holds for all $u_1, u_2 \in U_x$, $v_1, v_2 \in V_y$, $x \in X$ and $y \in Y$.

Let us show that given a converging net in $\mathcal{U} \times \mathcal{V}$, $(u_\lambda, v_\lambda) \to (u,v)$, it follows that $\Phi(u_\lambda, v_\lambda) \to \Phi(u,v)$. Suppose $u_\lambda \in U_{x_\lambda}$, $v_\lambda \in U_{y_\lambda}$, $u \in U_x$ and $v \in V_y$. Obviously, $\Phi(u_\lambda, v_\lambda) \in V_{f(x_\lambda, y_\lambda)}$ and $f(x_\lambda, y_\lambda) \to f(x,y)$. Given $\varepsilon > 0$, choose $\delta \in (0, \varepsilon)$ such that $k(\|u\| + \|v\| + 2\delta)2\delta < \varepsilon$. Then use condition (iii) to find $\xi \in \Gamma_{\mathcal{U}}$ and $\eta \in \Gamma_{\mathcal{V}}$ such that $\|u - \xi(x)\| < \delta$ and $\|v - \eta(y)\| < \delta$. Then equation (3.1) implies $\|\Phi(\xi(x), \eta(y)) - \Phi(u,v)\| < \varepsilon$. A direct continuity argument implies that there exists $\lambda_0 \in \Lambda$ such that $\|u_\lambda - \xi(x_\lambda)\| < \delta$ and $\|v_\lambda - \xi(y_\lambda)\| < \delta$, for all $\lambda \geqslant \lambda_0$. Using inequality (3.1) again we get $\|\Phi(\xi(x_\lambda), \eta(y_\lambda)) - \Phi(u_\lambda, v_\lambda)\| < \varepsilon$, for all $\lambda \geqslant \lambda_0$. In case $\Phi(\xi(x_\lambda), \eta(y_\lambda)) \to \Phi(\xi(x), \eta(y))$, II 13.12 of [10] implies that $\Phi(u_\lambda, v_\lambda) \to \Phi(u,v)$.

Now we show $\Phi(\xi(x_\lambda), \eta(y_\lambda)) \to \Phi(\xi(x), \eta(y))$. Fix $\varepsilon > 0$, condition (iii) implies the existence of $\zeta \in \Gamma_{\mathcal{W}}$ such that $\|\Phi(\xi(x), \eta(y)) - \zeta(f(x,y))\| < \varepsilon$. Then there exists λ_0 such that $\|\Phi(\xi(x_\lambda), \eta(y_\lambda)) - \zeta(f(x_\lambda, y_\lambda))\| < \varepsilon$ for all $\lambda \geqslant \lambda_0$. It is clear that $\zeta(f(x_\lambda, y_\lambda)) \to \zeta(f(x,y))$, thus II 13.12 of [10] implies $\Phi(\xi(x_\lambda), \eta(y_\lambda)) \to \Phi(\xi(x), \eta(y))$. ∎

THEOREM 3.2. *Given an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle, $\mathcal{X}$, there is a unique Fell bundle $\mathbb{L}(\mathcal{X}) = \{L_t\}_{t \in G}$ such that:*

(i) *for all $t \in G$, $L_t = \begin{pmatrix} A_t & X_t \\ \widetilde{X}_{t^{-1}} & B_t \end{pmatrix}$ with entrywise vector space operations;*

(ii) *product and involution are given by*

$$\begin{pmatrix} a & x \\ \widetilde{y} & b \end{pmatrix} \begin{pmatrix} c & u \\ \widetilde{v} & d \end{pmatrix} = \begin{pmatrix} ac + {}_{\mathcal{A}}\langle x, v \rangle & au + xd \\ \widetilde{c^* y + v b^*} & \langle y, u \rangle_{\mathcal{B}} + bd \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & x \\ \widetilde{y} & b \end{pmatrix}^* = \begin{pmatrix} a^* & y \\ \widetilde{x} & b^* \end{pmatrix};$$

(iii) *given* $\xi \in C_c(\mathcal{A})$, $\eta \in C_c(\mathcal{B})$ *and* $f, g \in C_c(\mathcal{X})$ *the function*

$$\begin{pmatrix} \xi & f \\ \widetilde{g} & \eta \end{pmatrix} : G \to \mathbb{L}(\mathcal{X}), \quad t \mapsto \begin{pmatrix} \xi(t) & f(t) \\ \widetilde{g(t^{-1})} & \eta(t) \end{pmatrix}$$

is a continuous section.

Proof. All the necessary algebraic verifications follow from Lemma 2.7, and will be ommited. To define a C^*-norm on $\mathbb{L}(\mathcal{X})$ and to (automatically) show $u^*u \geqslant 0$ in L_e, for all $u \in \mathbb{L}(\mathcal{X})$, we will provide a representation of $\mathbb{L}(\mathcal{X})$. We proceed as follows. First consider the Hilbert B_e-module direct sums:

$$\ell^2(\mathcal{B}) := \bigoplus_{t \in G} B_t \quad \text{and} \quad \ell^2(\mathcal{X}) := \bigoplus_{t \in G} X_t.$$

Given $x_r \in X_r$ and a section $\xi \in \ell^2(\mathcal{B})$, define the section $\Omega_{x_r}\xi : G \to \mathcal{X}$ by $\Omega_{x_r}\xi(s) := x_r\xi(r^{-1}s)$. Note that $x_r\xi(r^{-1}s) \in X_r B_{r^{-1}s} \subseteq X_s$, as needed. Besides,

$$\begin{aligned} \langle \Omega_{x_r}\xi, \Omega_{x_r}\xi \rangle_{\ell^2(\mathcal{X})} &= \sum_s \langle \Omega_{x_r}\xi(s), \Omega_{x_r}\xi(s) \rangle_{\mathcal{B}} = \sum_s \langle x_r\xi(r^{-1}s), x_r\xi(r^{-1}s) \rangle_{\mathcal{B}} \\ &= \sum_s \xi(r^{-1}s)^* \langle x_r, x_r \rangle_{\mathcal{B}} \xi(r^{-1}s) \leqslant \|x_r\|^2 \langle \xi, \xi \rangle_{\ell^2(\mathcal{B})}. \end{aligned}$$

Therefore there exists a unique bounded operator $\Omega_{x_r} : \ell^2(\mathcal{B}) \to \ell^2(\mathcal{X})$, $\xi \mapsto \Omega_{x_r}\xi$, which is easily seen to be adjointable with adjoint given by:

$$\Omega_{x_r}^*\eta(t) = \langle x_r, \eta(rt) \rangle_{\mathcal{B}}, \quad \forall \eta \in \ell^2(\mathcal{X}),\, t \in G.$$

In the particular case $\mathcal{X} = \mathcal{B}$, for each $b_r \in \mathcal{B}$ we have an adjointable map $\Lambda_{b_r} \in \mathbb{B}(\ell^2(\mathcal{B}))$, such that $\Lambda_{b_r}\xi(s) = b_r\xi(r^{-1}s)$, $\forall \xi \in \ell^2(\mathcal{B})$, $s \in G$ (note that Λ so defined is nothing but the left regular representation of $\mathcal{B}$ as a Fell bundle over the group G with the discrete topology). Similarly, we also have a map $\Lambda' : \mathcal{A} \to \mathbb{B}(\ell^2(\mathcal{X}))$, such that $\Lambda'_{a_r}\eta(s) = a_r\eta(r^{-1}s)$. Now it is easy to check that the following relations hold:

$$\begin{gathered} \Omega_{ax} = \Lambda'_a\Omega_x, \quad \text{and} \quad \Omega_{xb} = \Omega_x\Lambda_b; \\ \Omega_x^*\Omega_y = \Lambda_{\langle x,y \rangle_{\mathcal{B}}} \in \mathbb{B}(\ell^2(\mathcal{B})), \quad \forall x, y \in \mathcal{X}; \\ \Omega_x\Omega_y^* = \Lambda'_{{}_{\mathcal{A}}\langle x,y \rangle} \in \mathbb{B}(\ell^2(\mathcal{X})), \quad \forall x, y \in \mathcal{X}. \end{gathered}$$

Define $\phi : \mathbb{L}(\mathcal{X}) \to \mathbf{C} := \begin{pmatrix} \mathbb{B}(\ell^2(\mathcal{X})) & \mathbb{B}(\ell^2(\mathcal{B}), \ell^2(\mathcal{X})) \\ \mathbb{B}(\ell^2(\mathcal{X}), \ell^2(\mathcal{B})) & \mathbb{B}(\ell^2(\mathcal{B})) \end{pmatrix}$ by

$$\phi \begin{pmatrix} a & x \\ \widetilde{y} & b \end{pmatrix} = \begin{pmatrix} \Lambda'_a & \Omega_x \\ \Omega_y^* & \Lambda_b \end{pmatrix}.$$

Then ϕ is multiplicative, preserves adjoints and is linear in each fiber of $\mathbb{L}(\mathcal{X})$. Moreover, $\phi|_{L_e}$ is a faithful homomorphism into the C^*-algebra $\mathbf{C}$. Thus if we define $\|\ell\| := \|\phi(\ell)\|_{\mathbf{C}}$, we get a C^*-norm on $\mathbb{L}(\mathcal{X})$. Of course this C^*-norm is equivalent to the norm $\|\cdot\|_\infty$, defined as $\|\ell\|_\infty := \max\{\|a\|, \|x\|, \|\widetilde{y}\|, \|b\|\}$ for $\ell = \begin{pmatrix} a & x \\ \widetilde{y} & b \end{pmatrix}$.

To endow $\mathbb{L}(\mathcal{X})$ with a bundle topology, note that

$$\Gamma := \left\{ \begin{pmatrix} \xi & f \\ \widetilde{g} & \eta \end{pmatrix} : f, g \in C_c(\mathcal{X}), \xi \in C_c(\mathcal{A}), \eta \in C_c(\mathcal{B}) \right\}$$

is a subspace of sections such that $\{u(t) : u \in \Gamma\} = L_t$, for all $t \in G$ (recall the definition of $\begin{pmatrix} \xi & f \\ \widetilde{g} & \eta \end{pmatrix}$ made in part (iii) of the statement of the theorem). Moreover, for every $u \in \Gamma$ the entries of $t \mapsto u(t)^* u(t) \in L_e$ are continuous functions. Then $t \mapsto \|u(t)\| = \|u(t)^* u(t)\|^{1/2}$ is continuous, so there is a unique Banach bundle structure on $\mathbb{L}(\mathcal{X})$ such that $\Gamma \subset C_c(\mathbb{L}(\mathcal{X}))$.

Using similar arguments it can be shown that, for $u, v, w \in \Gamma$, the functions $G \times G \to \mathbb{R} : (r, s) \mapsto \|u(r)v(s) - w(rs)\|$ and $G \times G \to \mathbb{R} : (r, s) \mapsto \|u(s)^* - v(s^{-1})\|$ are continuous. For example, note that

$$(r, s) \mapsto (u(r)v(s) - w(rs))^* (u(r)v(s) - w(rs))$$

has continuous entries. Then Proposition 3.1 implies the involution and multiplication of $\mathbb{L}(\mathcal{X})$ are continuous. ∎

It can be shown that the continuous sections described in condition (iii) of the previous theorem are all the continuous sections of compact support of $\mathbb{L}(\mathcal{X})$.

REMARK 3.3. We can construct the direct sum $\mathcal{A} \oplus \mathcal{X}$ as the Banach subbundle

$$\left\{ \begin{pmatrix} a & x \\ 0 & 0 \end{pmatrix} : a \in A_t,\ x \in X_t,\ t \in G \right\} \subset \mathbb{L}(\mathcal{X}).$$

Then $\mathcal{A} \oplus \mathcal{X}$ becomes an $\mathcal{A}$-$\mathbb{L}(\mathcal{X})$-equivalence bundle. Moreover, in a similar way we can define $\mathcal{X} \oplus \mathcal{B}$ and make it into a $\mathbb{L}(\mathcal{X})$-$\mathcal{B}$-equivalence bundle.

As we expected, linking partial actions give rise to linking bundles.

PROPOSITION 3.4. *Let X be an equivalence A-B-bimodule and γ a partial action on X (see Section 2.2.2). If $\mathcal{X}_\gamma$ is the Fell bundle associated to γ, then $\mathbb{L}(\mathcal{X}_\gamma)$ is isomorphic to $\mathcal{B}_{\mathbb{L}(\gamma)}$.*

Proof. Define $\beta := \gamma^{\mathrm{r}}$ as in Section 2.2.2, and let $\mathcal{Y} := \mathcal{X}_\gamma \oplus \mathcal{B}_\beta$ and

$$\mathcal{Z} := \left\{ \begin{pmatrix} 0 & x \\ 0 & b \end{pmatrix} \delta_t : x \in X_t,\ b \in B_t,\ t \in G \right\} \subset \mathcal{B}_{\mathbb{L}(\gamma)}.$$

Then $\mathcal{Y}$ is a $\mathbb{L}(\mathcal{X}_\gamma)$-$\mathcal{B}_\beta$-equivalence bundle and $\mathcal{Z}$ a $\mathcal{B}_{\mathbb{L}(\gamma)}$-$\mathcal{B}_\beta$-equivalence bundle. Since the map

$$\rho : \mathcal{Y} \to \mathcal{Z} \quad \text{given by } \rho \begin{pmatrix} 0 & x\delta_t \\ 0 & b\delta_t \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & b \end{pmatrix} \delta_t$$

satisfies the hypotheses of Theorem 2.11(iv), $\rho^{\mathrm{l}} : \mathbb{L}(\mathcal{X}) \to \mathcal{B}_{\mathbb{L}(\gamma)}$ is an isomorphism of Fell bundles. ∎

THEOREM 3.5. *Let $\rho : \mathcal{X} \to \mathcal{Y}$ be a morphism from the $\mathcal{A}$-$\mathcal{B}$-equivalence bundle $\mathcal{X}$ to the $\mathcal{C}$-$\mathcal{D}$-equivalence bundle $\mathcal{Y}$. Let $\mathbb{L}(\rho) : \mathbb{L}(\mathcal{X}) \to \mathbb{L}(\mathcal{Y})$ be the map given by (recall ρ^{l} and ρ^{r} were defined in* (2.11)*):*

$$\mathbb{L}(\rho)\begin{pmatrix} a & x \\ \widetilde{y} & b \end{pmatrix} := \begin{pmatrix} \rho^{\mathrm{l}}(a) & \rho(x) \\ \widetilde{\rho(y)} & \rho^{\mathrm{r}}(b) \end{pmatrix}, \quad \forall \begin{pmatrix} a & x \\ \widetilde{y} & b \end{pmatrix} \in \mathbb{L}(\mathcal{X}).$$

Then $\mathbb{L}(\rho)$ is a morphism of Fell bundles, and $(\mathcal{X} \xrightarrow{\rho} \mathcal{Y}) \longmapsto (\mathbb{L}(\mathcal{X}) \xrightarrow{\mathbb{L}(\rho)} \mathbb{L}(\mathcal{Y}))$ is a functor from the category $\mathscr{E}$ of equivalence bundles to the category $\mathscr{F}$ of Fell bundles.

Proof. We just need to verify that the map $\mathbb{L}(\rho)$ defined in the statement is a morphism of Fell bundles over G. The routine algebraic verifications are left to the reader. Now note that the map $\mathbb{L}(\mathcal{X})_e \to \mathbb{L}(\mathcal{Y})_e$, $S \mapsto \mathbb{L}(\rho)S$, is contractive because it is a homomorphism of C^*-algebras. Then, for all $S \in \mathbb{L}(\mathcal{X})$, we have

$$\|\mathbb{L}(\rho)S\| = \|\mathbb{L}(\rho)[S^*S]\|^{1/2} \leqslant \|S^*S\|^{1/2} = \|S\|.$$

Now observe that given $\begin{pmatrix} \xi & f \\ \widetilde{g} & \eta \end{pmatrix} \in C_{\mathrm{c}}(\mathbb{L}(\mathcal{X}))$ as in Theorem 3.2, we have

$$\mathbb{L}(\rho) \circ \begin{pmatrix} \xi & f \\ \widetilde{g} & \eta \end{pmatrix} = \begin{pmatrix} \rho^{\mathrm{l}} \circ \xi & \rho \circ f \\ \widetilde{\rho \circ g} & \rho^{\mathrm{r}} \circ \eta \end{pmatrix} \in C_{\mathrm{c}}(\mathbb{L}(\mathcal{Y})). \tag{3.2}$$

Using II 13.16 of [10] we conclude $\mathbb{L}(\rho)$ is continuous, so it is a morphism of Fell bundles over G. ∎

3.2. THE BUNDLE OF GENERALIZED COMPACT OPERATORS. Assume $\mathcal{X}$ is a right Hilbert $\mathcal{B}$-bundle. We will construct a Fell bundle $\mathbb{K}(\mathcal{X})$ in such a way that $\mathcal{X}$ is a $\mathbb{K}(\mathcal{X})$-$\mathcal{B}$-equivalence bundle. In view of Theorem 2.11, this Fell bundle is uniquely determined by the right Hilbert bundle structure of $\mathcal{X}$.

DEFINITION 3.6. An adjointable operator of order $t \in G$ of $\mathcal{X}$ is a continuous map $S : \mathcal{X} \to \mathcal{X}$ with the following properties:

(i) there exists $c \in \mathbb{R}$ such that $\|Sx\| \leqslant c\|x\|$, for all $x \in \mathcal{X}$;

(ii) $S(X_r) \subset X_{tr}$, for all $r \in G$;

(iii) there exists $S^* : \mathcal{X} \to \mathcal{X}$ such that $\langle Sx, y\rangle_{\mathcal{B}} = \langle x, S^*y\rangle_{\mathcal{B}}, \forall x, y \in \mathcal{X}$.

The set of adjointable operators of order t will be denoted $\mathbb{B}_t(\mathcal{X})$.

If $S_1, S_2 \in \mathbb{B}_t(\mathcal{X})$ and $\alpha \in \mathbb{C}$, is clear that $\alpha S_1 + S_2$ is also an adjointable operator of order t. We define a norm on $\mathbb{B}_t(\mathcal{X})$ by $\|S\| := \sup\{\|Sx\| : \|x\| \leqslant 1\}$. Then we have an isometric map $\mathbb{B}_t(\mathcal{X}) \to \bigoplus_{r\in G} \mathbb{B}(X_r, X_{tr})$, given by $S \mapsto (S_r)_{r\in G}$, where $S_r : X_r \to X_{tr}$ is such that $S_r(x) = S(x)$, $\forall x \in X_r$. Since for all $x \in X_r$ and $y \in X_{tr}$ we have $\langle Sx, y\rangle_{\mathcal{B}} = \langle S_r x, y\rangle_{\mathcal{B}} = \langle x, S_r^* y\rangle_{\mathcal{B}}$, we see that the map S^* is determined by S, and $(S^*)_r = (S_r)^*$, $\forall r \in G$. We call S^* the adjoint of S.

Note that, using Proposition 3.1 and Definition 3.6, it can be shown that S^* is continuous. Then S^* is an adjointable map of order t^{-1}. Conversely, for every $(T_r) \in \bigoplus_{r \in G} \mathbb{B}(X_r, X_{tr})$ there exists a unique $S : \mathcal{X} \to \mathcal{X}$ such that $S_r = T_r, \forall r \in G$. However, this S does not need to be a continuous map. Despite this fact we have the following lemma.

LEMMA 3.7. *$\mathbb{B}_t(\mathcal{X})$ is a Banach space.*

Proof. Let $(S^{(n)})$ be a Cauchy sequence in $\mathbb{B}_t(\mathcal{X})$. Then $(S_r^{(n)})_{r \in G}$ is a Cauchy sequence in the complete space $\bigoplus_{r \in G} \mathbb{B}(X_r, X_{tr})$, so it has a limit $(S_r) \in \mathbb{B}(X_r, X_{tr})$. Let $S : \mathcal{X} \to \mathcal{X}$ be given by $Sx := S_r x$, $\forall x \in X_r$, $r \in G$. It is enough to show that S is continuous, what we do next. Take a net $\{x_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{X}$ converging to $x_0 \in \mathcal{X}$. Suppose $x_\lambda \in X_{r_\lambda}$ and $x_0 \in X_{r_0}$ and fix $\varepsilon > 0$. We can find $n \in \mathbb{N}$ such that $\|Sx - S_n x\| < \varepsilon$, for all $x \in \mathcal{X}$ with $\|x\| \leqslant 1$. Since $S^{(n)}$ is continuous, $S^{(n)} x_\lambda \to_\lambda S^{(n)} x_0$. Hence II 13.12 of [10] implies $Sx_\lambda \to Sx_0$. ∎

COROLLARY 3.8. *Let G_d be the group G endowed with the discrete topology, and let $\mathbb{B}(\mathcal{X}) := (\mathbb{B}_t(\mathcal{X}))_{t \in G_d}$. Then $\mathbb{B}(\mathcal{X})$ is a Fell bundle over G_d, with the product given by the composition of maps.*

Proof. Observe that if S_1 is of order t_1 and S_2 is of order t_2, then $S_1 S_2$ is of order $t_1 t_2$. On the other hand, if $S \in \mathbb{B}_t(\mathcal{X})$ corresponds to $(S_r) \in \bigoplus_{r \in G} \mathbb{B}(X_r, X_{tr})$, then S^*S corresponds to $(S_r^* S_r) \in \bigoplus_{r \in G} \mathbb{B}(X_r)$. Since $S_r^* S_r$ is a positive element of $\mathbb{B}(X_r)$, we have $S_r^* S_r = T_r^* T_r$, for some $T_r \in \mathbb{B}(X_r)$, $\forall r \in G$. Thus $S^*S = T^*T$, where $T \in \mathbb{B}_e(\mathcal{X})$ corresponds to the element $(T_r) \in \bigoplus_{r \in G} \mathbb{B}(X_r)$. Therefore S^*S is a positive element of the C^*-algebra $\mathbb{B}_e(\mathcal{X})$. The remaining verifications are routine and we ommit them. ∎

THEOREM 3.9. *Let $\mathcal{B}$ be a Fell bundle over the group G. Given a right Hilbert $\mathcal{B}$-bundle $\mathcal{X}$ there exists a unique Fell bundle over G, which we denote by $\mathbb{K}(\mathcal{X})$, such that:*

(i) *for all $t \in G$ the fiber $\mathbb{K}(\mathcal{X})_t$ is, as a Banach space, the closure in $\mathbb{B}_t(\mathcal{X})$ of*

$$\operatorname{span}\{[x, y] : x \in X_{ts},\ y \in X_s,\ s \in G\}$$

where $[x, y] : \mathcal{X} \to \mathcal{X}$ is defined to be $[x, y]z := x\langle y, z\rangle_{\mathcal{B}}$;

(ii) *given $f, g \in C_c(\mathcal{X})$ and $s \in G$, the function $[f, g, s] : G \to \mathbb{K}(\mathcal{X})$ given by $[f, g, s](t) = [f(ts), g(s)]$, is a continuous section of $\mathbb{K}(\mathcal{X})$.*

Proof. Note first that, if $x \in X_{ts}$ and $y \in X_s$, then $[x, y]X_r = x\langle y, X_r\rangle_{\mathcal{B}} \subseteq X_{ts}\mathcal{B}_{s^{-1}r} \subseteq X_{tr}$, so $[x, y] \in \mathbb{B}_t(\mathcal{X})$. Since $[x, y][z, w] = [x\langle y, z\rangle_{\mathcal{B}}, w]$ and $[x, y]^* = [y, x]$, $\mathbb{K}(\mathcal{X})$ is closed under multiplication and involution. We want to define a topology on $\mathbb{K}(\mathcal{X})$ such that $\mathbb{K}(\mathcal{X})$ is a Banach bundle with it, and the space

$\Gamma := \operatorname{span}\{[f,g,s] : f,g \in C_c(\mathcal{X}),\ s \in G\}$ is contained in the subspace of continuous sections of that bundle. By II13.18 of [10] there exists at most one such topology and, to prove its existence we must show that given $n \in \mathbb{N}$, $f_1, g_1, \ldots, f_n, g_n \in C_c(\mathcal{X})$, and $s_1, \ldots, s_n \in G$, the function $h : G \to \mathbb{R}$, given by $h(t) = \Big\| \sum_{j=1}^{n} [f_j, g_j, s_j](t) \Big\|$, is continuous. Now if $k(t) := \sum_{j=1}^{n} [f_j, g_j, s_j](t)$, we have $h(t) = \|k(t)k(t)^*\|^{1/2}$, so it suffices to show that the map $G \to \mathbb{B}_e(\mathcal{X})$ $t \mapsto k(t)k(t)^*$, is continuous. In fact we just need to show that $t \mapsto [k(t)(g_j(s_j)), f_j(ts_j)]$ is continuous (for all $j = 1, \ldots, n$) because

$$k(t)k(t)^* = \sum_{j=1}^{n} [k(t)(g_j(s_j)), f_j(ts_j)].$$

Fix $j = 1, \ldots, n$ and let $u, v : G \to \mathcal{X}$ be defined as $u(t) := k(t)(g_j(s_j))$ and $v(t) := f_j(ts_j)$. Then u and v are continuous and, for all $z \in \mathcal{X}$ with $\|z\| \leqslant 1$, we have

$$\begin{aligned}
&\|u(t)\langle v(t), z\rangle_{\mathcal{B}} - u(r)\langle v(r), z\rangle_{\mathcal{B}}\|^2 \\
&\quad = \|\langle u(t)\langle v(t), z\rangle_{\mathcal{B}} - u(r)\langle v(r), z\rangle_{\mathcal{B}}, u(t)\langle v(t), z\rangle_{\mathcal{B}} - u(r)\langle v(r), z\rangle_{\mathcal{B}}\rangle_{\mathcal{B}}\| \\
&\quad \leqslant \|\langle z, v(t)\rangle_{\mathcal{B}}\langle v(t)\langle u(t), u(t)\rangle_{\mathcal{B}} - v(r)\langle u(r), u(t)\rangle_{\mathcal{B}}, z\rangle_{\mathcal{B}}\| \\
&\qquad\quad + \|\langle z, v(r)\rangle_{\mathcal{B}}\langle v(t)\langle u(t), u(r)\rangle_{\mathcal{B}} - v(r)\langle u(r), u(r)\rangle_{\mathcal{B}}\rangle_{\mathcal{B}}\| \\
&\quad \leqslant \|v(t)\|\|v(t)\langle u(t), u(t)\rangle_{\mathcal{B}} - v(r)\langle u(r), u(t)\rangle_{\mathcal{B}}\| \\
&\qquad\quad + \|v(r)\|\|v(t)\langle u(t), u(r)\rangle_{\mathcal{B}} - v(r)\langle u(r), u(r)\rangle_{\mathcal{B}}\|.
\end{aligned}$$

The right member of the above inequality is the sum of two terms that do not depend on z and have limit 0 when $r \to t$. Hence $t \mapsto [k(t)(g(s_j)), f_j(ts_j)]$ is continuous.

We still have to show that multiplication and involution are continuous, for which we use Proposition 3.1. As for the multiplication we need to show that, given $u, v, w \in \Gamma$, the function $G \times G \to \mathbb{R}$, $(r,s) \mapsto \|u(r)v(s) - w(rs)\|$, is continuous. It is enough to prove that $(r,s) \mapsto (u(r)v(s) - w(rs))^*(u(r)v(s) - w(rs))$ is a continuous function from $G \times G$ to $\mathbb{K}(\mathcal{X})_e$, and this can be done by using the same arguments we have used in the previous paragraphs.

To prove that involution is continuous, let $\mathcal{V}$ be the Banach bundle over $\{e\}$ with fiber $\mathbb{C}$, and define $\Phi : \mathbb{K}(\mathcal{X}) \times \mathcal{V} \to \mathbb{K}(\mathcal{X})$ such that $\Phi(b, \lambda) = \lambda b^*$. The map Φ is continuous because of Proposition 3.1. Then the involution $\mathbb{K}(\mathcal{X}) \to \mathbb{K}(\mathcal{X})$, $b \mapsto \Phi(b, 1)$, also is continuous. ∎

COROLLARY 3.10. *Every right Hilbert $\mathcal{B}$-bundle, $\mathcal{X}$, is a $\mathbb{K}(\mathcal{X})$-$\mathcal{B}$-equivalence bundle with the action $\mathbb{K}(\mathcal{X}) \times \mathcal{X} \to \mathcal{X}$ given by $(b, x) \mapsto b(x)$, and the left inner product $\mathcal{X} \times \mathcal{X} \to \mathbb{K}(\mathcal{X})$ given by $(x, y) \mapsto [x, y]$.*

The proof is a straightforward consequence of Theorem 3.9.

COROLLARY 3.11. *If $\mathcal{X}$ is an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle, then there exists a unique isomorphism of Fell bundles $\pi : \mathcal{A} \to \mathbb{K}(\mathcal{X})$ such that $\pi({}_{\mathcal{A}}\langle x, y\rangle) = [x, y]$.*

Proof. Let $\mathcal{Y}$ be the bundle $\mathcal{X}$ considered as a $\mathbb{K}(\mathcal{X})$-$\mathcal{B}$-equivalence bimodule and let $\mathrm{id} : \mathcal{X} \to \mathcal{Y}$ be the identity. Then, by Theorem 2.11, $\pi := \mathrm{id}^1 : \mathbb{K}(\mathcal{X}) \to \mathcal{A}$ is the isomorphism we are looking for. ∎

REMARK 3.12. With the notation of the previous Corollary, $\mathbb{K}(\mathcal{X} \oplus \mathcal{B})$ is isomorphic to $\mathbb{L}(\mathcal{X})$ because $\mathcal{X} \oplus \mathcal{B}$ is a $\mathbb{L}(\mathcal{X})$-$\mathcal{B}$-equivalence bundle.

4. MORITA–RIEFFEL EQUIVALENCE OF CROSS-SECTIONAL C^*-ALGEBRAS

It is well known (see [7]) that equivalent actions on C^*-algebras have Morita–Rieffel equivalent crossed products (full and reduced), and the same can be said about equivalent partial actions ([1] and [3]). We will show in this section that, more generally, any $\mathcal{A}$-$\mathcal{B}$-equivalence bundle $\mathcal{X}$ gives rise to a $C^*(\mathcal{A})$-$C^*(\mathcal{B})$- and also a $C^*_{\mathrm{r}}(\mathcal{A})$-$C^*_{\mathrm{r}}(\mathcal{B})$-equivalence bimodule. We consider first the case of the full C^*-algebras, for which we will construct an equivalence bimodule contained in $C^*(\mathbb{L}(\mathcal{X}))$. We will make use of the fact that $\mathcal{A}$ and $\mathcal{B}$ are hereditary in $\mathbb{L}(\mathcal{X})$ in the following sense.

DEFINITION 4.1. Given a Fell bundle $\mathcal{C}$ and a Fell subbundle $\mathcal{A} \subset \mathcal{C}$, we say $\mathcal{A}$ is hereditary (in $\mathcal{C}$) if $\mathcal{A}\mathcal{C}\mathcal{A} \subset \mathcal{A}$.

The condition $A_e\mathcal{C}A_e \subset \mathcal{A}$ may look weaker than $\mathcal{A}\mathcal{C}\mathcal{A} \subset \mathcal{A}$, but in fact they are equivalent. Indeed, suppose the former condition holds and take $a, c \in \mathcal{A}$ and $b \in \mathcal{C}$. Let $\{d_\lambda\}_{\lambda\in\Lambda} \subset A_e$ be an approximate unit of A_e and assume $abc \in C_t$. Then the net $\{ad_\lambda bd_\lambda c\}_{\lambda\in\Lambda}$ converges to abc and is contained in A_t. This implies $abc \in \mathcal{A}$ because A_t is closed in C_t.

We may obtain hereditary subbundles by considering partial actions and restriction of them to ideals.

PROPOSITION 4.2. *Let $\beta = (\{B_t\}_{t\in G}, \{\beta_t\}_{t\in G})$ be a partial action of the group G on the C^*-algebra B. Then for every ideal A of B there exists a unique partial action of G on A, $\beta|_A := (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$, such that: $A_{t^{-1}} = A \cap \beta_{t^{-1}}(B_t \cap A)$ and $\alpha_t(b) = \beta_t(b)$, for all $t \in G$ and $b \in A_{t^{-1}}$. Moreover, $\mathcal{B}_{\beta|_A}$ is hereditary in $\mathcal{B}_\beta$.*

Proof. Note that $\beta_t(A_{t^{-1}}) = \beta_t(B_{t^{-1}} \cap A \cap \beta_{t^{-1}}(B_t \cap A)) = \beta_t(B_{t^{-1}} \cap A) \cap A = A_t$, so there exists a unique isomorphism of C^*-algebras $\alpha_t : A_{t^{-1}} \to A_t$ such that $\alpha_t(a) = \beta_t(a)$. Clearly, α_e is the identity on A. If $a \in A_{t^{-1}}$ and $\alpha_t(a) \in A_{s^{-1}}$, then

$$a \in \beta_{t^{-1}}(B_t \cap \beta_{s^{-1}}(B_s \cap A)) \subset \beta_{t^{-1}}(B_t \cap B_{s^{-1}}) \subset B_{t^{-1}} \cap B_{t^{-1}s^{-1}}.$$

This implies $\beta_{st}(a) = \beta_s(\beta_t(a)) = \beta_s(\alpha_t(a)) \in \beta_s(\beta_{s^{-1}}(B_s \cap A)) \subset A$. Putting all this together we conclude that $a \in A \cap \beta_{t^{-1}s^{-1}}(B_{st} \cap A) = A_{t^{-1}s^{-1}}$ and $\alpha_{st}(a) = \beta_{st}(a) = \beta_s(\beta_t(a)) = \alpha_s(\alpha_t(a))$. Thus $\beta|_A$ is a set theoretic partial action.

To show that $\beta|_A$ is a continuous partial action on A, it suffices to prove that $\{A_t\}_{t\in G}$ is a continuous family. To see this fix $t \in G$ and $b \in A_t$. It suffices to find $g \in C(G, A)$ such that $g(t) = b$ and $g(r) \in A_r$, for all $r \in G$. The Cohen–Hewitt theorem provides $x, y \in A$ and $z \in B_{t^{-1}}$ such that $b = x\beta_t(yz)$. Now pick $f \in C(G, B)$ such that $f(r) \in B_r$ (for all $r \in G$) and $f(t^{-1}) = z$, what we can do because $\{B_r\}_{r\in G}$ is a continuous family. Then the function $g : G \to A$ defined as $g(r) = x\beta_r(yf(r^{-1}))$ is continuous, $g(t) = b$ and $g(r) \in A_r$ for all $r \in G$.

To show $\mathcal{B}_{\beta|_A}$ is hereditary in $\mathcal{B}_\beta$ fix $a, c \in A_e$ and $b \in B_r$ and observe that

$$a\delta_e b\delta_r c\delta_e = a\delta_e(b\delta_r c\delta_e) = a\beta_r(\beta_{r^{-1}}(b)c)\delta_r.$$

Clearly, $a\beta_r(\beta_{r^{-1}}(b)c) \in A \cap \beta_r(B_{r^{-1}} \cap A) = A_r$, so $a\delta_e b\delta_r c\delta_e \in \mathcal{B}_{\beta|_A}$. ∎

THEOREM 4.3. *If $\mathcal{A}$ is an hereditary Fell subbundle of $\mathcal{B}$, then $C^*(\mathcal{A})$ is the closure of $L^1(\mathcal{A})$ in $C^*(\mathcal{B})$, and it is an hereditary subalgebra of $C^*(\mathcal{B})$.*

Proof. Let $\mathcal{X}$ be the Banach subbundle of $\mathcal{B}$ such that, for each $t \in G$, $X_t = \overline{\operatorname{span}}\{ab : a \in A_r,\ b \in B_{r^{-1}t},\ r \in G\}$. Now let $\mathcal{C}$ be the Banach subbundle of $\mathcal{B}$ such that, for each $t \in G$, $C_t = \overline{\operatorname{span}}\{x^*y : x \in X_r,\ y \in X_{rt},\ r \in G\}$. In fact $\mathcal{C}$ is a Fell subbundle of $\mathcal{B}$ and $\mathcal{CB} \cup \mathcal{BC} \subset \mathcal{C}$; in other words $\mathcal{C}$ is an ideal of $\mathcal{B}$. Then we can think of $L^1(\mathcal{A})$ as a $*$-Banach subalgebra of $L^1(\mathcal{C})$ and of $L^1(\mathcal{C})$ as a closed $*$-ideal of $L^1(\mathcal{B})$.

Using Theorem 1.1 of [3] and Corollary 5.3 of [2] with $\mathcal{E} = \mathcal{X}$, we conclude that $C^*(\mathcal{A})$ is the closure of $L^1(\mathcal{A})$ in $C^*(\mathcal{C})$. Let $\pi : C^*(\mathcal{C}) \to C^*(\mathcal{B})$ be the unique $*$-homomorphism extending the natural inclusion of $L^1(\mathcal{C})$ in $L^1(\mathcal{B})$. To show that π is injective take a non-degenerate faithful representation $\rho : C^*(\mathcal{C}) \to \mathbb{B}(\mathcal{H})$. In this situation we know from VI 19.11 of [10] that $\rho|_{L^1(\mathcal{C})}$ can be extended in a unique way to a representation defined on all of $L^1(\mathcal{B})$. Then there exists a unique representation $\overline{\rho} : C^*(\mathcal{B}) \to \mathbb{B}(\mathcal{H})$ such that $\overline{\rho} \circ \pi(f) = \rho(f)$, for all $f \in L^1(\mathcal{C})$. This implies that π is injective because $\rho = \overline{\rho} \circ \pi$. Putting all this together we conclude that the maximal C^*-norm of $L^1(\mathcal{A})$ is the restriction of the maximal C^*-norm of $L^1(\mathcal{B})$. The last assertion of the statement is clear. ∎

COROLLARY 4.4. *If $\mathcal{X}$ is an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle, then $C^*(\mathcal{A})$ and $C^*(\mathcal{B})$ are the closure of $L^1(\mathcal{A})$ and of $L^1(\mathcal{B})$ in $C^*(\mathbb{L}(\mathcal{X}))$, respectively.*

For the proof just note that $A_e\mathbb{L}(\mathcal{X})A_e \subset \mathcal{A}$ and $B_e\mathbb{L}(\mathcal{X})B_e \subset \mathcal{B}$.

From now on we will think of the cross-sectional C^*-algebras $C^*(\mathcal{A})$ and $C^*(\mathcal{B})$ as C^*-subalgebras of $C^*(\mathbb{L}(\mathcal{X}))$.

THEOREM 4.5. *For every $\mathcal{A}$-$\mathcal{B}$-equivalence bundle, $\mathcal{X}$, the closure of $C_c(\mathcal{X})$ in $C^*(\mathbb{L}(\mathcal{X}))$, $C^*(\mathcal{X})$, is a $C^*(\mathcal{A})$-$C^*(\mathcal{B})$-equivalence bimodule with the bimodule structure inherited from $C^*(\mathbb{L}(\mathcal{X}))$.*

Proof. Given $f \in C_c(\mathcal{X})$ and $g \in C_c(\mathcal{B})$ we have $f * u \in C_c(\mathcal{X})$ because $f * u \in C_c(\mathbb{L}(\mathcal{X}))$, $f * u(t) = \int_G f(r)u(r^{-1}t)\,dr$ and $u(r)f(r^{-1}t) \in X_t$, for all

$r, t \in G$. Using the continuity of the product we see that $C^*(\mathcal{X})C^*(\mathcal{B}) \subset C^*(\mathcal{X})$. In a similar way we can show that $C^*(\mathcal{X})^*C^*(\mathcal{X}) \subset C^*(\mathcal{B})$, so the right inner product $C^*(\mathcal{X}) \times C^*(\mathcal{X}) \to C^*(\mathcal{B})$, $(f, g) \mapsto f^* * g$, is defined. Moreover, this inner product is positive because $f^* * f$ is positive in $C^*(\mathbb{L}(\mathcal{X}))$.

To prove that $C^*(\mathcal{X})$ is a full Hilbert $C^*(\mathcal{B})$-module it suffices to prove that every element of the form $f^* * g$ ($f, g \in C_c(\mathcal{B})$) can be approximated, in the inductive limit topology, by a sum of (right) inner products. Given $b \in B_e$ define $bg \in C_c(\mathcal{B})$ as $[bg](r) := bg(r)$. Let $\{b_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit of B_e as the one given in Lemma 2.8. Then $b_\lambda g \to g$ and $f^* * (b_\lambda g) \to f^* * g$ in the inductive limit topology. For every $\lambda \in \Lambda$, the function $f^* * (b_\lambda g)$ is a sum of elements of the form $f^* * (\langle x, x\rangle_{\mathcal{B}} g)$, which we will prove are inner products. Given $x \in X_s$, consider $xf \in C_c(\mathcal{X})$ given by $(xf)(r) := xf(s^{-1}r)$, and note that $f^* * (\langle x, x\rangle_{\mathcal{B}} g) = \langle xf, xg\rangle_{C^*(\mathcal{B})}$ because, for all $t \in G$,

$$\begin{aligned} f^* * (\langle x, x\rangle_{\mathcal{B}} g)(t) &= \int_G \Delta(r)^{-1} \langle xf(r^{-1}), xg(r^{-1}t)\rangle_{\mathcal{B}} \, \mathrm{d}r \\ &= \int_G \langle xf(s^{-1}r), xg(s^{-1}rt)\rangle_{\mathcal{B}} \, \mathrm{d}r = \langle xf, xg\rangle_{C^*(\mathcal{B})}(t). \end{aligned}$$

By symmetry all the claims concerning the $C^*(\mathcal{A})$-valued inner product hold. Finally, the compatibility of the operations is immediate because all the computations are performed within $C^*(\mathbb{L}(\mathcal{X}))$. ∎

The construction of a C^*-algebra from a Fell bundle [10] motivates the following definition.

DEFINITION 4.6. The *(full) cross-sectional Hilbert bimodule* of the $\mathcal{A}$-$\mathcal{B}$-equivalence bundle $\mathcal{X}$ is the $C^*(\mathcal{A})$-$C^*(\mathcal{B})$-equivalence bimodule $C^*(\mathcal{X})$ of Theorem 4.5.

It is important to recall that $C^*(\mathcal{X})$ is the closure of $C_c(\mathcal{X})$ in $C^*(\mathbb{L}(\mathcal{X}))$ and that we regard $C^*(\mathcal{A})$ and $C^*(\mathcal{B})$ as C^*-subalgebras of $C^*(\mathbb{L}(\mathcal{X}))$. These representations will be used without explicit mention in the rest of the text.

REMARK 4.7. The inductive limit topology of $C_c(\mathcal{X})$, $\tau^{\mathcal{X}}_{\mathrm{ilt}}$, contains the topology relative to $\tau^{\mathbb{L}(\mathcal{X})}_{\mathrm{ilt}}$ (see the universal property described in II 14.3 of [10]). Besides, the topology of $C_c(\mathcal{X})$ relative to the norm topology of $C^*(\mathcal{X})$, $\tau^{\mathcal{X}}_{*}$, is the one relative to $\tau^{\mathbb{L}(\mathcal{X})}_{*}$. Since $\tau^{\mathbb{L}(\mathcal{X})}_{*} \subset \tau^{\mathbb{L}(\mathcal{X})}_{\mathrm{ilt}}$, we have $\tau^{\mathcal{X}}_{*} \subset \tau^{\mathcal{X}}_{\mathrm{ilt}}$.

COROLLARY 4.8. *If $\mathcal{X}$ is a right Hilbert $\mathcal{B}$-bundle and we construct $C^*(\mathcal{X})$ considering $\mathcal{X}$ as a $\mathbb{K}(\mathcal{X})$-$\mathcal{B}$-equivalence bundle, then $\mathbb{K}(C^*(\mathcal{X}))$ is isomorphic to $C^*(\mathbb{K}(\mathcal{X}))$.*

Proof. By [17], if ${}_A X_B$ is an A-B-equivalence bimodule then A is isomorphic to $\mathbb{K}(X_B)$. To get the desired result consider the bimodule ${}_{C^*(\mathbb{K}(\mathcal{X}))}C^*(\mathcal{X})_{C^*(\mathcal{B})}$. ∎

The notation adopted for the cross-sectional equivalence bimodule is justified by the following corollary of Theorem 4.5.

COROLLARY 4.9. *If the Fell bundle $\mathcal{B}$ is regarded as a $\mathcal{B}$-$\mathcal{B}$-equivalence bundle (Example 2.5) then the cross-sectional equivalence bimodule of $\mathcal{B}$ is the cross-sectional C^*-algebra of $\mathcal{B}$ (regarded as an equivalence bimodule).*

Proof. Suppose $\mathcal{A} := \mathcal{B}$, and denote by $\mathcal{X}$ the bundle $\mathcal{B}$ when regarding it as an $\mathcal{A}$-$\mathcal{B}$-equivalence bimodule. Then $C^*(\mathcal{B})$ is the cross-sectional C^*-algebra of $\mathcal{B}$ and $C^*(\mathcal{X})$ is the cross-sectional equivalence bimodule of $\mathcal{X} = \mathcal{B}$.

We claim that the identity $\mathrm{id} : C_c(\mathcal{X}) \to C_c(\mathcal{B})$ has a unique extension to a unitary $U : C^*(\mathcal{X}) \to C^*(\mathcal{B})$. It suffices to show that, for all $f, g \in C_c(\mathcal{B})$, the element $f^* * g$ computed in $C^*(\mathbb{L}(\mathcal{X}))$ agrees with $f^* * g$ computed in $C^*(\mathcal{B})$. Recall that we may think of $\mathcal{B}$ and $\mathcal{X}$ as Banach subbundles of $\mathbb{L}(\mathcal{X})$. To avoid complicated notation we make the following identifications, for all $b \in \mathcal{B}$ and $x \in \mathcal{X}$,

$$b = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \in \mathbb{L}(\mathcal{X}) \quad \text{and} \quad x = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \mathbb{L}(\mathcal{X}).$$

If we compute $f^* * g$ in $C^*(\mathbb{L}(\mathcal{X}))$ we obtain, for all $t \in G$,

$$\begin{aligned} f^* * g(t) &= \int_G \Delta(s)^{-1} \begin{pmatrix} 0 & 0 \\ \widetilde{f(s^{-1})} & 0 \end{pmatrix} \begin{pmatrix} 0 & g(s^{-1}t) \\ 0 & 0 \end{pmatrix} \mathrm{d}s \\ &= \int_G \begin{pmatrix} 0 & 0 \\ 0 & f(s)^* g(st) \end{pmatrix} \mathrm{d}s = \int_G f(s)^* g(st)\, \mathrm{d}s. \end{aligned}$$

On the other hand, computing $f^* * g$ in $C^*(\mathcal{B})$ we obtain

$$f^* * g(t) = \int_G \Delta(s)^{-1} f(s^{-1})^* g(s^{-1}t)\, \mathrm{d}s = \int_G f(s)^* g(st)\, \mathrm{d}s.$$

Hence the claim follows. ∎

COROLLARY 4.10. *If $\mathcal{X}$ is an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle, then $\mathbb{L}(C^*(\mathcal{X}))$ is isomorphic to $C^*(\mathbb{L}(\mathcal{X}))$.*

Proof. Let $C^*(\mathcal{X}) \oplus C^*(\mathcal{B})$ be considered as a Hilbert $C^*(\mathcal{B})$-module with the inner product $\langle x \oplus y, u \oplus v \rangle = \langle x, u \rangle_{C^*(\mathcal{B})} + y^* * v$. We identify $C_c(\mathcal{X}) \oplus C_c(\mathcal{B})$ with $C_c(\mathcal{X} \oplus \mathcal{B})$ in the natural way, and represent by M the closure of $C_c(\mathcal{X} \oplus \mathcal{B})$ in $C^*(\mathbb{L}(\mathcal{X}))$. Let $U : C^*(\mathcal{X}) \oplus C^*(\mathcal{B}) \to M$ be the unitary extending the identification $C_c(\mathcal{X}) \oplus C_c(\mathcal{B}) = C_c(\mathcal{X} \oplus \mathcal{B})$.

We claim that, as a Hilbert module, M is $C^*(\mathcal{X} \oplus \mathcal{B})$. In fact, if we take $f \in C_c(\mathcal{X} \oplus \mathcal{B})$ and compute $f^* * f$ using the product and involution of $C^*(\mathbb{L}(\mathcal{X}))$ and of $C^*(\mathbb{L}(\mathcal{X} \oplus \mathcal{B}))$, we obtain the same element of $C^*(\mathcal{B})$. Moreover, at the level of $C_c(\mathcal{X} \oplus \mathcal{B})$ and $C_c(\mathcal{B})$, it does not matter whether we use $C^*(\mathbb{L}(\mathcal{X}))$ or $C^*(\mathbb{L}(\mathcal{X} \oplus \mathcal{B}))$ to compute the right inner products and the action. Then M is unitary equivalent, as a right Hilbert module, to $C^*(\mathcal{X} \oplus \mathcal{B})$.

Finally, recall that $\mathcal{X} \oplus \mathcal{B}$ is an $\mathbb{L}(\mathcal{X})$-$\mathcal{B}$-equivalence bimodule, thus we may think of $C^*(\mathbb{L}(\mathcal{X}))$ as the algebra of generalized compact operators of $C^*(\mathcal{X}) \oplus C^*(\mathcal{B})$. Thus (up to canonical isomorphisms)

$$\mathbb{L}(C^*(\mathcal{X})) = \mathbb{K}(C^*(\mathcal{X}) \oplus C^*(\mathcal{B})) = \mathbb{K}(C^*(\mathcal{X} \oplus \mathcal{B})) = C^*(\mathbb{L}(\mathcal{X})). \quad \blacksquare$$

REMARK 4.11. In the proof above we showed that $C^*(\mathcal{X} \oplus \mathcal{B})$ can be regarded as the completion of $C_c(\mathcal{X} \oplus \mathcal{B})$ in $C^*(\mathbb{L}(\mathcal{X}))$. This representation of $C^*(\mathcal{X} \oplus \mathcal{B})$ will be used instead of the representation in $C^*(\mathbb{L}(\mathcal{X} \oplus \mathcal{B}))$. Similar considerations apply for $C^*(\mathcal{A} \oplus \mathcal{B})$.

4.1. INDUCTION OF IDEALS THROUGH CROSS-SECTIONAL HILBERT BIMODULES. All the constructions we have carried out can be performed using reduced cross-sectional C^*-algebras. In fact we can use other quotients of the full cross-sectional C^*-algebra, as the ones defined in [4]. In fact we will give an alternative (and equivalent) way of extending exotic crossed products to the realm of Fell bundles.

Suppose $\mu : \mathscr{F} \to \mathscr{C}$ is a functor, from the category of Fell bundles to the category of C^*-algebras, that associates to each Fell bundle $\mathcal{B}$ a quotient $C^*_\mu(\mathcal{B})$ of $C^*(\mathcal{B})$, such that $C^*_{\mathrm{r}}(\mathcal{B})$ is in turn a quotient of $C^*_\mu(\mathcal{B})$, in such a way that the collections of the corresponding quotient maps

$$q^{\mathcal{B}}_\mu : C^*(\mathcal{B}) \to C^*_\mu(\mathcal{B}) \quad \text{and} \quad p^{\mathcal{B}}_\mu : C^*_\mu(\mathcal{B}) \to C^*_{\mathrm{r}}(\mathcal{B})$$

are natural transformations $C^* \overset{q_\mu}{\to} \mu \overset{p_\mu}{\to} C^*_{\mathrm{r}}$ satisfying $pq = \Lambda$, where Λ is the regular representation. In other words, for every morphism $\phi : \mathcal{A} \to \mathcal{B}$ the following diagram is commutative:

$$\begin{tikzcd} C^*(\mathcal{A}) \arrow[r, "q^{\mathcal{A}}_\mu"] \arrow[d, "\phi^*"'] \arrow[rr, bend left, "\Lambda^{\mathcal{A}}"] & C^*_\mu(\mathcal{A}) \arrow[r, "p^{\mathcal{A}}_\mu"] \arrow[d, "\phi_\mu"'] & C^*_{\mathrm{r}}(\mathcal{A}) \arrow[d, "\phi_{\mathrm{r}}"] \\ C^*(\mathcal{B}) \arrow[r, "q^{\mathcal{B}}_\mu"'] \arrow[rr, bend right, "\Lambda^{\mathcal{B}}"'] & C^*_\mu(\mathcal{B}) \arrow[r, "p^{\mathcal{B}}_\mu"'] & C^*_{\mathrm{r}}(\mathcal{B}). \end{tikzcd} \tag{4.1}$$

For instance, the functors C^* and C^*_{r} given by taking the universal and the reduced cross-sectional algebras respectively, satisfy the property above. Following [5], we call any such functor μ a *crossed product functor*, and we refer to $C^*_\mu(\mathcal{B})$ as the μ-crossed product of $\mathcal{B}$. When only the functors C^* and μ are involved, as well as the natural transformation q, and the left square is commutative in the above diagram, we say that μ a *pseudo crossed product functor*, and we refer to $C^*_\mu(\mathcal{B})$ as the μ-pseudo crossed product of $\mathcal{B}$. Also following [5], μ is said to be an exotic crossed product functor when it is neither the full crossed product functor C^* nor the reduced crossed product functor C^*_{r}.

If we define $I_\mu^{\mathcal{B}}$ to be the kernel of $q_\mu^{\mathcal{B}}$, the assignment $\mathcal{B} \mapsto I_\mu^{\mathcal{B}}$ is another functor $\mathscr{F} \to \mathscr{C}$, because the diagram above implies $\phi_*(I_\mu^{\mathcal{A}}) \subseteq I_\mu^{\mathcal{B}}$. Observe that if $\mathcal{A}$ is a Fell subbundle of $\mathcal{B}$ such that $C^*(\mathcal{A}) \subseteq C^*(\mathcal{B})$ (e.g. $\mathcal{A}$ is an hereditary subbundle of $\mathcal{B}$, according to Theorem 4.3), then $C^*_\mu(\mathcal{A})$ is a C^*-subalgebra of $C^*_\mu(\mathcal{B})$ if and only if $I_\mu^{\mathcal{A}} = C^*(\mathcal{A}) \cap I_\mu^{\mathcal{B}}$. We are interested in those functors which satisfy the above properties for any hereditary subbundle $\mathcal{A}$ of $\mathcal{B}$.

DEFINITION 4.12. A pseudo crossed product functor μ is said to have the hereditary subbundle property if for every Fell bundle $\mathcal{B}$ and every hereditary Fell subbundle $\mathcal{A}$ of $\mathcal{B}$, it follows that $I_\mu^{\mathcal{A}} = C^*(\mathcal{A}) \cap I_\mu^{\mathcal{B}}$.

Of course the functor C^* has the hereditary subbundle property. Since, according to Proposition 3.2 of [1], $C^*_{\mathrm{r}}(\mathcal{A}) \subseteq C^*_{\mathrm{r}}(\mathcal{B})$ for every Fell subbundle $\mathcal{A}$ of $\mathcal{B}$, also the reduced crossed product functor C^*_{r} has the hereditary subbundle property.

PROPOSITION 4.13. *Let μ be a pseudo crossed product functor with the hereditary subbundle property, and $\mathcal{X}$ an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle. Then $C^*_\mu(\mathcal{X}) := q_\mu^{\mathbb{L}(\mathcal{X})}(C^*(\mathcal{X}))$ is a $C^*_\mu(\mathcal{A})$-$C^*_\mu(\mathcal{B})$-equivalence bimodule.*

Proof. Recall from Section 2.1 that the image of a homomorphism of C^*-trings is a C^*-tring. Then $C^*_\mu(\mathcal{X})$ is a C^*-subtring of $C^*_\mu(\mathbb{L}(\mathcal{X}))$. Besides, Theorem 4.5 implies $C^*_\mu(\mathcal{X})$ is a $q_\mu^{\mathbb{L}(\mathcal{X})}(C^*(\mathcal{A}))$-$q_\mu^{\mathbb{L}(\mathcal{X})}(C^*(\mathcal{B}))$-equivalence bimodule. Finally, since $\mathcal{A}$ and $\mathcal{B}$ are hereditary Fell subbundles of $\mathbb{L}(\mathcal{X})$,

$$q_\mu^{\mathbb{L}(\mathcal{X})}(C^*(\mathcal{A})) = C^*_\mu(\mathcal{A}) \quad \text{and} \quad q_\mu^{\mathbb{L}(\mathcal{X})}(C^*(\mathcal{B})) = C^*_\mu(\mathcal{B}),$$

which ends the proof. ∎

Crossed product functors give rise to new cross-sectional Hilbert modules.

DEFINITION 4.14. In the conditions and notation of Proposition 4.13, we say that $C^*_\mu(\mathcal{X})$ is the μ-cross-sectional Hilbert bimodule of $\mathcal{X}$; the map $q_\mu^{\mathcal{X}} : C^*(\mathcal{X}) \to C^*_\mu(\mathcal{X})$ is the restriction of $q_\mu^{\mathbb{L}(\mathcal{X})}$ to $C^*(\mathcal{X})$, and the ideal $I_\mu^{\mathcal{X}}$ (of $C^*(\mathcal{X})$ regarded as a C^*-tring) is defined to be $\ker(q_\mu^{\mathcal{X}})$.

PROPOSITION 4.15. *Let μ be a pseudo crossed product functor with the hereditary subbundle property, and suppose $\mathcal{X}$ is an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle. Then $C^*(\mathcal{X})$ induces $I_\mu^{\mathcal{B}}$ to $I_\mu^{\mathcal{A}}$. Moreover, the submodule of $C^*(\mathcal{X})$ corresponding to these ideals* ([17], *Theorem* 3.22*) is $I_\mu^{\mathcal{X}}$.*

Proof. It suffices to show that $I_\mu^{\mathcal{X}} = I_\mu^{\mathcal{A}} C^*(\mathcal{X}) = C^*(\mathcal{X}) I_\mu^{\mathcal{B}}$. We prove $I_\mu^{\mathcal{X}} = C^*(\mathcal{X}) I_\mu^{\mathcal{B}}$ and leave the remaining identity to the reader.

We regard $C^*(\mathcal{B})$ as a C^*-subalgebra of $C^*(\mathbb{L}(\mathcal{X}))$ (as we are allowed by Theorem 4.3). Since $I_\mu^{\mathcal{B}} = C^*(\mathcal{B}) \cap I_\mu^{\mathbb{L}(\mathcal{X})}$, we have $C^*(\mathcal{X}) I_\mu^{\mathcal{B}} \subset I_\mu^{\mathcal{X}}$ and $\langle I_\mu^{\mathcal{X}}, I_\mu^{\mathcal{X}} \rangle_{C^*(\mathcal{B})}$

is contained in $I^{\mathcal{B}}_\mu$. Since $I^{\mathcal{X}}_\mu$ is closed we have

$$I^{\mathcal{X}}_\mu = I^{\mathcal{X}}_\mu \overline{\operatorname{span}}\, \langle I^{\mathcal{X}}_\mu, I^{\mathcal{X}}_\mu \rangle_{C^*_\mu(\mathcal{B})} \subset C^*(\mathcal{X}) I^{\mathcal{B}}_\mu \subset I^{\mathcal{X}}_\mu. \quad \blacksquare$$

One can guess that the μ-cross-sectional C^*-algebra of a Fell bundle, considered as a bimodule over itself, is the same as its μ-cross-sectional equivalence bundle. This is precisely the case.

COROLLARY 4.16. *Let $\mathcal{B}$ be a Fell bundle, and denote it $\mathcal{X}$ when regarded as a $\mathcal{B}$-$\mathcal{B}$-equivalence bundle. If we identify $C^*(\mathcal{X})$ with $C^*(\mathcal{B})$ as in Corollary 4.9, then $I^{\mathcal{B}}_\mu = I^{\mathcal{X}}_\mu$, for every pseudo crossed product functor μ with the hereditary subbundle property. In particular, $C^*_\mu(\mathcal{X})$ is $C^*_\mu(\mathcal{B})$ regarded as an equivalence bimodule.*

Proof. By Proposition 4.15, $C^*(\mathcal{X}) I^{\mathcal{B}}_\mu = I^{\mathcal{X}}_\mu$. Then, considering $C^*(\mathcal{X}) = C^*(\mathcal{B})$, we get $I^{\mathcal{X}}_\mu = C^*(\mathcal{B}) I^{\mathcal{B}}_\mu = I^{\mathcal{B}}_\mu$. $\blacksquare$

DEFINITION 4.17. Let $\mathcal{X}$ and $\mathcal{Y}$ be an $\mathcal{A}$-$\mathcal{B}$- and a $\mathcal{C}$-$\mathcal{D}$-equivalence bundles respectively. We say that $\mathcal{X}$ is an equivalence subbundle of $\mathcal{Y}$ if:

(i) $\mathcal{X}$ is a Banach subbundle of $\mathcal{Y}$, $\mathcal{A}$ is a Fell subbundle of $\mathcal{C}$ and $\mathcal{B}$ a Fell subbundle of $\mathcal{D}$;

(ii) the equivalence bundle structure of $\mathcal{X}$ agrees with that inherited from the equivalence bundle structure of $\mathcal{Y}$.

Besides, we say $\mathcal{X}$ is hereditary in $\mathcal{Y}$ if $\mathcal{X}\langle \mathcal{Y}, \mathcal{X} \rangle_{\mathcal{D}} \subset \mathcal{X}$ (note this is equivalent to the condition ${}_{\mathcal{C}}\langle \mathcal{X}, \mathcal{Y} \rangle \mathcal{X} \subseteq \mathcal{X}$).

PROPOSITION 4.18. *Let $\mathcal{X}$ and $\mathcal{Y}$ be an $\mathcal{A}$-$\mathcal{B}$- and a $\mathcal{C}$-$\mathcal{D}$-equivalence bundle, respectively, such that $\mathcal{X}$ is an equivalence subbundle of $\mathcal{Y}$. Then the following are equivalent:*

(i) *$\mathcal{X}$ is hereditary in $\mathcal{Y}$;*

(ii) *$\mathbb{L}(\mathcal{X})$ is an hereditary Fell subbundle of $\mathbb{L}(\mathcal{Y})$.*

Besides, if μ is a pseudo crossed product functor with the hereditary subbundle property and the conditions above are satisfied, then $C^*(\mathcal{X})$ is (isomorphic to) the closure of $C_c(\mathcal{X})$ in $C^*(\mathcal{Y})$ and $I^{\mathcal{X}}_\mu = C^*(\mathcal{X}) \cap I^{\mathcal{Y}}_\mu$. In particular, $C^*_\mu(\mathcal{X})$ is isomorphic (as a C^*-tring) to $q^{\mathcal{Y}}_\mu(C^*(\mathcal{X}))$.

Proof. Assume $\mathcal{X}$ is hereditary in $\mathcal{Y}$. We can regard $\mathbb{L}(\mathcal{X})$ as a Banach subbundle of $\mathbb{L}(\mathcal{Y})$ because every continuous section of $\mathcal{X}$ ($\mathcal{A}, \mathcal{B}$) is a continuous section of $\mathcal{Y}$ ($\mathcal{C}, \mathcal{D}$, respectively). Besides, the product and involution of $\mathbb{L}(\mathcal{X})$ are the ones inherited from $\mathbb{L}(\mathcal{Y})$ because they are defined in terms of the equivalence bundle structure of $\mathcal{X}$, which is inherited from $\mathcal{Y}$. Then $\mathbb{L}(\mathcal{X})$ is a Fell subbundle of $\mathbb{L}(\mathcal{Y})$.

Now fix $x \in \mathbb{L}(\mathcal{X})\mathbb{L}(\mathcal{Y})\mathbb{L}(\mathcal{X})$. Then

$$x_{1,1} \in \mathcal{A}\mathcal{C}\mathcal{A} + {}_{\mathcal{C}}\langle \mathcal{X}, \mathcal{Y} \rangle \mathcal{A} + \mathcal{A}\,{}_{\mathcal{C}}\langle \mathcal{Y}, \mathcal{X} \rangle + {}_{\mathcal{C}}\langle \mathcal{X}\mathcal{D}, \mathcal{X} \rangle.$$

Firstly, we claim that $\mathcal{A}\mathcal{C}\mathcal{A} \subset \mathcal{A}$. By Lemma 2.7, it suffices to prove that ${}_{\mathcal{C}}\langle \mathcal{X}, \mathcal{X}\rangle \mathcal{C}{}_{\mathcal{C}}\langle \mathcal{X}, \mathcal{X}\rangle \subset \mathcal{A}$, which is true because

$$\begin{aligned}{}_{\mathcal{C}}\langle \mathcal{X}, \mathcal{X}\rangle \mathcal{C}{}_{\mathcal{C}}\langle \mathcal{X}, \mathcal{X}\rangle &= {}_{\mathcal{C}}\langle \mathcal{X}, \mathcal{C}\mathcal{X}\rangle_{\mathcal{C}}\langle \mathcal{X}, \mathcal{X}\rangle \subset {}_{\mathcal{C}}\langle {}_{\mathcal{C}}\langle \mathcal{X}, \mathcal{Y}\rangle \mathcal{X}, \mathcal{X}\rangle \\ &= {}_{\mathcal{C}}\langle \mathcal{X}\langle \mathcal{Y}, \mathcal{X}\rangle_{\mathcal{D}}, \mathcal{X}\rangle \subset {}_{\mathcal{C}}\langle \mathcal{X}, \mathcal{X}\rangle \subset \mathcal{A}.\end{aligned}$$

Secondly, since the computations above also imply that ${}_{\mathcal{C}}\langle \mathcal{X}, \mathcal{Y}\rangle_{\mathcal{C}}\langle \mathcal{X}, \mathcal{X}\rangle \subset \mathcal{A}$, another invocation of Lemma 2.7 shows that ${}_{\mathcal{C}}\langle \mathcal{X}, \mathcal{Y}\rangle \mathcal{A} \subset \mathcal{A}$ and that $\mathcal{A}{}_{\mathcal{C}}\langle \mathcal{Y}, \mathcal{X}\rangle \subset \mathcal{A}$. Finally, we can use again Lemma 2.7, which together with the identity $\mathcal{X} = \mathcal{X}\mathcal{A}$ allows us to deduce that ${}_{\mathcal{C}}\mathcal{X}\langle \mathcal{X}\mathcal{D}, \mathcal{X}\rangle \subset \mathcal{A}$, because $\mathcal{X}\mathcal{D}\langle \mathcal{X}, \mathcal{X}\rangle_{\mathcal{D}} \subset \mathcal{X}$.

Putting all together we conclude that $x_{1,1} \in \mathcal{A}$. Using similar arguments it can be shown that $x_{1,2} \in \mathcal{X}$, $x_{2,1} \in \widetilde{\mathcal{X}}$ and $x_{2,2} \in \mathcal{B}$. Thus $\mathbb{L}(\mathcal{X})$ is hereditary in $\mathbb{L}(\mathcal{Y})$.

Conversely, every continuous section of $\mathcal{X}$ $(\mathcal{A}, \mathcal{B})$ is a continuous section of $\mathbb{L}(\mathcal{X})$ and so one of $\mathbb{L}(\mathcal{Y})$. This implies that $\mathcal{X}$ $(\mathcal{A}, \mathcal{B})$ is a Banach subbundle of $\mathcal{Y}$ $(\mathcal{C}, \mathcal{D}$, respectively). Also, observe that the equivalence bundle structure of $\mathcal{X}$ is the one inherited from $\mathcal{Y}$ because it is defined in terms of the product and involution of $\mathbb{L}(\mathcal{X})$, whose operations are inherited from $\mathbb{L}(\mathcal{Y})$. Furthermore, $\mathcal{X}$ is hereditary in $\mathcal{Y}$ because

$$\mathcal{X}\langle \mathcal{Y}, \mathcal{X}\rangle_{\mathcal{D}} \subset [\mathbb{L}(\mathcal{X})\mathbb{L}(\mathcal{Y})\mathbb{L}(\mathcal{X})] \cap \mathcal{Y} \subset \mathcal{X}.$$

If $\mathbb{L}(\mathcal{X})$ is an hereditary Fell subbundle of $\mathbb{L}(\mathcal{Y})$, then we can regard $C^*(\mathbb{L}(\mathcal{X}))$ as a C^*-subalgebra of $C^*(\mathbb{L}(\mathcal{Y}))$. Considering the canonical representation of $C^*(\mathcal{X})$ and $C^*(\mathcal{Y})$ in $C^*(\mathbb{L}(\mathcal{X}))$ and $C^*(\mathbb{L}(\mathcal{Y}))$, respectively, we conclude that $C^*(\mathcal{X})$ is a C^*-subtring of $C^*(\mathcal{Y})$. Thus $q^{\mathcal{Y}}_{\mu}(C^*(\mathcal{X}))$ is a C^*-subtring of $C^*_{\mu}(\mathcal{Y})$. Besides,

$$\begin{aligned}I^{\mathcal{X}}_{\mu} &= C^*(\mathcal{X}) \cap I^{\mathbb{L}(\mathcal{X})}_{\mu} = C^*(\mathcal{X}) \cap C^*(\mathbb{L}(\mathcal{X})) \cap I^{\mathbb{L}(\mathcal{Y})}_{\mu} \\ &= C^*(\mathcal{X}) \cap I^{\mathbb{L}(\mathcal{Y})}_{\mu} = C^*(\mathcal{X}) \cap I^{\mathcal{Y}}_{\mu}.\end{aligned}$$

Hence $\ker(q^{\mathcal{Y}}_{\mu}|_{C^*(\mathcal{X})}) = I^{\mathcal{X}}_{\mu}$ and there exists a unique isomorphism of C^*-trings

$$C^*_{\mu}(\mathcal{X}) \to q^{\mathcal{Y}}_{\mu}(C^*(\mathcal{X})), \quad x + I^{\mathcal{X}}_{\mu} \mapsto q^{\mathcal{Y}}_{\mu}(x). \quad \blacksquare$$

DEFINITION 4.19. Given a pseudo crossed product functor μ, with the hereditary subbundle property, and an equivalence bundle $\mathcal{X}$, we say that $\mathcal{X}$ is μ-amenable if $I^{\mathcal{X}}_{\mu} = \{0\}$. Similarly, a Fell bundle $\mathcal{B}$ will be called μ-amenable if $I^{\mathcal{B}}_{\mu} = \{0\}$.

COROLLARY 4.20. *Let $\mathcal{X}$ be an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle and μ a pseudo crossed product functor with the hereditary property. Then the following are equivalent:*

(i) *$\mathcal{X}$ is μ-amenable;*
(ii) *$\mathcal{A}$ is μ-amenable;*
(iii) *$\mathcal{B}$ is μ-amenable;*
(iv) *$\mathbb{L}(\mathcal{X})$ is μ-amenable;*

(v) $\mathbb{K}(\mathcal{X})$ *is μ-amenable.*
In particular, μ-amenability is preserved by equivalence of Fell bundles.

Proof. By Proposition 4.15 and the correspondence of ideals via equivalence bimodules, $I^{\mathcal{A}}_\mu = \{0\} \Leftrightarrow I^{\mathcal{B}}_\mu = \{0\} \Leftrightarrow I^{\mathcal{X}}_\mu = \{0\}$. Then (i), (ii) and (iii) are equivalent. Besides, that equivalence together with Remark 4.11 and Corollary 3.10 implies the last four claims are equivalent to each other. ∎

4.2. EXTENSION OF PSEUDO CROSSED PRODUCT FUNCTORS. Every Fell bundle $\mathcal{B}$ can be considered as a $\mathcal{B}$-$\mathcal{B}$-equivalence bundle, and every C^*-algebra can be considered as a C^*-tring. Then the following result says that the crossed product functor $C^* : \mathscr{F} \to \mathscr{C}$ can be extended to a functor $C^* : \mathscr{E} \to \mathscr{T}$ from the category $\mathscr{E}$ of equivalence bundles to the category $\mathscr{T}$ of C^*-trings.

THEOREM 4.21. *Assume $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{A}$-$\mathcal{B}$- and $\mathcal{C}$-$\mathcal{D}$-equivalence bundles over G, respectively. Then for every morphism of equivalence bundles, $\rho : \mathcal{X} \to \mathcal{Y}$, there exists a unique homomorphism of C^*-trings, $\rho^* : C^*(\mathcal{X}) \to C^*(\mathcal{Y})$, such that $\rho^*(f) = \rho \circ f$, for all $f \in C_c(\mathcal{X})$. This homomorphism satisfies:*
(i) $(\rho^*)^r = (\rho^{\mathrm{r}})^*$ *and* $(\rho^*)^l = (\rho^{\mathrm{l}})^*$*;*
(ii) $\mathbb{L}(\rho)^* = \mathbb{L}(\rho^*)$*, under the isomorphism provided by Corollary 4.10;*
(iii) *if μ is a pseudo crossed product functor then $\rho^*(I^{\mathcal{X}}_\mu) \subset I^{\mathcal{Y}}_\mu$.*

Proof. Uniqueness is clear because $C_c(\mathcal{X})$ is dense in $C^*(\mathcal{X})$ and every homomorphism of C^*-trings is contractive. To prove existence let $\mathbb{L}(\rho) : \mathbb{L}(\mathcal{X}) \to \mathbb{L}(\mathcal{Y})$ be the morphism given by Theorem 3.5. If we think of $C^*(\mathcal{X})$ as a C^*-subtring of $C^*(\mathbb{L}(\mathcal{X}))$, then it follows that $\rho \circ f = \mathbb{L}(\rho) \circ f$, for all $f \in C_c(\mathcal{X})$. This implies

$$\mathbb{L}(\rho)^*(C^*(\mathcal{X})) = \overline{\mathbb{L}(\rho)^*(C_c(\mathcal{X}))} \subset \overline{C_c(\mathcal{Y})} = C^*(\mathcal{Y}),$$

so it is enough to define $\rho^* := \mathbb{L}(\rho)^*|_{C^*(\mathcal{X})}$.

Note that (i) follows from the fact that, for all $f, g \in C_c(\mathcal{X})$:

$$(\rho^*)^r(f^* * g) = (\rho \circ f)^* * (\rho \circ g) = \mathbb{L}(\rho)^*(f^* * g) = (\rho^{\mathrm{r}})^*(f^* * g).$$

Similarly $(\rho^*)^l(f * g^*) = (\rho^{\mathrm{l}})^*(f * g^*)$.

To prove the second statement, note first that if $\begin{pmatrix} \xi & f \\ \widetilde{g} & \eta \end{pmatrix} \in C_c(\mathbb{L}(\mathcal{X}))$, then

$$\mathbb{L}(\rho)^* \begin{pmatrix} \xi & f \\ \widetilde{g} & \eta \end{pmatrix} = \mathbb{L}(\rho) \circ \begin{pmatrix} \xi & f \\ \widetilde{g} & \eta \end{pmatrix} = \begin{pmatrix} \rho^{\mathrm{l}} \circ \xi & \rho \circ f \\ \widetilde{\rho \circ g} & \rho^{\mathrm{r}} \circ \eta \end{pmatrix} = \begin{pmatrix} (\rho^{\mathrm{l}})^*(\xi) & \rho^*(f) \\ \widetilde{\rho^*(g)} & (\rho^{\mathrm{r}})^*(\eta) \end{pmatrix},$$

and therefore by (i) we have:

$$\mathbb{L}(\rho)^* \begin{pmatrix} \xi & f \\ \widetilde{g} & \eta \end{pmatrix} = \begin{pmatrix} (\rho^*)^l(\xi) & \rho^*(f) \\ \widetilde{\rho^*(g)} & (\rho^*)^r(\eta) \end{pmatrix} = \mathbb{L}(\rho^*) \begin{pmatrix} \xi & f \\ \widetilde{g} & \eta \end{pmatrix}.$$

Then $\mathbb{L}(\rho)^* = \mathbb{L}(\rho^*)$ on a dense subset, so they agree.

As for the last statement, we have:

$$\rho^*(I_\mu^{\mathcal{X}}) = \mathbb{L}(\rho)^*(C^*(\mathcal{X}) \cap I_\mu^{\mathbb{L}(\mathcal{X})}) \subset C^*(\mathcal{Y}) \cap I_\mu^{\mathbb{L}(\mathcal{Y})} = I_\mu^{\mathcal{Y}}. \qquad \blacksquare$$

COROLLARY 4.22. *Let $\mu : \mathscr{F} \to \mathscr{C}$ be a pseudo crossed product functor with the hereditary subbundle property. Then μ can be extended to a functor $\mu : \mathscr{E} \to \mathscr{T}$, from the category of equivalence bundles to the category of C^*-trings.*

Proof. Given an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle $\mathcal{X}$, we have $C^*_\mu(\mathcal{X}) = C^*(\mathcal{X})/I_\mu^{\mathcal{X}}$, where $I_\mu^{\mathcal{X}} = I_\mu^{\mathcal{A}} C^*(\mathcal{X}) = C^*(\mathcal{X}) I_\mu^{\mathcal{B}}$ (recall Definition 4.14 and the proof of Proposition 4.15).

Now if $\rho : \mathcal{X} \to \mathcal{Y}$ is a morphism of equivalence bundles, we have $\rho^*(I_\mu^{\mathcal{X}}) \subseteq I_\mu^{\mathcal{Y}}$ by the third statement of Theorem 4.21, so ρ^* induces a unique homomorphism of C^*-trings $\rho_\mu : C^*_\mu(\mathcal{X}) \to C^*_\mu(\mathcal{Y})$. It is easy to check that

$$(\mathcal{X} \xrightarrow{\rho} \mathcal{Y}) \longmapsto (C^*_\mu(\mathcal{X}) \xrightarrow{\rho_\mu} C^*_\mu(\mathcal{Y}))$$

is a functor that extends μ. $\blacksquare$

REMARK 4.23. Note that the extended functor also has the hereditary subbundle property, in the sense that if $\mathcal{X}$ is hereditary in $\mathcal{Y}$, then $C^*_\mu(\mathcal{X})$ is hereditary in $C^*_\mu(\mathcal{Y})$ (recall Definition 4.17, and also note that any positive C^*-tring can be thought of as an equivalence bundle over the trivial group).

REMARK 4.24. After Corollary 4.22 a natural question arises: given the extension of a crossed product functor with the hereditary subbundle property, μ, do we obtain commutative diagrams like (4.1) (see diagram (4.3) below) if we consider morphisms of equivalence bundles instead of morphism of Fell bundles? To answer this question affirmatively we need to choose the map $p_\mu^{\mathcal{X}}$: $C^*_\mu(\mathcal{X}) \to C^*_{\mathrm{r}}(\mathcal{X})$, for every equivalence bundle $\mathcal{X}$.

Given a morphism of equivalence bundles, $\phi : \mathcal{X} \to \mathcal{Y}$, the commutative diagram associated to $\mathbb{L}(\phi) : \mathbb{L}(\mathcal{X}) \to \mathbb{L}(\mathcal{Y})$ is

(4.2)
$$\begin{tikzcd}
C^*(\mathbb{L}(\mathcal{X})) \arrow[r, "q_\mu^{\mathbb{L}(\mathcal{X})}"] \arrow[rr, bend left, "q_r^{\mathbb{L}(\mathcal{X})} = \Lambda^{\mathbb{L}(\mathcal{X})}"] \arrow[d, "\mathbb{L}(\phi)^*"'] & C^*_\mu(\mathbb{L}(\mathcal{X})) \arrow[r, "p_\mu^{\mathbb{L}(\mathcal{X})}"] \arrow[d, "\mathbb{L}(\phi)_\mu"'] & C^*_{\mathrm{r}}(\mathbb{L}(\mathcal{X})) \arrow[d, "\mathbb{L}(\phi)_r"] \\
C^*(\mathbb{L}(\mathcal{Y})) \arrow[r, "q_\mu^{\mathbb{L}(\mathcal{Y})}"'] \arrow[rr, bend right, "q_r^{\mathbb{L}(\mathcal{Y})} = \Lambda^{\mathbb{L}(\mathcal{Y})}"'] & C^*_\mu(\mathbb{L}(\mathcal{Y})) \arrow[r, "p_\mu^{\mathbb{L}(\mathcal{Y})}"'] & C^*_{\mathrm{r}}(\mathbb{L}(\mathcal{Y})).
\end{tikzcd}$$

Since $C^*_\bullet(\mathcal{X}) = q_\bullet^{\mathcal{X}}(C^*(\mathcal{X}))$, for $\bullet = r, \mu$, we have

$$p_\mu^{\mathbb{L}(\mathcal{X})}(C^*_\mu(\mathcal{X})) = p_\mu^{\mathbb{L}(\mathcal{X})} \circ q_\mu^{\mathbb{L}(\mathcal{X})}(C^*(\mathcal{X})) = q_r^{\mathbb{L}(\mathcal{X})}(C^*(\mathcal{X})) = C^*_{\mathrm{r}}(\mathcal{X}).$$

Then it is natural to define $p_\mu^{\mathcal{X}} : C^*_\mu(\mathcal{X}) \to C^*_{\mathrm{r}}(\mathcal{X})$ to be the restriction of $p_r^{\mathbb{L}(\mathcal{X})}$ to $C^*_\mu(\mathcal{X})$. Adding to the commutative diagram (4.2) the fact that $\phi_\bullet : C^*_\bullet(\mathcal{X}) \to C^*_\bullet(\mathcal{Y})$ is the corresponding restriction of $\mathbb{L}(\phi)_\bullet$ (for $\bullet = r, \mu$) we obtain the commutative diagram

$$\begin{tikzcd} C^*(\mathcal{X}) \arrow[r, "q_\mu^{\mathcal{X}}"] \arrow[rr, bend left, "q_r^{\mathcal{X}}"] \arrow[d, "\phi^*"'] & C^*_\mu(\mathcal{X}) \arrow[r, "p_\mu^{\mathcal{X}}"] \arrow[d, "\phi_\mu"'] & C^*_{\mathrm{r}}(\mathcal{X}) \arrow[d, "\phi_{\mathrm{r}}"] \\ C^*(\mathcal{Y}) \arrow[r, "q_\mu^{\mathcal{Y}}"'] \arrow[rr, bend right, "q_r^{\mathcal{Y}}"'] & C^*_\mu(\mathcal{Y}) \arrow[r, "p_\mu^{\mathcal{Y}}"'] & C^*_{\mathrm{r}}(\mathcal{Y}), \end{tikzcd} \tag{4.3}$$

which is the diagram mentioned in our question.

5. INTERNAL TENSOR PRODUCTS AND TRANSITIVITY

This last section is devoted to proving that equivalence of Fell bundles is an equivalence relation. To this end we will define internal tensor products of Fell bundles.

Suppose we are given an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle, $\mathcal{X}$, and a $\mathcal{B}$-$\mathcal{C}$-equivalence bundle, $\mathcal{Y}$. Then we can form the internal tensor product $C^*(\mathcal{X}) \otimes_{C^*(\mathcal{B})} C^*(\mathcal{Y})$, which establishes a Morita–Rieffel equivalence between $C^*(\mathcal{A})$ and $C^*(\mathcal{C})$. One could expect this equivalence to come from an $\mathcal{A}$-$\mathcal{C}$-equivalence bundle $\mathcal{Z}$ in such a way that $C^*(\mathcal{Z})$ is isomorphic to $C^*(\mathcal{X}) \otimes_{C^*(\mathcal{B})} C^*(\mathcal{Y})$. This bundle $\mathcal{Z}$ should then be denoted $\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y}$, for then we would have

$$C^*(\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y}) = C^*(\mathcal{X}) \otimes_{C^*(\mathcal{B})} C^*(\mathcal{Y}).$$

In what follows we will construct such a bundle $\mathcal{Z}$. The construction is a bit complicated, and will be done along several steps. The final part of the process will consist in obtaining an equivalence bundle from a kind of a pre equivalence bundle. The following two results will serve to this purpose.

PROPOSITION 5.1. *Let $\mathcal{B}$ be a Fell bundle over G. Assume there is a bundle of normed vector spaces $\mathcal{X} := \{X_t\}_{t\in G}$, sets of sections $\Gamma_{\mathcal{X}}$ and $\Gamma_{\mathcal{B}}$ (of $\mathcal{X}$ and $\mathcal{B}$ respectively) and maps*

$$\mathcal{X} \times \mathcal{B} \to \mathcal{X} : (x,b) \mapsto xb, \quad \textit{and} \quad \mathcal{X} \times \mathcal{X} \to \mathcal{B} : (x,y) \mapsto \langle x,y\rangle_{\mathcal{B}}, \tag{5.1}$$

with the following properties:

(i) *conditions* (1R)–(7R) *from Definition* 2.1 *hold;*

(ii) *$\Gamma_{\mathcal{X}}$ is a complex vector space with pointwise operations and $\Gamma_{\mathcal{B}} \subset C_{\mathrm{c}}(\mathcal{B})$;*

(iii) *for all* $t \in G$ *and* $\mathcal{F} \in \{\mathcal{B}, \mathcal{X}\}$, $\{u(t) : u \in \Gamma_{\mathcal{F}}\}$ *is dense in* F_t, *where the norm considered on* X_t *is* $\|x\| = \|\langle x, x\rangle_{\mathcal{B}}\|^{1/2}$;

(iv) *for all* $u, v \in \Gamma_{\mathcal{X}}$ *and* $f, g \in \Gamma_{\mathcal{B}}$ *the maps*

$$G \times G \to \mathbb{R}, \quad (r,s) \mapsto \|\langle u(r), v(s)\rangle_{\mathcal{B}} - f(r^{-1}s)\|,$$
$$G \times G \to B_e, \quad (r,s) \mapsto \langle u(r)f(s) - g(rs), u(r)f(s) - g(rs)\rangle_{\mathcal{B}},$$

and $G \to B_e$, $r \mapsto \langle u(r), u(r)\rangle_{\mathcal{B}}$, *are continuous.*

Then there exists a unique right Hilbert $\mathcal{B}$*-bundle* $\overline{\mathcal{X}} = \{\overline{X}_t\}_{t\in G}$ *such that:*

(a) *for all* $t \in G$, $\overline{X}_t$ *is the completion of* X_t;

(b) $\Gamma_{\mathcal{X}}$ *is a set of continuous sections of* $\overline{\mathcal{X}}$;

(c) *the inner product and action of* $\overline{\mathcal{X}}$ *extend those of* $\mathcal{X}$.

Proof. First note that each fiber X_t is a right pre Hilbert B_e-module with positive definite inner product $(x,y) \mapsto \langle x, y\rangle_{\mathcal{B}}$.

Given $t \in G$, let $\overline{X}_t$ be the completion of X_t, and consider $\overline{\mathcal{X}} = \{\overline{X}_t\}_{t\in G}$ as an untopologized bundle over G. It follows from II 13.18 of [10] and condition (iv) that there exists a unique Banach bundle structure on $\overline{\mathcal{X}}$ such that $\Gamma_{\mathcal{X}}$ is a set of continuous $\overline{\mathcal{X}}$ sections.

Using linearity and continuity arguments we can easily prove that the action of $\mathcal{B}$ on $\mathcal{X}$, as well as the inner product, can be extended in a unique way to an action and an inner product on $\overline{\mathcal{X}}$. The same sort of arguments can be used to prove these new operations satisfy conditions (1R)–(7R) from Definition 2.1. Finally, by Proposition 3.1 and condition (iv), the inner product and the action are continuous. ∎

We also have a bilateral version of the previous result: we just need to show that the norms coming from the left and right structures agree.

PROPOSITION 5.2. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be a Fell bundles over* G. *Assume there is a bundle of complex vector spaces* $\mathcal{X} := \{X_t\}_{t\in G}$, *sets of sections* $\Gamma_{\mathcal{F}}$ *for* $\mathcal{F} \in \{\mathcal{A}, \mathcal{B}, \mathcal{X}\}$ *and maps*

$$\mathcal{A} \times \mathcal{X} \to \mathcal{X}, \quad (a,x) \mapsto ax, \quad \mathcal{X} \times \mathcal{X} \to \mathcal{A}, \quad (x,y) \mapsto {}_{\mathcal{A}}\langle x, y\rangle, \tag{5.2}$$
$$\mathcal{X} \times \mathcal{B} \to \mathcal{X}, \quad (x,b) \mapsto xb, \quad \mathcal{X} \times \mathcal{X} \to \mathcal{B}, \quad (x,y) \mapsto \langle x, y\rangle_{\mathcal{B}}, \tag{5.3}$$

with the following properties:

(i) *conditions* (1R)–(5R), (1L)–(5L), (7R) *and* (7L) *of Definitions* 2.1 *hold and, for all* $x, y, z \in \mathcal{X}$, ${}_{\mathcal{A}}\langle x, y\rangle z = x\langle y, z\rangle_{\mathcal{B}}$;

(ii) $\Gamma_{\mathcal{X}}$ *is a complex vector space with pointwise operations and* $\Gamma_{\mathcal{F}} \subset C_c(\mathcal{F})$, *for* $\mathcal{F} \in \{\mathcal{A}, \mathcal{B}\}$;

(iii) *for all* $t \in G$ *and* $\mathcal{F} \in \{\mathcal{A}, \mathcal{B}, \mathcal{X}\}$, $\{u(t) : u \in \Gamma_{\mathcal{F}}\}$ *is dense in* F_t, *where the norm considered on* X_t *is* $\|x\|_{\mathcal{B}} = \|\langle x, x\rangle_{\mathcal{B}}\|^{1/2}$;

(iv) *condition* (iv) *of Proposition* 5.1 *holds and, analogously, for all* $u, v \in \Gamma_{\mathcal{X}}$ *and* $f, g \in \Gamma_{\mathcal{A}}$ *the maps*

$$G \times G \to \mathbb{R}, \quad (r,s) \mapsto \|{}_{\mathcal{A}}\langle u(r), v(s)\rangle - f(r^{-1}s)\|,$$
$$G \times G \to A_e, \quad (r,s) \mapsto {}_{\mathcal{A}}\langle u(r)f(s) - g(r^{-1}s), u(r)f(s) - g(r^{-1}s)\rangle,$$

and $G \to A_e$, $r \mapsto {}_{\mathcal{A}}\langle u(r), u(r)\rangle$, *are continuous.*

Then there exists a unique $\mathcal{A}$-$\mathcal{B}$*-equivalence bundle* $\overline{\mathcal{X}} = \{\overline{X}_t\}_{t\in G}$ *such that:*

(a) *for all* $t \in G$, $\overline{X}_t$ *is the completion of* X_t;

(b) $\Gamma_{\mathcal{X}}$ *is a set of continuous sections of* $\overline{\mathcal{X}}$;

(c) *the inner products and actions of* $\overline{\mathcal{X}}$ *extend those of* $\mathcal{X}$.

Proof. Forgetting the left structure, and defining $\|x\|_{\mathcal{B}} = \|\langle x, x\rangle_{\mathcal{B}}\|^{1/2}$, $\forall x \in \mathcal{X}$, we are in the hypotheses of Proposition 5.1. Let $\overline{\mathcal{X}}$ be the right Hilbert $\mathcal{B}$-bundle given by Proposition 5.1. We will show that this bundle can be made into an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle preserving the right structure.

Given $t \in G$ define $I_t := \operatorname{span}{}_{\mathcal{A}}\langle X_t, X_t\rangle \subseteq A_e$. Note that conditions (1L)–(5L) imply I_t is an algebraic $*$-ideal of A_e. The compatibility of the left and right structure on $\mathcal{X}$ ensures that given $a \in I_t$ there exists a unique operator $\rho_t(a) \in \mathbb{K}(\overline{X}_t)$ such that $\rho_t(a)x = ax$, for all $x \in X_t$. Then there exists a unique $*$-homomorphism $\phi_t : I_t \to \mathbb{K}(\overline{X}_t)$, $a \mapsto \rho_t(a)$. According to VI 19.11 of [10] the homomorphism ϕ_t is norm continuous, so it has a unique extension to a $*$-representation of the C^*-ideal $\overline{I}_t$. As this last representation is contractive, for all $x \in X_t$ we have

$$\|\langle x, x\rangle_{\mathcal{B}}\| = \|{}_{\mathbb{K}(\overline{X}_t)}\langle x, x\rangle\| = \|\rho_t({}_{\mathcal{A}}\langle x, x\rangle)\| \leqslant \|{}_{\mathcal{A}}\langle x, x\rangle\|.$$

Now reverse the arguments: take the bundle of complex conjugate normed spaces $\widetilde{\mathcal{X}} = \{\widetilde{X}_{t^{-1}}\}_{t\in G}$ with the natural action of $\mathcal{A}$ on the right and the $\mathcal{A}$-valued inner product, letting $\mathcal{B}$ act on the left. Then we conclude that for all $x \in \mathcal{X}$:

$$\|{}_{\mathcal{A}}\langle x, x\rangle\| = \|\langle \widetilde{x}, \widetilde{x}\rangle_{\mathcal{A}}\| \leqslant \|{}_{\mathcal{B}}\langle \widetilde{x}, \widetilde{x}\rangle\| = \|\langle x, x\rangle_{\mathcal{B}}\|.$$

Moreover the adjoint bundle of $\overline{\widetilde{\mathcal{X}}}$ is equal to $\overline{\mathcal{X}}$ (as a Banach bundle). Then $\overline{\mathcal{X}}$ is, at the same time, a left Hilbert $\mathcal{A}$-bundle and a right Hilbert $\mathcal{B}$-bundle. We also know that ${}_{\mathcal{A}}\langle x, y\rangle z = x\langle y, z\rangle_{\mathcal{B}}$ holds for all $x, y, z \in \mathcal{X}$, and a simple continuity argument implies the same identity also holds for all $x, y, z \in \overline{\mathcal{X}}$. ∎

5.1. A TENSOR PRODUCT OF EQUIVALENCE BUNDLES. Fix, for the rest of this section, three Fell bundles $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ over G, an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle, $\mathcal{X}$, and a $\mathcal{B}$-$\mathcal{C}$-equivalence bundle, $\mathcal{Y}$. From these data we want to construct an $\mathcal{A}$-$\mathcal{C}$-equivalence bundle.

There are three bundles we will use in our construction of the $\mathcal{A}$-$\mathcal{C}$-equivalence bundle. First we define a bundle $\mathcal{Z}$ over $G \times G$, whose fibers are tensor products of the form $X_r \otimes_{B_e} Y_s$. Then we construct a bundle $\mathcal{U}$ over G by defining U_t as the set of continuous sections of compact support of the reduction of $\mathcal{Z}$ to

$\{(r,s) \in G \times G : rs = t\}$. Finally, the fibers of the bundle $[\mathcal{U}]$ will be quotients of the fibers of $\mathcal{U}$. The desired $\mathcal{A}$-$\mathcal{C}$-equivalence bundle will then be obtained from $[\mathcal{U}]$ using Proposition 5.2.

The Banach bundle $\mathcal{Z}$. Given $(r,s) \in G \times G$ let $Z_{(r,s)}$ be the Hilbert C_e-module $X_r \otimes_{B_e} Y_s$ and consider the bundle $\mathcal{Z} := \{Z_w\}_{w \in G\times G}$. We want to endow $\mathcal{Z}$ with a Banach bundle structure such that every function of the form $f \boxtimes g : G \times G \to \mathcal{Z}$, $f \boxtimes g(r,s) = f(r) \otimes g(s)$ ($f \in C_c(\mathcal{X})$, $g \in C_c(\mathcal{Y})$), is a continuous section of $\mathcal{Z}$. To this end we use II 13.18 of [10]. Let $\Gamma_{\mathcal{Z}} := \operatorname{span}\{f \boxtimes g : f \in C_c(\mathcal{X}),\ g \in C_c(\mathcal{Y})\}$. It is clear that for each $(r,s) \in G \times G$ the set $\{\xi(s,t) : \xi \in \Gamma_{\mathcal{Z}}\}$ is dense in $X_r \otimes Y_s$. For $\xi = \sum_{j=1}^{n} f_j \boxtimes g_j \in \Gamma_{\mathcal{Z}}$, the map $(r,s) \mapsto \|\xi(r,s)\|$ is continuous because so is the map

$$G \times G \to C_e, \quad (r,s) \mapsto \langle \xi(r,s), \xi(r,s)\rangle_{C_e} = \sum_{j,k=1}^{n} \langle g_j(s), \langle f_j(r), f_k(s)\rangle_{\mathcal{B}} g_k(s)\rangle_{\mathcal{C}}.$$

Then, by II 13.18 of [10], there exists a unique topology on $\mathcal{Z}$ making it a Banach bundle such that every $f \boxtimes g$ is a continuous section of $\mathcal{Z}$.

The bundle $\mathcal{U}$. To construct $\mathcal{U}$, for each $t \in G$ let $\mathcal{Z}^t$ be the reduction ([10], II 13.3) of $\mathcal{Z}$ to $H_t := \{(r,s) \in G \times G : rs = t\}$ and set $U_t := C_c(\mathcal{Z}^t)$. Let $\mathcal{U}$ be the (untopologized) bundle $\{U_t\}_{t\in G}$. Every section $\xi \in C_c(\mathcal{Z})$ defines a section of $\mathcal{U}$, $\xi|_t : G \to \mathcal{U}$, given by $t \mapsto \xi|_t$, where $\xi|_t$ is the restriction of ξ to H_t.

LEMMA 5.3. *For each $t \in G$ and $u \in U_t$, there exists a compact set $K \subset G$ such that: for all $\varepsilon > 0$ there exists $\xi \in \Gamma_{\mathcal{Z}}$ with $\|u - \xi|_t\|_\infty < \varepsilon$ and $\operatorname{supp}(\xi) \subset K \times K$. In particular, $\{\xi|_t : \xi \in \Gamma_{\mathcal{Z}}\}$ is dense in U_t in the inductive limit topology of $C_c(\mathcal{Z}^t)$.*

Proof. From Tietze's extension theorem for Banach bundles ([10], II 14.8) we know that there exists $\eta \in C_c(\mathcal{Z})$ such that $\eta|_t = u$. Since $C_c(G) \otimes C_c(G)\Gamma_{\mathcal{Z}} \subset \Gamma_{\mathcal{Z}}$, we can use Lemma 5.1 of [1] to deduce that $\Gamma_{\mathcal{Z}}$ is dense in $C_c(\mathcal{Z})$ in the inductive limit topology. Thus there exists a net $\{\xi_j\}_{j\in J} \subset \Gamma_{\mathcal{Z}}$ converging to η in the inductive limit topology, and so uniformly on compact sets. Take a compact set $K \subset G$ such that $K \times K$ contains the support of η in its interior, and take $\phi \in C_c(G)$ with: $0 \leqslant \phi \leqslant 1$, $\phi \otimes \phi|_{\operatorname{supp}\eta} \equiv 1$ and $\phi|_{G\setminus K} \equiv 0$. Then $\{\phi \otimes \phi\xi_j\}_{j\in J} \subset \Gamma_{\mathcal{Z}}$, and $\{(\phi \otimes \phi\xi_j)|_t\}_{j\in J}$ converges uniformly to $\eta|_t = u$. Then there exists $j_0 \in J$ such that $\|(\phi \otimes \phi\xi_{j_0})|_t - u\| < \varepsilon$. Finally, note that $\operatorname{supp}(\phi \otimes \phi\xi_{j_0}) \subset K \times K$. ∎

The $\mathcal{C}$-valued inner product of $\mathcal{U}$, and so the seminorm of $\mathcal{U}$, will be described as the integral of a kind of inner product defined on $\mathcal{Z}$, which we now construct. Recall that a map between vector bundles is said to be quasi-linear if it is linear when restricted to each fiber ([10], page 790). Quasi-bilinear maps are defined analogously.

LEMMA 5.4. *There exist unique continuous maps*

$$\mathcal{Z} \times \mathcal{Z} \to \mathcal{C} : (u,v) \mapsto u \triangleright v \quad \textit{and} \quad \mathcal{Z} \times \mathcal{Z} \to \mathcal{A} : (u,v) \mapsto u \triangleleft v,$$

such that:

(i) $\rhd$ *($\lhd$) is quasi-linear in the second (first) variable and conjugate quasi-linear in the first (second) variable;*

(ii) $Z_{(r,s)} \rhd Z_{(p,q)} \subset C_{(rs)^{-1}pq}$ *and* $Z_{(r,s)} \lhd Z_{(p,q)} \subset A_{rs(pq)^{-1}}$*, for all* $r,s,p,q \in G$*;*

(iii) $\|u \rhd v\| \leqslant \|u\|\|v\|$ *and* $\|u \lhd v\| = \|u\|\|v\|$*, for all* $u, v \in \mathcal{Z}$*;*

(iv) $(x \otimes y) \rhd (z \otimes w) = \langle y, \langle x, z\rangle_{\mathcal{B}} w\rangle_{\mathcal{C}}$ *and* $(x \otimes y) \lhd (z \otimes w) = {}_{\mathcal{A}}\langle x_{\mathcal{B}}\langle y, w\rangle, z\rangle$*, for all* $x \otimes y, z \otimes w \in \mathcal{Z}$*.*

Proof. Take $u = \sum_{i=1}^{n} x_i \otimes y_i \in Z_{(r,s)}$ and $v = \sum_{j=1}^{n} z_j \otimes w_j \in Z_{(p,q)}$. To satisfy (i) and (iv), $u \rhd v$ must be given by: $u \rhd v := \sum_{j,k=1}^{n} \langle y_j, \langle x_j, z_k\rangle_{\mathcal{B}} w_k\rangle_{\mathcal{C}}$. In order to see that this is really a definition, it suffices to show that

$$\Big\| \sum_{j,k=1}^{n} \langle y_j, \langle x_j, z_k\rangle_{\mathcal{B}} w_k\rangle_{\mathcal{C}} \Big\| \leqslant \|u\|\,\|v\|.$$

Now if $\{a_\lambda\}_{\lambda\in\lambda}$ is an approximate unit of A_e as the one given by Lemma 2.8, then

$$\Big\| \sum_{j,k=1}^{n} \langle y_j, \langle x_j, z_k\rangle_{\mathcal{B}} w_k\rangle_{\mathcal{C}} \Big\| = \lim_{\lambda} \Big\| \sum_{j,k=1}^{n} \langle y_j, \langle a_\lambda x_j, z_k\rangle_{\mathcal{B}} w_k\rangle_{\mathcal{C}} \Big\| \tag{5.4}$$

where $a_\lambda = \sum_{l=1}^{n_\lambda} {}_{\mathcal{A}}\langle \xi_l^\lambda, \xi_l^\lambda\rangle$, for some $\xi_l^\lambda \in X_{t_l^\lambda}$ $(l = 1, \ldots, n_\lambda)$.

From Lemma 2.7 it follows that

$$\langle y_j, \langle a_\lambda x_j, z_k\rangle_{\mathcal{B}} w_k\rangle_{\mathcal{C}} = \sum_{l=1}^{n_\lambda} \langle \langle \xi_l^\lambda, x_j\rangle_{\mathcal{B}} y_j, \langle \xi_l^\lambda, z_k\rangle_{\mathcal{B}} w_k\rangle_{\mathcal{C}}. \tag{5.5}$$

Given $l = 1, \ldots, n_\lambda$, for all $j = 1, \ldots, n$ we have $\langle \xi_l^\lambda, x_j\rangle_{\mathcal{B}} y_j \in Y_{(t_l^\lambda)^{-1}rs}$ and $\langle \xi_l^\lambda, z_j\rangle_{\mathcal{B}} w_j \in Y_{(t_l^\lambda)^{-1}pq}$. Define

$$\eta_l^\lambda := \sum_{j=1}^{n} \langle \xi_l^\lambda, x_j\rangle_{\mathcal{B}} y_j, \quad \zeta_l^\lambda := \sum_{j=1}^{n} \langle \xi_l^\lambda, z_j\rangle_{\mathcal{B}} w_j.$$

From (5.5) and (5.4) we obtain

$$\Big\| \sum_{j,k=1}^{n} \langle y_j, \langle x_j, z_k\rangle_{\mathcal{B}} w_k\rangle_{\mathcal{C}} \Big\| = \lim_{\lambda} \Big\| \sum_{k=1}^{n} \langle \eta_k^\lambda, \zeta_k^\lambda\rangle_{\mathcal{C}} \Big\|. \tag{5.6}$$

When $v = u$ we get

$$\|u\|^2 = \Big\| \sum_{j,k=1}^{n} \langle y_j, \langle x_j, x_k\rangle_{\mathcal{B}} y_k\rangle_{\mathcal{C}} \Big\| = \lim_{\lambda} \Big\| \sum_{k=1}^{n} \langle \eta_k^\lambda, \eta_k^\lambda\rangle_{\mathcal{C}} \Big\|$$

and, analogously, we obtain $\|v\|^2 = \lim_\lambda \Big\| \sum_{k=1}^n \langle \zeta_k^\lambda, \zeta_k^\lambda \rangle_{\mathcal{C}} \Big\|$. Therefore the inequality $\Big\| \sum_{j,k=1}^n \langle y_j, \langle x_j, z_k \rangle_{\mathcal{B}} w_k \rangle_{\mathcal{C}} \Big\| \leqslant \|u\| \|v\|$ follows from the inequality

$$\Big\| \sum_{k=1}^{n_\lambda} \langle \eta_k^\lambda, \zeta_k^\lambda \rangle_{\mathcal{C}} \Big\| \leqslant \Big\| \sum_{k=1}^{n_\lambda} \langle \eta_k^\lambda, \eta_k^\lambda \rangle_{\mathcal{C}} \Big\| \Big\| \sum_{k=1}^{n_\lambda} \langle \zeta_k^\lambda, \zeta_k^\lambda \rangle_{\mathcal{C}} \Big\|,$$

which holds (for all λ) by Lemma 2.8.

To show $\triangleright$ is continuous we use Proposition 3.1. Take $\xi, \eta \in \Gamma_{\mathcal{Z}}$ and $f \in C_c(\mathcal{C})$. It suffices to show that $(r,s,p,q) \mapsto \|\xi(r,s) \triangleright \eta(p,q) - f((rs)^{-1}pq)\|$ is continuous. But, since $(r,s,p,q)) \mapsto f((rs)^{-1}pq)$ is continuous, it suffices to show that $(r,s,p,q) \mapsto \xi(r,s) \triangleright \eta(p,q)$ is continuous. It is enough to consider $\xi = f \boxtimes g$ and $\eta = h \boxtimes k$. In this case we have $\xi(r,s) \triangleright \eta(p,q) = \langle g(s), \langle f(r), h(p) \rangle_{\mathcal{B}} k(q) \rangle_{\mathcal{C}}$, which is clearly a continuous function of (r,s,p,q).

The existence of the operator $\triangleleft$ can be inferred from the previous arguments applied to the adjoint bundles of $\mathcal{X}$ and $\mathcal{Y}$. ∎

Now we define two maps we will use to construct the actions of $\mathcal{A}$ and $\mathcal{C}$ on the (still not precisely defined) bundle $[\mathcal{U}]$.

LEMMA 5.5. *There are unique quasi-bilinear and continuous maps*

$$\mathcal{A} \times \mathcal{Z} \to \mathcal{Z}, \quad (a,z) \mapsto az, \quad \text{and} \quad \mathcal{Z} \times \mathcal{C} \to \mathcal{Z}, \quad (z,c) \to zc,$$

such that:

(i) $A_r Z_{(s,t)} \subset Z_{(rs,t)}$ *and* $Z_{(s,t)} C_r \subset Z_{(s,tr)}$, *for all* $r,s,t \in G$;

(ii) $\|az\| \leqslant \|a\| \|z\|$ *and* $\|zc\| \leqslant \|z\| \|c\|$, *for all* $a \in \mathcal{A}$, $z \in \mathcal{Z}$ *and* $c \in \mathcal{C}$;

(iii) $a(x \otimes y) = (ax) \otimes y$ *and* $(x \otimes y)c = x \otimes (yc)$, *for all* $a \in \mathcal{A}$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$ *and* $c \in \mathcal{C}$.

Proof. Take $u = \sum_{i=1}^n x_i \otimes y_i \in Z_{(r,s)}$ and $a \in \mathcal{A}$. Recall that there exists a natural representation of A_e, $\psi : A_e \to \mathbb{B}(Z_{(r,s)})$, such that $\psi(b)(x \otimes y) = (bx) \otimes y$. Now observe that

$$\Big\| \sum_{i=1}^n (ax_i) \otimes y_i \Big\|^2 = \Big\| \sum_{i,j=1}^n \langle y_i, \langle ax_i, ax_j \rangle_{\mathcal{B}} y_j \rangle_{\mathcal{C}} \Big\| = \Big\| \sum_{i,j=1}^n \langle y_i, \langle x_i, a^* a x_j \rangle_{\mathcal{B}} y_j \rangle_{\mathcal{C}} \Big\|$$
$$= \| \langle \psi(a^* a) u, u \rangle_{C_e} \| \leqslant \|a\|^2 \|u\|^2.$$

On the other hand, for every $c \in \mathcal{C}$ we have

$$\Big\| \sum_{i=1}^n x_i \otimes (y_i c) \Big\|^2 = \Big\| \sum_{i,j=1}^n \langle y_i c, \langle x_i, x_j \rangle_{\mathcal{B}} y_j c \rangle_{\mathcal{C}} \Big\| = \Big\| c^* \sum_{i,j=1}^n \langle y_i, \langle x_i, x_j \rangle_{\mathcal{B}} y_j \rangle_{\mathcal{C}} c \Big\|$$
$$\leqslant \|c\|^2 \| \langle u, u \rangle_{C_e} \| = \|c\|^2 \|u\|^2.$$

With these inequalities we can define the left action of $\mathcal{A}$ and the right action of $\mathcal{C}$ on $\mathcal{Z}$ on each product of fibers ($A_r \times Z_{(s,t)}$ and $Z_{(s,t)} \times C_r$). To prove that the resulting map is continuous it suffices to use Proposition 3.1 (adapting the arguments we gave during the construction of $\triangleright$ and $\triangleleft$). ∎

The identities we prove in the following lemma will be used to show the compatibility of the left and right Hilbert bundle structures of our $\mathcal{A}$-$\mathcal{C}$-equivalence bundle.

LEMMA 5.6. *For all $z_1, z_2, z_3, z_4 \in \mathcal{Z}$, $a \in \mathcal{A}$ and $c \in \mathcal{C}$ we have:*
(i) $a(z_1 \triangleleft z_2) = (az_1) \triangleleft z_2$ *and* $(z_1 \triangleright z_2)c = z_1 \triangleright (z_2 c)$;
(ii) $(z_1 \triangleleft z_2)^* = z_2 \triangleleft z_1$ *and* $(z_1 \triangleright z_2)^* = z_2 \triangleright z_1$;
(iii) $((z_1 \triangleleft z_2)z_3) \triangleright z_4 = (z_1(z_2 \triangleright z_3)) \triangleright z_4$.

Proof. By linearity it suffices to consider elementary tensors z_i. Assume $z_i = x_i \otimes y_i$, for $i = 1, 2, 3, 4$. Then

$$a(z_1 \triangleleft z_2) = a\,{}_{\mathcal{A}}\langle x_1\,{}_{\mathcal{B}}\langle y_1, y_2\rangle, x_2\rangle = {}_{\mathcal{A}}\langle a x_1\,{}_{\mathcal{B}}\langle y_1, y_2\rangle, x_2\rangle = (az_1) \triangleleft z_2.$$

The second identity in (i) and the two of (ii) are left to the reader. Besides, (iii) follows from

$$\begin{aligned}
((z_1 \triangleleft z_2)z_3) \triangleright z_4 &= ({}_{\mathcal{A}}\langle x_1\,{}_{\mathcal{B}}\langle y_1, y_2\rangle, x_2\rangle x_3 \otimes y_3) \triangleright z_4 \\
&= \langle y_3, \langle {}_{\mathcal{A}}\langle x_1\,{}_{\mathcal{B}}\langle y_1, y_2\rangle, x_2\rangle x_3, x_4\rangle_{\mathcal{B}}\, y_4\rangle_{\mathcal{C}} \\
&= \langle y_3, \langle x_1\,{}_{\mathcal{B}}\langle y_1, y_2\rangle \langle x_2, x_3\rangle_{\mathcal{B}}, x_4\rangle_{\mathcal{B}}\, y_4\rangle_{\mathcal{C}} \\
&= \langle {}_{\mathcal{B}}\langle y_1, y_2\rangle \langle x_2, x_3\rangle_{\mathcal{B}}\, y_3, \langle x_1, x_4\rangle_{\mathcal{B}}\, y_4\rangle_{\mathcal{C}} \\
&= \langle y_1 \langle y_2, \langle x_2, x_3\rangle_{\mathcal{B}}\, y_3\rangle_{\mathcal{C}}, \langle x_1, x_4\rangle_{\mathcal{B}}\, y_4\rangle_{\mathcal{C}} \\
&= (z_1 \langle y_2, \langle x_2, x_3\rangle_{\mathcal{B}}\, y_3\rangle_{\mathcal{C}}) \triangleright z_4 = (z_1(z_2 \triangleright z_3)) \triangleright z_4. \quad ∎
\end{aligned}$$

To define the pre-inner products and actions on $\mathcal{U}$ take $u \in U_r$, $v \in U_s$, $a \in A_t$ and $c \in C_t$. Note that $G \times G \to C_{r^{-1}s} : (p, q) \mapsto u(p, p^{-1}r) \triangleleft v(q, q^{-1}s)$, and $G \times G \to A_{rs^{-1}} : (p, q) \mapsto u(p, p^{-1}r) \triangleright v(q, q^{-1}s)$ are continuous maps. Then we can define

$$\text{(5.7)} \qquad {}_{\mathcal{A}}^{\mathcal{U}}\langle u, v\rangle := \iint_{G\times G} u(p, p^{-1}r) \triangleleft v(q, q^{-1}s)\, \mathrm{d}p\mathrm{d}q,$$

$$\text{(5.8)} \qquad \langle u, v\rangle_{\mathcal{C}}^{\mathcal{U}} := \iint_{G\times G} u(p, p^{-1}r) \triangleright v(q, q^{-1}s)\, \mathrm{d}p\mathrm{d}q,$$

(5.9) $au \in U_{tr}$ by the formula $(au)(p, p^{-1}tr) := au(t^{-1}p, p^{-1}tr)$, and

(5.10) $uc \in U_{rt}$ by the formula $(uc)(p, p^{-1}rt) := u(p, p^{-1}r)c$.

REMARK 5.7. Some straightforward arguments together with Lemma 5.6 imply ${}_{\mathcal{A}}^{\mathcal{U}}\langle \cdot, \cdot\rangle$ behaves like a left pre-inner product, that is: it is quasi-linear in the first variable, ${}_{\mathcal{A}}^{\mathcal{U}}\langle au, v\rangle = a\,{}_{\mathcal{A}}^{\mathcal{U}}\langle u, v\rangle$ and ${}_{\mathcal{A}}^{\mathcal{U}}\langle u, v\rangle^* = {}_{\mathcal{A}}^{\mathcal{U}}\langle v, u\rangle$. Also $\langle \cdot, \cdot\rangle_{\mathcal{C}}^{\mathcal{U}}$ behaves like a right pre-inner product with respect to $\mathcal{C}$.

REMARK 5.8. For every compact set $K \subset G$ and $u, v \in \mathcal{U}$ supported in $K \times K$, we have $\|\langle u, v\rangle_{\mathcal{C}}^{\mathcal{U}}\| \leqslant M^2 \|u\|_\infty \|v\|_\infty$, where M is the measure of K.

LEMMA 5.9. *For all $u, v, w, x \in \mathcal{U}$ and $a \in \mathcal{A}$ we have*

$$\langle {}_{\mathcal{A}}^{\mathcal{U}}\langle u, v\rangle w, x\rangle_{\mathcal{C}}^{\mathcal{U}} = \langle u\langle v, w\rangle_{\mathcal{C}}^{\mathcal{U}}, x\rangle_{\mathcal{C}}^{\mathcal{U}}, \tag{5.11}$$

$$0 \leqslant \langle u, u\rangle_{\mathcal{C}}^{\mathcal{U}}, \tag{5.12}$$

$$\langle au, au\rangle_{\mathcal{C}}^{\mathcal{U}} \leqslant \|a\|^2 \langle u, u\rangle_{\mathcal{C}}^{\mathcal{U}}. \tag{5.13}$$

Proof. To prove the first identity assume $u \in U_r$, $v \in U_s$, $w \in U_t$ and $x \in U_q$. Then, by Lemma 5.6,

$$\begin{aligned}
\langle {}_{\mathcal{A}}^{\mathcal{U}}\langle u, v\rangle w, x\rangle_{\mathcal{C}}^{\mathcal{U}} &= \int_{G^2} [{}_{\mathcal{A}}^{\mathcal{U}}\langle u, v\rangle w](p_1, {p_1}^{-1}rs^{-1}t) \triangleright x(p_2, {p_2}^{-1}q)\, \mathrm{d}(p_1, p_2) \\
&= \int_{G^2} [{}_{\mathcal{A}}^{\mathcal{U}}\langle u, v\rangle w(sr^{-1}p_1, {p_1}^{-1}rs^{-1}t)] \triangleright x(p_2, {p_2}^{-1}q)\, \mathrm{d}(p_1, p_2) \\
&= \int_{G^4} [(u(p_3, {p_3}^{-1}r) \triangleleft v(p_4, {p_4}^{-1}s)) w(sr^{-1}p_1, {p_1}^{-1}rs^{-1}t)] \triangleright \\
&\qquad x(p_2, {p_2}^{-1}q)\, \mathrm{d}(p_1, p_2, p_3, p_4) \\
&= \int_{G^4} [u(p_3, {p_3}^{-1}r)(v(p_4, {p_4}^{-1}s) \triangleright w(sr^{-1}p_1, {p_1}^{-1}rs^{-1}t))] \triangleright \\
&\qquad x(p_2, {p_2}^{-1}q)\, \mathrm{d}(p_1, p_2, p_3, p_4) \\
&= \int_{G^4} [u(p_3, {p_3}^{-1}r)(v(p_4, {p_4}^{-1}s) \triangleright w(p_1, {p_1}^{-1}t))] \triangleright \\
&\qquad x(p_2, {p_2}^{-1}q)\, \mathrm{d}(p_1, p_2, p_3, p_4) \\
&= \int_{G^2} [u(p_3, {p_3}^{-1}r)\langle v, w\rangle_{\mathcal{C}}^{\mathcal{U}}] \triangleright x(p_2, {p_2}^{-1}q)\, \mathrm{d}(p_2, p_3) \\
&= \int_{G^2} [u\langle v, w\rangle_{\mathcal{C}}^{\mathcal{U}}](p_3, {p_3}^{-1}rs^{-1}t) \triangleright x(p_2, {p_2}^{-1}q)\, \mathrm{d}(p_2, p_3) \\
&= \langle u\langle v, w\rangle_{\mathcal{C}}^{\mathcal{U}}, x\rangle_{\mathcal{C}}^{\mathcal{U}}.
\end{aligned}$$

From Remarks 5.8 and 5.3 we conclude that it suffices to show (5.12) and (5.13) hold for $u = \xi|_r$, with $\xi \in \Gamma_{\mathcal{Z}}$. Assume $\xi = \sum_{i=1}^{n} f_i \boxtimes g_i$. Then $\langle u, u\rangle_{\mathcal{C}}^{\mathcal{U}}$ and $\langle au, au\rangle_{\mathcal{C}}^{\mathcal{U}}$ are respectively the integrals in $G \times G$ of the functions $\eta, \theta : G \times G \to C_e$ given by

$$\eta(p, q) = \sum_{i,j=1}^{n} \langle g_i(p^{-1}r), \langle f_i(p), f_j(q)\rangle_{\mathcal{B}} g_j(q^{-1}r)\rangle_{\mathcal{C}},$$

$$\theta(p,q) = \sum_{i,j=1}^{n} \langle g_i(p^{-1}r), \langle af_i(t^{-1}p), af_j(t^{-1}q)\rangle_{\mathcal{B}} g_j(q^{-1}r)\rangle_{\mathcal{C}}.$$

Take a compact set $K \subset G$ such that $\mathrm{supp}(\eta) \cup \mathrm{supp}(\theta) \subset K \times K$. Given a compact neighborhood W of $e \in G$, take $p_1^W, \ldots, p_{n_W}^W \in G$ such that K is contained in the interior of $p_1^W W \cup \cdots \cup p_{n_W}^W W$. Now let $\{\phi_1^W, \ldots, \phi_{n_W}^W\} \subset C_c(G)^+$ be a partition of the unit of K subordinated to the covering $\{p_1^W W, \ldots, p_{n_W}^W W\}$. Define

$$\eta_W(p,q) := \sum_{j,k=1}^{n_W} \eta(p_j^W, p_k^W)\phi_j^W(p)\phi_k^W(q),$$

$$\theta_W(p,q) := \sum_{j,k=1}^{n_W} \theta(p_j^W, p_k^W)\phi_j^W(p)\phi_k^W(q).$$

We order the family $\mathcal{N}$ of compact neighborhoods of e by decreasing inclusion. Then we have nets $\{\eta_W\}_{W\in\mathcal{N}}$ and $\{\theta_W\}_{W\in\mathcal{N}}$ that can be shown to converge to η and θ, respectively, in the inductive limit topology. The inequalities $0 \leqslant \langle u, u\rangle_{\mathcal{C}}^{\mathcal{U}}$ and $\langle au, au\rangle_{\mathcal{C}}^{\mathcal{U}} \leqslant \|a\|^2\langle u, u\rangle_{\mathcal{C}}^{\mathcal{U}}$ follow by taking limit in W after we show that

$$0 \leqslant \int_G \int_G \eta_W(p,q)\, \mathrm{d}p\mathrm{d}q, \tag{5.14}$$

$$\int_G \int_G \theta_W(p,q)\, \mathrm{d}p\mathrm{d}q \leqslant \|a\|^2 \int_G \int_G \eta_W(p,q)\, \mathrm{d}p\mathrm{d}q. \tag{5.15}$$

Consider W fixed and put $m := n_\lambda$, $\lambda_j := \int_G \phi_j^W(p)\, \mathrm{d}p$ and $p_k := p_k^W$. Then

$$\begin{aligned}\int_G \int_G \theta_W(p,q)\, \mathrm{d}p\mathrm{d}q &= \sum_{k,l=1}^{m} \theta(p_k, p_l)\lambda_k\lambda_l \\ &= \sum_{k,l=1}^{m} \sum_{i,j=1}^{n} \langle \lambda_k g_i({p_k}^{-1}r), \langle af_i(t^{-1}p_k), af_j(t^{-1}p_l)\rangle_{\mathcal{B}} \lambda_l g_j({p_l}^{-1}r)\rangle_{\mathcal{C}}.\end{aligned}$$

The key is to interpret the latter sum as an inner product, which we do next.

Let $\mathbb{M}_{\mathbf{p}}(\mathcal{B})$ be the C^*-algebra provided by Lemma 2.8 for $\mathbf{p} = (p_1, \ldots, p_m) \in G^m$, and let $\mathbb{X}'_{\mathbf{p}} = X_{p_1} \oplus \cdots \oplus X_{p_m}$, where the direct sum is as left Hilbert A_e-modules. The left Hilbert A_e-module $\mathbb{X}'_{\mathbf{p}}$ can be given an A_e-$\mathbb{M}_{\mathbf{p}}(\mathcal{B})$ Hilbert bimodule structure in the following way. Write the elements of $\mathbb{X}'_{\mathbf{p}}$ as row matrices and define the right action by matrix multiplication; the right inner product is defined to be

$$\langle (x_1, \ldots, x_m), (y_1, \ldots, y_m)\rangle_{\mathbb{M}_{\mathbf{p}}(\mathcal{B})} = (\langle x_i, y_j\rangle_{\mathcal{B}})_{i,j=1}^m.$$

(the positivity of this inner product is shown in the same way as done in the proof of Lemma 2.8 for the inner product ${}_{\mathbb{M}_{\mathbf{r}^{-1}}(\mathcal{A})}\langle \cdot, \cdot \rangle$).

Now let $\mathbb{Y}_{\mathbf{p}^{-1}t}$ be the direct sum $Y_{{p_1}^{-1}t} \oplus \cdots \oplus Y_{{p_m}^{-1}t}$, considered as a right Hilbert C_e-module. Writing the elements of $\mathbb{Y}_{\mathbf{p}^{-1}t}$ as column matrices, the matrix multiplication by elements of $\mathbb{M}_{\mathbf{p}}(\mathcal{B})$ defines a $*$-homomorphism of $\mathbb{M}_{\mathbf{p}}(\mathcal{B})$ into $\mathbb{B}(\mathbb{Y}_{\mathbf{p}^{-1}t})$.

Now, if we define

$$\mathbf{f}_i := (f_i(t^{-1}p_1), \ldots, f_i(t^{-1}p_m)) \in \mathbb{X}'_{t^{-1}\mathbf{p}},$$
$$\mathbf{g}_j := (\lambda_1 g_j({p_1}^{-1}r), \ldots, \lambda_m g_j({p_m}^{-1}r)) \in \mathbb{Y}_{\mathbf{p}^{-1}r},$$

then we have

$$\int_G \int_G \theta_W(p,q)\,\mathrm{d}p\mathrm{d}q = \sum_{i,j=1}^{n} \langle \mathbf{g}_i, \langle a\mathbf{f}_i, a\mathbf{f}_j\rangle_{\mathbb{M}_{\mathbf{p}}(\mathcal{B})}\mathbf{g}_j\rangle_{C_e}.$$

We shall interpret the latter double sum as an inner product. Consider $\mathbb{X}^n_{t^{-1}\mathbf{p}}$ as a $\mathbb{M}_n(A_e)$-$\mathbb{M}_n(\mathbb{M}_{\mathbf{p}}(\mathcal{B}))$ Hilbert bimodule in the usual way. Considering $\mathbb{Y}^n_{\mathbf{p}^{-1}r}$ as a Hilbert C_e-module, and thinking of its elements as column matrices, matrix multiplication provides us with a representation $\mathbb{M}_n(\mathbb{M}_{\mathbf{p}}(\mathcal{B})) \to \mathbb{B}(\mathbb{Y}^n_{\mathbf{p}^{-1}r})$.

If $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_n)$, $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_n)$ and $D_{a^*a} \in \mathbb{M}_n(A_e)$ is the diagonal matrix with value a^*a in the diagonal, then

$$\int_G \int_G \theta_W(p,q)\,\mathrm{d}p\mathrm{d}q = \langle \mathbf{g}, \langle D_{a^*a}\mathbf{f}, \mathbf{f}\rangle_{\mathbb{M}_n(\mathbb{M}_{\mathbf{p}}(\mathcal{B}))}\mathbf{g}\rangle_{C_e} \leqslant \|a\|^2 \langle \mathbf{g}, \langle \mathbf{f}, \mathbf{f}\rangle_{\mathbb{M}_n(\mathbb{M}_{\mathbf{p}}(\mathcal{B}))}\mathbf{g}\rangle_{C_e}.$$

Using the interpretation of the double integral of η_W as an inner product we conclude that

$$0 \leqslant \langle \mathbf{g}, \langle \mathbf{f}, \mathbf{f}\rangle_{\mathbb{M}_n(\mathbb{M}_{\mathbf{p}}(\mathcal{B}))}\mathbf{g}\rangle_{C_e} = \int_G \int_G \eta_W(p,q)\,\mathrm{d}p\mathrm{d}q.$$

Putting the last two inequalities together we get (5.14) and (5.15). ∎

LEMMA 5.10. *For every $\xi, \eta \in \Gamma_{\mathcal{Z}}$ the following maps are continuous:*
(i) $G \times G \to \mathcal{C}$, $(p,q) \mapsto \langle \xi|_p, \eta|_q\rangle^{\mathcal{U}}_{\mathcal{C}}$;
(ii) $G \times G \to C_e$, $(p,q) \mapsto \langle g(p)\xi|_q - \eta|_{pq}, g(p)\xi|_q - \eta|_{pq}\rangle^{\mathcal{U}}_{\mathcal{C}}$.

Proof. Let θ be the map in (i). It suffices to consider the case $\xi = f \boxtimes g$ and $\eta = h \boxtimes k$. Then

$$\theta(p,q) = \iint_{G\times G} \langle g(r^{-1}p), \langle f(r), h(s)\rangle_{\mathcal{B}} k(s^{-1}q)\rangle_{\mathcal{C}}\,\mathrm{d}r\mathrm{d}s.$$

Fix $(p_0, q_0) \in G \times G$ and take a compact neighborhood V of $e \in G$. We show θ is continuous in $W := p_0V \times q_0V$.

Let $\mathcal{V}$ be the retraction of $\mathcal{C}$ by $W \to G$, $(p,q) \mapsto p^{-1}q$, and define $\eta : G \times G \to C(\mathcal{V})$ as $\eta(r,s)(p,q) = \langle g(r^{-1}p), \langle f(r), h(s)\rangle_{\mathcal{B}} k(s^{-1}q)\rangle_{\mathcal{C}}$. Note that η has compact support and $C(\mathcal{V})$ is a Banach space with the supremum norm because W is compact. Moreover, if $\{(r_i, s_i, p_i, q_i)\}_{i\in I} \subset G \times G \times W$ is a net converging to

(r,s,p,q), then $\eta(r_i,s_i)(p_i,q_i)\to\eta(r,s)(p,q)$. This implies η is continuous. Then θ is continuous because $\theta=\iint\limits_{G\times G}\eta(r,s)\,\mathrm{d}r\mathrm{d}s$.

We just give an indication of how to prove the map defined in (ii) is continuous. The trick here is to think of that map as the integral of a continuous map from $G\times G$ to the space of continuous sections of the trivial Banach bundle over W with constant fiber C_e. ∎

REMARK 5.11. The last two lemmas and their proofs can be carried out with $\langle\cdot,\cdot\rangle^{\mathcal{U}}_{\mathcal{C}}$ replaced by ${}^{\mathcal{U}}_{\mathcal{A}}\langle\cdot,\cdot\rangle$ and the actions of $\mathcal{A}$ and $\mathcal{C}$ on $\mathcal{U}$ interchanged.

Now we enter the final phase of our construction of a tensor product bundle. Each fiber U_t is a pre Hilbert C_e-module with the inner product $\langle\cdot,\cdot\rangle^{\mathcal{U}}_{\mathcal{C}}$, so $\|u\|_{\mathcal{C}}:=\|\langle u,u\rangle^{\mathcal{U}}_{\mathcal{C}}\|^{1/2}$ defines a seminorm on U_t. We also know that ${}_{\mathcal{A}}\|u\|:=\|{}^{\mathcal{U}}_{\mathcal{A}}\langle u,u\rangle\|^{1/2}$ is a seminorm on U_t. If in (5.13) we put $a={}^{\mathcal{U}}_{\mathcal{A}}\langle u,u\rangle$, then use Remark 5.7, the relations (5.12)–(5.13) and, finally, take norms in $\mathcal{C}$, we get

$$\|u\|_{\mathcal{C}}{}^6=\|\langle u,u\rangle^{\mathcal{U}}_{\mathcal{C}}{}^3\|\leqslant{}_{\mathcal{A}}\|u\|^4\|\langle u,u\rangle^{\mathcal{U}}_{\mathcal{C}}\|={}_{\mathcal{A}}\|u\|^4\|u\|_{\mathcal{C}}{}^2.$$

Then $\|u\|_{\mathcal{C}}\leqslant{}_{\mathcal{A}}\|u\|$ and, by symmetry, it must be $\|u\|_{\mathcal{C}}={}_{\mathcal{A}}\|u\|$.

Let $U^0_t:=\{u\in U_t:\|u\|_{\mathcal{C}}=0\}$ and $[U]_t:=U_t/U^0_t$. We denote $u\mapsto[u]$ the quotient map of all fibers. Then form a bundle $[\mathcal{U}]:=\{[U]_t\}_{t\in G}$ and consider the set of sections

$$\Gamma_{[\mathcal{U}]}:=\{[\xi]:\xi\in\Gamma_{\mathcal{Z}}\},\quad\text{where }[\xi](t):=[\xi|_t],\ \forall\,\xi\in\Gamma_{\mathcal{Z}},\ t\in G. \tag{5.16}$$

The action of $\mathcal{C}$ on the left is

$$[\mathcal{U}]\times\mathcal{C}\to[\mathcal{U}],\quad([u],c)\mapsto[uc],$$

which is defined because for all $u\in\mathcal{U}$ and $c\in C$

$$\|uc\|_{\mathcal{C}}{}^2=\|c^*{}^{\mathcal{U}}_{\mathcal{A}}\langle u,u\rangle c\|\leqslant\|c\|^2\|u\|_{\mathcal{C}}{}^2.$$

The $\mathcal{C}$-valued inner product is

$$[\mathcal{U}]\times[\mathcal{U}]\to\mathcal{C},\quad([u],[v])\mapsto\langle u,v\rangle^{\mathcal{U}}_{\mathcal{C}}.$$

To show this operation is defined note that for $u\in U_r$ and $v\in U_s$ we have $v,u\langle u,v\rangle^{\mathcal{U}}_{\mathcal{C}}\in U_s$. So it follows that

$$\|\langle u,v\rangle^{\mathcal{U}}_{\mathcal{C}}\|^2=\|\langle u\langle u,v\rangle^{\mathcal{U}}_{\mathcal{C}},v\rangle^{\mathcal{U}}_{\mathcal{C}}\|\leqslant\|u\|_{\mathcal{C}}\|\langle u,v\rangle^{\mathcal{U}}_{\mathcal{C}}\|\|v\|_{\mathcal{C}},$$

which in turn implies $\|\langle u,v\rangle^{\mathcal{U}}_{\mathcal{C}}\|\leqslant\|u\|_{\mathcal{C}}\|v\|_{\mathcal{C}}$.

The left hand side operations are

$$\mathcal{A}\times[\mathcal{U}]\to[\mathcal{U}],\quad(a,[u])\mapsto[au]\quad\text{and}\quad[\mathcal{U}]\times[\mathcal{U}]\to\mathcal{A},\quad([u],[v])\mapsto{}^{\mathcal{U}}_{\mathcal{A}}\langle u,v\rangle.$$

Now we use Proposition 5.2 to construct an $\mathcal{A}$-$\mathcal{C}$-equivalence bundle from $[\mathcal{U}]$. We already have the operations and inner products. Take $\Gamma_{\mathcal{A}}=C_c(\mathcal{A})$, $\Gamma_{\mathcal{C}}=C_c(\mathcal{C})$ as sets of sections, and $\Gamma_{[\mathcal{U}]}$ as we have defined in (5.16) above.

Conditions (1R)–(5R) and (1L)–(5L) follow by construction, Remark 5.7 and symmetry. Now we show (7R), and by symmetry we will have (7L). Take $x_1, x_2 \in X_r$, $y_1, y_2 \in Y_s$ and $\varepsilon > 0$. It suffices to find $u, v \in \mathcal{U}$ such that

$$\|\langle y_1, \langle x_1, x_2\rangle_{\mathcal{B}} y_2\rangle_{\mathcal{C}} - \langle u, v\rangle^{\mathcal{U}}_{\mathcal{C}}\| < \varepsilon.$$

Choose $f_i \in C_c(\mathcal{X})$ and $g_i \in C_c(\mathcal{Y})$ such that $f_i(r) = x_1$ and $g_i(s) = y_i$ $(i = 1, 2)$. Now choose a compact neighborhood of $e \in G$, W, and $\phi_W \in C_c(G)^+$ with $\int_G \phi_W(t)\phi_W(s^{-1}t^{-1}s)\,dt = 1$. Let $\xi^W_i \in \Gamma_{\mathcal{Z}}$ be defined as

$$\xi^W_i(p, q) := (\phi_W(r^{-1}p) f_i(p)) \boxtimes (\phi_W(s^{-1}q) g_i(q)).$$

Then

$$\langle \xi^W_1|_{rs}, \xi^W_2|_{rs}\rangle^{\mathcal{U}}_{\mathcal{C}} = \iint_{G\times G} \phi_W(r^{-1}p)\phi_W(s^{-1}p^{-1}rs)\phi_W(r^{-1}q)\phi_W(s^{-1}q^{-1}rs) \\ \langle g_1(p^{-1}rs), \langle f_1(p), f_2(p)\rangle_{\mathcal{B}} g_2(q^{-1}rs)\rangle_{\mathcal{C}}\,dp dq.$$

The function inside the integral is zero outside $rW \times sW$. With W small enough we can arrange the expression in the bottom of the equation (without $dpdq$) to be at most at distance $\varepsilon/2$ from $c := \langle g_1(s), \langle f_1(r), f_2(r)\rangle_{\mathcal{B}} g_2(s)\rangle_{\mathcal{C}} = \langle y_1, \langle x_1, x_2\rangle_{\mathcal{B}} y_2\rangle_{\mathcal{C}}$, for all $(p, q) \in W$. Using the identity

$$\iint_{G\times G} \phi_W(r^{-1}p)\phi_W(s^{-1}p^{-1}rs)\phi_W(r^{-1}q)\phi_W(s^{-1}q^{-1}rs)c\,dp dq = c,$$

it follows that $\|\langle \xi^W_1|_{rs}, \xi^W_2|_{rs}\rangle^{\mathcal{U}}_{\mathcal{C}} - c\| < \varepsilon$.

Once we have verified (1R)–(5R), (7R), (1L)–(5L), and (7L), we deal with the compatibility of the left and right operations. We have

$${}^{\mathcal{U}}_{\mathcal{A}}\langle [u], [u]\rangle [w] = [u]\langle [v], [w]\rangle^{\mathcal{U}}_{\mathcal{C}}$$

because, if $x := {}^{\mathcal{U}}_{\mathcal{A}}\langle [u], [u]\rangle [w] - [u]\langle [v], [w]\rangle^{\mathcal{U}}_{\mathcal{C}}$, then Lemma 5.9 implies

$$\langle x, x\rangle^{\mathcal{U}}_{\mathcal{C}} = \langle {}^{\mathcal{U}}_{\mathcal{A}}\langle [u], [u]\rangle [w], x\rangle^{\mathcal{U}}_{\mathcal{C}} - \langle [u]\langle [v], [w]\rangle^{\mathcal{U}}_{\mathcal{C}}, x\rangle^{\mathcal{U}}_{\mathcal{C}} = 0.$$

Thus $x = 0$.

Note that hypothesis (ii) of Proposition 5.2 is immediate in the present situation. Besides, hypothesis (iii) follows immediately from Remark 5.8 and Lemma 5.3. Finally (iv) follows from Lemma 5.10 and symmetry.

DEFINITION 5.12. The internal tensor product of the equivalence bundles $\mathcal{X}$ and $\mathcal{Y}$ is the equivalence bundle given by Proposition 5.2 for $[\mathcal{U}]$. This tensor product bundle is denoted $\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y}$.

The existence of $\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y}$ proves the transitivity of the relation of equivalence of Fell bundles. So we get the following theorem.

THEOREM 5.13. *Equivalence of Fell bundles is an equivalence relation.*

Proof. From Example 2.5 and Remark 2.6 we know that equivalence of Fell bundles is reflexive and symmetric. It is also transitive because of the above construction: if $\mathcal{X}$ is an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle and $\mathcal{Y}$ is a $\mathcal{B}$-$\mathcal{C}$-equivalence bundle, then $\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y}$ is an $\mathcal{A}$-$\mathcal{C}$-equivalence bundle. ∎

COROLLARY 5.14. *If $\mathcal{X}$ is an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle, then $\mathcal{A}$, $\mathcal{B}$, $\mathbb{K}(\mathcal{X})$ and $\mathbb{L}(\mathcal{X})$ are equivalent Fell bundles.*

Proof. Since equivalence of Fell bundles is an equivalence relation, it suffices to note that $\mathbb{K}(\mathcal{X})$ is isomorphic to $\mathcal{A}$, $\mathcal{A} \oplus \mathcal{X}$ is an $\mathcal{A}$-$\mathbb{L}(\mathcal{X})$-equivalence bundle and that $\mathcal{X} \oplus \mathcal{B}$ is a $\mathbb{L}(\mathcal{X})$-$\mathcal{B}$-equivalence bundle. ∎

COROLLARY 5.15. *Every Fell bundle associated to a partial action is equivalent to a Fell bundle associated to a global action, thus to a saturated Fell bundle.*

Proof. By Theorem 6.1 of [1], every partial action α has a Morita enveloping action β, that is, the partial action α is equivalent to a partial action α' that has an enveloping action β. The equivalence between α and α' provides a $\mathcal{B}_\alpha$-$\mathcal{B}_{\alpha'}$-equivalence bundle $\mathcal{X}$ (see Examples 2.2.2 and 2.4), and the enveloping action β of α' provides a $\mathcal{B}_{\alpha'}$-$\mathcal{B}_\beta$-equivalence bundle $\mathcal{Y}$ (see Examples 2.2.1 and 2.3). Therefore $\mathcal{X} \otimes_{\mathcal{B}_{\alpha'}} \mathcal{Y}$ is a $\mathcal{B}_\alpha$-$\mathcal{B}_\beta$-equivalence bundle. ∎

5.2. TENSOR PRODUCTS AND CROSS-SECTIONAL HILBERT BIMODULES.

THEOREM 5.16. *Assume $\mathcal{X}$ is an $\mathcal{A}$-$\mathcal{B}$-equivalence bundle and $\mathcal{Y}$ a $\mathcal{B}$-$\mathcal{C}$-equivalence bundle. Let $\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y}$ be the equivalence bundle of Definition* 5.12 *(see also the construction in Section* 5.1*). Then there exists a unique unitary*

$$U : C^*(\mathcal{X}) \otimes_{C^*(\mathcal{B})} C^*(\mathcal{Y}) \to C^*(\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y})$$

such that $U(f \otimes g) = [f \boxtimes g]$, for all $f \in C_c(\mathcal{X})$ and $g \in C_c(\mathcal{Y})$.

Proof. To prove the existence of the linear isometry U it is enough to show that, for all $f, f' \in C_c(\mathcal{X})$ and $g, g' \in C_c(\mathcal{Y})$, we have:

$$\langle f \otimes g, f' \otimes g' \rangle_{C^*(\mathcal{C})} = \langle [f \boxtimes g], [f' \boxtimes g'] \rangle_{C^*(\mathcal{C})},$$

where the inner product in the left member of the equality above corresponds to $C^*(\mathcal{X}) \otimes_{C^*(\mathcal{B})} C^*(\mathcal{Y})$, while the inner product in the right member is that of $C^*(\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y})$. On the one hand we have, for $r \in G \times G$:

$$\langle f \otimes g, f' \otimes g' \rangle_{C^*(\mathcal{C})}(r) = \iiint_{G^3} \langle g(s), \langle f(p), f'(psrt) \rangle_{\mathcal{B}} g'(t^{-1}) \rangle_{\mathcal{C}} \, \mathrm{d}p \mathrm{d}t \mathrm{d}s.$$

On the other hand

$$\langle [f \boxtimes g], [f' \boxtimes g'] \rangle_{C^*(\mathcal{C})}(r) = \iiint_{G^3} \langle g(p^{-1}s), \langle f(p), f'(t) \rangle_{\mathcal{B}} g'(t^{-1}sr) \rangle_{\mathcal{C}} \, \mathrm{d}p \mathrm{d}t \mathrm{d}s.$$

These triple integrals agree because the second one is obtained form the first one with the following substitutions (consecutively): $s = p^{-1}s'$ and $t' = s'rt$.

A procedure analogous to the preceding one allows us to see that U also preserves the left inner product.

Let us show that U is surjective by proving that

$$S := \operatorname{span}\{[f \boxtimes g] : f \in C_c(\mathcal{X}),\ g \in C_c(\mathcal{Y})\}$$

is dense in the inductive limit topology in $C_c(\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y})$ (see Remark 4.7). Let $\overline{S}$ be the closure of S in the inductive limit topology. We already know that $\{u(t) : u \in \overline{S}\}$ is dense in $(\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y})_t$, for all $t \in G$ (Lemma 5.3). Then from [10] we conclude it suffices to show that $C_c(G)\overline{S} \subset \overline{S}$ or, equivalently, that $\phi[f \boxtimes g] \in \overline{S}$ for all $\phi \in C_c(G)$, $f \in C_c(\mathcal{X})$ and $g \in C_c(\mathcal{Y})$.

Choose compact sets $K_1, K_2 \subset G$ such that K_1 is contained in the interior of K_2 and K_1 contains the supports of f and g in its interior. Then take $\psi \in C_c(G)$ such that $\psi f = f$, $\psi g = g$ and $\operatorname{supp}\psi \subset K_1$. The function $\Phi : G \times G \to \mathbb{C}$, $(p,q) \mapsto \psi(p)\psi(q)\phi(pq)$, is continuous, has compact support and vanishes outside $K_1 \times K_1$. Then for every $\varepsilon > 0$ there exist $\varphi_j^{h,\varepsilon} \in C_c(G)$ ($h = f, g$ and $j = 1, \dots, n_\varepsilon$) such that $\left\| \Phi - \sum_{j=1}^{n_\varepsilon} \varphi_j^{f,\varepsilon} \otimes \varphi_j^{g,\varepsilon} \right\|_\infty < \varepsilon$. Moreover, we may assume $\operatorname{supp}(\varphi_j^{h,\varepsilon}) \subset K_2$, for all h, j, ε. Now set $\xi^\varepsilon := \sum_{j=1}^{n_\varphi} (\varphi_j^{f,\varepsilon} f) \boxtimes (\varphi_j^{g,\varepsilon} g)$.

Note that $[\xi^\varepsilon] \in S$ and $\operatorname{supp}(\xi^\varepsilon) \subset K_2 \times K_2$, for all $\varepsilon > 0$. Also $\operatorname{supp}(\phi[f \boxtimes g]) \subset K_2 \times K_2$. Besides, if M_2 is the measure of K_2, then Remark 5.8 implies that for all $t \in G$ we have

$$\|\phi(t)[f \boxtimes g](t) - [\xi^\varepsilon](t)\|_{\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y}} \leqslant M_2 \|\phi(t) f \boxtimes g|_{H_t} - \xi^\varepsilon|_{H_t}\|_\infty.$$

For all $r \in G$ we have

$$\phi(t) f \boxtimes g|_{H_t}(r, r^{-1}t) = \phi(t) f(r) \boxtimes g(r^{-1}t) = \Phi(r, r^{-1}t) f(r) \otimes g(r^{-1}t).$$

Then

$$\begin{aligned} \|\phi(t) f(r) \boxtimes g(r^{-1}t) - \xi^\varepsilon(r, r^{-1}t)\| &\leqslant \left\| \Phi - \sum_{j=1}^{n_\varepsilon} \varphi_j^{f,\varepsilon} \otimes \varphi_j^{g,\varepsilon} \right\|_\infty \|f\|_\infty \|g\|_\infty \\ &\leqslant \varepsilon \|f\|_\infty \|g\|_\infty. \end{aligned}$$

Putting all together we conclude that $\|\phi[f \boxtimes g] - [\xi^\varepsilon]\|_\infty \leqslant M_2 \varepsilon \|f\|_\infty \|g\|_\infty$. Thus we have that $\phi[f \boxtimes g] \in \overline{S}$. ∎

COROLLARY 5.17. *In the hypotheses of Theorem* 5.16 *and for every pseudo crossed product μ with the hereditary subbundle property, there exists a unique unitary*

$$U_\mu : C^*_\mu(\mathcal{X}) \otimes_{C^*_\mu(\mathcal{B})} C^*_\mu(\mathcal{Y}) \to C^*_\mu(\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y})$$

such that $U_\mu(q_\mu^{\mathcal{X}}(f) \otimes q_\mu^{\mathcal{Y}}(g)) = q_\mu^{\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y}}[f \boxtimes g]$, for all $f \in C_c(\mathcal{X})$ and $g \in C_c(\mathcal{Y})$.

Proof. Uniqueness is clear, we deal with existence. Let I be the submodule of $C^*(\mathcal{X})\otimes_{C^*(\mathcal{B})}C^*(\mathcal{Y})$ corresponding to $I_\mu^{\mathcal{C}}$. Then $I = C^*(\mathcal{X})\otimes_{C^*(\mathcal{B})}C^*(\mathcal{Y})I_\mu^{\mathcal{C}}$ and, by Proposition 4.15,

$$U(I) = C^*(\mathcal{X}\otimes_{\mathcal{B}}\mathcal{Y})I_\mu^{\mathcal{C}} = I_\mu^{\mathcal{X}\otimes_{\mathcal{B}}\mathcal{Y}}.$$

Then there exists a unique unitary

$$V : C^*(\mathcal{X})\otimes_{C^*(\mathcal{B})}C^*(\mathcal{Y})/I \to C_\mu^*(\mathcal{X}\otimes_{\mathcal{B}}\mathcal{Y}),\quad \xi + I \mapsto q_\mu^{\mathcal{X}\otimes_{\mathcal{B}}\mathcal{Y}}\circ U(\xi),$$

where we have identified $C_\mu^*(\mathcal{X}\otimes_{\mathcal{B}}\mathcal{Y})$ with the quotient of $C^*(\mathcal{X}\otimes_{\mathcal{B}}\mathcal{Y})$ by $I_\mu^{\mathcal{X}\otimes_{\mathcal{B}}\mathcal{Y}}$.

To prove the existence of a unitary

$$W : C^*(\mathcal{X})\otimes_{C^*(\mathcal{B})}C^*(\mathcal{Y})/I \to C_\mu^*(\mathcal{X})\otimes_{C_\mu^*(\mathcal{B})}C_\mu^*(\mathcal{Y})$$

such that $W(f\otimes g + I) = q_\mu^{\mathcal{X}}(f)\otimes q_\mu^{\mathcal{Y}}(g)$, it suffices to prove that

$$\Big\|\sum_{i=1}^n q_\mu^{\mathcal{X}}(f_i)\otimes q_\mu^{\mathcal{Y}}(g_i)\Big\| = \Big\|\sum_{i=1}^n f_i\otimes g_i + I\Big\|$$

for all $f_1,\ldots,f_n\in C_c(\mathcal{X})$, $g_1,\ldots,g_n\in C_c(\mathcal{Y})$ and $n\in\mathbb{N}$. But, thinking of $C_\mu^*(\mathcal{Z})$ as $C^*(\mathcal{Z})/I_\mu^{\mathcal{Z}}$ for $\mathcal{Z}=\mathcal{X},\mathcal{Y},\mathcal{C}$, we obtain

$$\begin{aligned}
\Big\|\sum_{i=1}^n q_\mu^{\mathcal{X}}(f_i)\otimes q_\mu^{\mathcal{Y}}(g_i)\Big\|^2 &= \Big\|\sum_{i,j=1}^n \langle q_\mu^{\mathcal{Y}}(g_i), \langle q_\mu^{\mathcal{X}}(f_i), q_\mu^{\mathcal{X}}(f_j)\rangle_{C_\mu^*(\mathcal{B})} q_\mu^{\mathcal{Y}}(g_j)\rangle_{C_\mu^*(\mathcal{C})}\Big\| \\
&= \Big\|\sum_{i,j=1}^n \langle q_\mu^{\mathcal{Y}}(g_i), (\langle f_i, f_j\rangle_{C^*(\mathcal{B})} + I_\mu^{\mathcal{B}}) q_\mu^{\mathcal{Y}}(g_j)\rangle_{C_\mu^*(\mathcal{C})}\Big\| \\
&= \Big\|\sum_{i,j=1}^n \langle q_\mu^{\mathcal{Y}}(g_i), q_\mu^{\mathcal{Y}}(\langle f_i, f_j\rangle_{C^*(\mathcal{B})} g_j)\rangle_{C_\mu^*(\mathcal{C})}\Big\| \\
&= \Big\|\sum_{i,j=1}^n \langle g_i, \langle f_i, f_j\rangle_{C^*(\mathcal{B})} g_j\rangle_{C^*(\mathcal{C})} + I_\mu^{\mathcal{C}}\Big\| = \Big\|\sum_{i=1}^n f_i\otimes g_i + I\Big\|^2.
\end{aligned}$$

Then the unitary U_μ we are looking for is $V\circ W^*$ because, for all $f\in C_c(\mathcal{X})$ and $g\in C_c(\mathcal{Y})$,

$$V\circ W^*(q_\mu^{\mathcal{X}}(f)\otimes q_\mu^{\mathcal{Y}}(g)) = V(f\otimes g + I) = q_\mu^{\mathcal{X}\otimes_{\mathcal{B}}\mathcal{Y}}\circ U(f\otimes g) = q_\mu^{\mathcal{X}\otimes_{\mathcal{B}}\mathcal{Y}}[f\boxtimes g]. \quad ∎$$

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