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GEOMETRY OF JOINT SPECTRA AND DECOMPOSABLE OPERATOR TUPLES

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ABSTRACT. Joint spectra of tuples of operators are subsets in complex projective space. We investigate the relationship between the geometry of the spectrum and the properties of the operators in the tuple when these operators are self-adjoint. In the case when the spectrum contains an algebraic hypersurface passing through an isolated spectral point of one of the operators we give necessary and sufficient geometric conditions for the operators in the tuple to have a common reducing subspace. We also address spectral continuity and obtain a norm estimate for the commutant of a pair of self-adjoint matrices in terms of the Hausdorff distance of their joint spectrum to a family of lines.

KEYWORDS: *Joint spectrum, algebraic curves.*

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1. INTRODUCTION

When $A_1, \dots, A_n$ are $N \times N$ complex matrices, the determinant

$$\mathcal{S}(x_1, \dots, x_n) = \det(x_1 A_1 + \cdots + x_n A_n) \tag{1.1}$$

is a homogeneous polynomial of degree N in the variables $x_1, \dots, x_n$, and zeros of this polynomial determine a hypersurface in the projective space $\mathbb{CP}^{n-1}$. Conversely, given a hypersurface Γ of degree N in $\mathbb{CP}^{n-1}$, if there are $N \times N$ matrices $A_1, \dots, A_n$ such that

$$\Gamma = \{\det(x_1 A_1 + \cdots + x_n A_n) = 0\}$$

then the tuple $(A_1, \dots, A_n)$ is called a determinantal representation of Γ. A classical avenue of research in algebraic geometry with a long history, see e.g. [4], [9], [10], [18], [25], [31], is determining when a given hypersurface admits a determinantal representation, and classifying all such representations. We would like to mention specifically self-adjoint determinantal representations of real curves, and decomposable representations of reducible curves, since they are close to the subject of this paper. The former produce hyperbolic polynomials are important in relation to the Lax conjecture, cf. [16], [20], [21]. The latter have special meaning in operator theory, see [18].

The point of view from operator theory leads us to a second natural avenue of research, that seems to have attracted less attention in algebraic geometry: given that a hypersurface has a determinantal representation (or self-adjoint representation), what does the geometry of the hypersurface say about mutual relationships between the matrices $A_1, \dots, A_n$? In this direction we would like to mention the result of Motzkin and Taussky [24], which states that a real curve in $\mathbb{CP}^2$ with a self-adjoint determinantal representation satisfies the condition: the matrices of the corresponding tuple commute if and only if this curve is a union of projective lines (in [24] the result is stated in equivalent but different terms).

In 2009 R. Yang [32] started an investigation of what can be called infinite dimensional determinantal representations. Since well-known definitions of spectra of a tuple of operators such as Taylor spectrum, cf. [12], [29], exist for commuting tuples, Yang was looking for a good definition of joint spectrum for non-commuting operators and introduced the notion of joint spectrum of a tuple $(A_1, \dots, A_n)$ of operators acting on a Hilbert space H.

DEFINITION 1.1. The *joint spectrum* $\sigma(A_1, \dots, A_n)$ of $A_1, \dots, A_n$ consists of all $(x_1, \dots, x_n) \in \mathbb{C}^n$ such that $x_1A_1 + \cdots + x_nA_n$ is not invertible on H. If $A_n = I$, the identity operator, the *proper part* of the joint spectrum of $A_1, \dots, A_{n-1}$ is $\sigma_{\mathrm{p}}(A_1, \dots, A_{n-1}) = \sigma(A_1, \dots, A_{n-1}, I) \cap \{x_n = -1\}$.

Joint spectra were further investigated in [1], [3], [27]. It is easily seen that if $(x_1, \dots, x_n) \in \sigma(A_1, \dots, A_n)$, then the whole complex line $\{(cx_1, \dots, cx_n) : c \in \mathbb{C}\}$ lies in $\sigma(A_1, \dots, A_n)$, and, therefore, $\sigma(A_1, \dots, A_n)$ determines a set in $\mathbb{CP}^{n-1}$. By analogy with the finite dimensional case, given a set Γ in $\mathbb{CP}^{n-1}$, if there are operators $A_1, \dots, A_n$ acting on a Hilbert space H such that

$$\Gamma = \{[x_1 : \cdots : x_n] \in \mathbb{CP}^{n-1} : x_1A_1 + \cdots + x_nA_n \text{ is not invertible}\},$$

then it is natural to call the tuple $(A_1, \dots, A_n)$ a *spectral representation* of Γ. The main difference compared to the classical matrix case is that this set is not necessarily an analytic set. For example, if A_1 and A_2 are compact and of infinite rank, and A_3 is invertible, the whole line $\{[x_1 : x_2 : 0]\}$ in $\mathbb{CP}^2$ is contained in the joint spectrum and the spectrum is not an analytic set near each point of this line. It was shown in [27] that if $A_1, \dots, A_{n-1}$ are compact and A_n is invertible (and, therefore, can be considered to be identity) the part of the joint spectrum that lies in the chart $\{x_n \neq 0\}$ is an analytic set. When the operators $A_1, \dots, A_{n-1}$ are trace class, that part of the joint spectrum is given by the equation

$$S(x_1, \dots, x_{n-1}) = \det(x_1A_1 + \cdots + x_{n-1}A_{n-1} - I) = 0,$$

and we obtain that in this case the spectral representation is a "true" determinantal representation. In particular, when all the operators are of finite rank, the joint spectrum is a classical determinantal hypersurface in $\mathbb{CP}^{n-1}$. Of course, for infinite rank operators the analyticity holds only on an open subset of $\mathbb{CP}^{n-1}$ and that moves the problem of describing properties of the joint spectrum from the area of projective geometry to analytic geometry.

The main goal of our paper is to investigate the relationship between the geometry of the spectrum and the mutual behavior of the operators. There is a recent result [5] which generalizes to the infinite dimensional case the Motzkin–Taussky theorem mentioned above. It states that a tuple $(A_1, \dots, A_n)$ of self-adjoint compact operators acting on a separable Hilbert space commute pairwise if and only if their proper joint spectrum $\sigma_{\mathrm{p}}(A_1, \dots, A_n)$ is a locally finite union of affine hyperplanes (of course, local finiteness is not a condition but just a property coming from compactness). This suggests one needs to understand further the role that the degrees of the algebraic components of the joint spectrum play. Clearly, if the operators $A_1, \dots, A_n$ have a common eigenvector with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$, then the proper joint spectrum $\sigma_{\mathrm{p}}(A_1, \dots, A_n)$ contains a hyperplane $\{\lambda_1 x_1 + \cdots + \lambda_n x_n = 1\}$. More generally, if these operators have a common invariant subspace L of dimension k (so the corresponding tuple is decomposable with one block having dimension k), then the proper spectrum contains an algebraic hypersurface of degree k given by

$$\det(x_1(A_1|_L) + \cdots + x_n(A_n|_L) - I) = 0.$$

It is natural to ask when the converse holds (especially since in general it fails: taking for example

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 7 & 1 \\ 1 & 1 & \frac{1}{2} \end{bmatrix},$$

yields

$$\sigma_{\mathrm{p}}(A_1, A_2) = \{(x + y - 1)(5xy + 5y^2 - 15y - 10x + 2) = 0\}$$

but A_1 and A_2 have neither a common eigenvector nor a common two-dimensional invariant subspace). Our first main task is to give necessary and sufficient *geometric* conditions for the presence of an algebraic hypersurface in the proper joint spectrum to indicate decomposability, that is, the existence of a common invariant subspace of dimension equal to the degree of the hypersurface. The case of hypersurface of degree one turns out to be of fundamental importance and we address it in Theorem 2.1. The general case can then be derived from the degree one case and is addressed in Theorem 2.2 and Theorem 2.3. These results establish that the geometry of joint spectra plays a fundamental role in operator theory, and they have already found a very interesting application also to representation theory, see [8].

In the last part of the paper we address another important aspect of the geometry of joint spectra: the issue of spectral continuity, that is, if two hypersurfaces are close in a neighborhood of a point, and both have self-adjoint spectral representation of which one is decomposable, how far from being decomposable is the other? The specific question we are considering is: given that the proper joint spectrum of two operators is close to a line in a neighborhood of one of its points, does this mean that the operators have a common "almost eigenvector" (common "almost invariant" subspace)? Results in Sections 7 and 8 present conditions that guarantee that this is true, and give some norm estimates, see Theorem 2.4.

The structure of this paper is as follows. In Section 2 we give precise statements of our main results. Section 3 is devoted to determining functions. In Section 4 we derive key necessary conditions for an algebraic curve to be a component of the proper joint spectrum of two operators. These conditions are expressed in terms of holomorphy of a sequence of certain operator-valued functions. In Section 5 we prove Theorem 2.1. Theorems 2.2 and 2.3 are proved in Section 6. Section 7 is devoted to spectral continuity. Theorem 2.4 is proved in Section 8. Finally, Section 9 contains several concluding remarks and open questions.

2. STATEMENTS OF THE MAIN RESULTS

The first important case of the problem when an algebraic hypersurface in the proper joint spectrum is associated with a common invariant subspace is the case of a spectral affine hyperplane. This case turns out to be crucial for higher order spectral algebraic hypersurfaces. The following result is proved in Section 5 (here, as well as in the rest of the paper, we denote by $\Delta_\rho(x)$ the polydisk of radius ρ centered at $x \in \mathbb{C}^n$).

THEOREM 2.1. *Let $A_1, \dots, A_n$ be self-adjoint, $\lambda \neq 0$ be an isolated point of $\sigma(A_1)$, and suppose there exists $\rho > 0$ such that, up to multiplicity,*

$$\begin{aligned}\Delta_\rho(\tfrac{1}{\lambda}, 0, \dots, 0) \cap \{\lambda x_1 + a_2 x_2 + \cdots + a_n x_n = 1\} \\ = \Delta_\rho(\tfrac{1}{\lambda}, 0, \dots, 0) \cap \sigma_{\mathrm{p}}(A_1, \dots, A_n).\end{aligned}$$

The following are equivalent:

(i) *the eigensubspace of A_1 corresponding to eigenvalue λ is an eigensubspace for each of the operators $A_2, \dots, A_n$;*

(ii) *there exist an $\varepsilon \in \mathbb{R}, \varepsilon \neq 1$, and $\rho' > 0$ such that $A_1(\varepsilon, \lambda)$ is invertible and, up to multiplicity,*

$$\begin{aligned}&\Delta_{\rho'}(\lambda, 0, \dots, 0) \cap \{\tfrac{1}{\lambda} x_1 + a_2 x_2 + \cdots + a_n x_n = 1\} \\ &\qquad = \Delta_{\rho'}(\lambda, 0, \dots, 0) \cap \sigma_{\mathrm{p}}(A_1(\varepsilon, \lambda)^{-1}, A_2(\varepsilon, a_2), \dots, A_n(\varepsilon, a_n)),\end{aligned}$$

where $A(\varepsilon, b) = (1 + \varepsilon)A - b\varepsilon I$.

The most important case here is the one of two operators. Theorem 2.2 below is obtained from this case by passing to tensor powers of operators and considering their action on the exterior power of the corresponding Hilbert space. Here for an operator A acting on a Hilbert space H we write $\bigwedge^n A$ to indicate that we consider the action of $\bigotimes^n A$ on $\bigwedge^n H$. We say that a self-adjoint operator A on a separable Hilbert space H belongs to the class $\mathcal{E}(H)$ if $A = K + aI$ for some compact self-adjoint operator K and some $a \in \mathbb{R}$. It is shown in Section 6 that for operators A and B in $\mathcal{E}(H)$ one can always use an appropriate change of coordinates to reduce the search for common invariant subspaces to the "general position" setting considered in our next main result.

THEOREM 2.2. *Let $A = K_1 + aI$ and $B = K_2 + bI$ be self-adjoint operators in the class $\mathcal{E}(H)$, with A invertible. Let Γ be an algebraic curve of degree k which is a union of components of the proper joint spectrum $\sigma_{\mathrm{p}}(A, B)$, and which does not have the line $\{ax + by = 1\}$ as a reduced component. Suppose that the x-axis (respectively the y-axis) intersects Γ in the k points (counted with multiplicity) $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_k}$ (respectively $\frac{1}{\mu_1}, \dots, \frac{1}{\mu_k}$) such that each point $(\frac{1}{\lambda_i}, 0)$ belongs only to components of $\sigma_{\mathrm{p}}(A, B)$ contained in Γ. Set $\lambda = \lambda_1 \cdots \lambda_k$ and $\mu = \mu_1 \cdots \mu_k$, and suppose that λ is an isolated eigenvalue of multiplicity 1 in the spectrum of $\bigwedge^k A$. The following are equivalent:*

(i) *the eigenspace for A corresponding to $\lambda_1, \dots, \lambda_k$ is invariant for B;*

(ii) *there exists $\rho > 0$ such that the line segments*

$$\{\lambda x + \mu y = 1\} \cap \Delta_\rho(\tfrac{1}{\lambda}, 0) \quad \textit{and} \quad \{\tfrac{1}{\lambda} x + \mu y = 1\} \cap \Delta_\rho(\lambda, 0)$$

are contained in $\sigma_{\mathrm{p}}(\bigwedge^k A, \bigwedge^k B)$ and $\sigma_{\mathrm{p}}(\bigwedge^k A^{-1}, \bigwedge^k B)$, respectively;

(iii) *the lines*

$$\{\lambda x + \mu y = 1\} \quad \textit{and} \quad \{\tfrac{1}{\lambda} x + \mu y = 1\}$$

are contained in $\sigma_{\mathrm{p}}(\bigwedge^k A, \bigwedge^k B)$ and $\sigma_{\mathrm{p}}(\bigwedge^k A^{-1}, \bigwedge^k B)$, respectively.

An extension of this result to a tuple of arbitrary length holds here as well, and is derived from Theorem 2.2 exactly the same way as the result of Theorem 2.1 is derived from the coresponding result for two operators. For this reason its precise statement is omitted. Application to the classical setting when $A_1, \dots, A_n$ are self-adjoint operators on $\mathbb{C}^N$ yields the following theorem (here, as explained in Section 6, for an invertible self-adjoint operator A we consider $\bigwedge^{N-k} A$ in a natural way as an operator on $\bigwedge^k \mathbb{C}^N$, and again, we can always reduce to the "general position" situation considered below).

THEOREM 2.3. *Let C be a reducible real algebraic hypersurface of degree N in $\mathbb{C}^n$, and let Γ be a degree k hypersurface that is a union of components of C, such that for each i the x_i-axis intersects Γ in the k points $\frac{1}{\alpha_{i1}}, \dots, \frac{1}{\alpha_{ik}}$, counted with multiplicities. Let $a_i = \alpha_{i1} \cdots \alpha_{ik}$ and suppose also that each point $\frac{1}{\alpha_{1j}}$ belongs only to components of C contained in Γ. Let a tuple $(A_1, \dots, A_n)$ consisting of self-adjoint operators on $\mathbb{C}^N$,* *with A_1 invertible, be a determinantal representation of C, and suppose that a_1 is an eigenvalue of multiplicity 1 for $\bigwedge^k A_1$.*

This representation induces a determinantal representation of Γ if and only if the hypersurface

$$\left\{ \det\left(x_1 \bigwedge^k A_1 + \cdots + x_n \bigwedge^k A_n - I\right) = 0 \right\}$$

contains the hyperplane $\{a_1 x_1 + \cdots + a_n x_n = 1\}$, and the hypersurface

$$\left\{ \det\left(x_1 \bigwedge^{N-k} A_1 + x_2 \bigwedge^k A_2 + \cdots + x_n \bigwedge^k A_n - I\right) = 0 \right\}$$

contains the hyperplane $\{(\det \frac{A_1}{a_1})x_1 + a_2 x_2 + \cdots + a_n x_n = 1\}$.

Finally, we turn to spectral continuity. For a positive ε we say that a vector ξ is an ε-eigenvector of an operator A (almost eigenvector) if there exists λ such that

$$\|A\xi - \lambda\xi\| < \varepsilon\|\xi\|.$$

Our first result regarding spectral continuity, Theorem 7.2, states that, under some natural assumptions, if the joint spectrum of a pair (A_1, A_2) of self-adjoint operators, with A_1 invertible, is ε-close in the Hausdorff metric to a line $\{\alpha x + \beta y = 1\}$ in a neighborhood of an isolated spectral point of A_1, and the same is true for the joint spectrum of the pair (A_1^{-1}, A_2), then they have a common almost eigenvector of order $\sqrt{\varepsilon}$. If $|\beta| = \|A_2\|$, the condition on the joint spectrum of A_1^{-1} and A_2 can be omitted. As a corollary to this result we obtain the following estimate for the commutant of two self-adjoint matrices.

THEOREM 2.4. *Let A_1 and A_2 be two self-adjoint $N \times N$ matrices with eigenvalues $\alpha_1, \ldots, \alpha_N$ and $\beta_1, \ldots, \beta_N$ respectively, satisfying $|\alpha_1| > \cdots > |\alpha_N| > 0$ and $|\beta_1| > \cdots > |\beta_N| > 0$. Suppose that $\ell_1, \ldots, \ell_N$ is a family of lines,*

$$\ell_j = \{\alpha_{n(j)} x + \beta_j y = 1\}, \quad 1 \leqslant j, n(j) \leqslant N,$$

such that:

(i) *each of the points $(\frac{1}{\alpha_k}, 0), 1 \leqslant k \leqslant N$ belongs to one of these lines;*

(ii) *there exist $0 < \rho < 1$ and $0 < \varepsilon \ll \rho$ such that conditions* (i) *and* (ii) *of Theorem 7.4 are true for $\sigma_{\mathrm{p}}(A_1, A_2)$ and each ℓ_j.*

Then if ε is small enough, the norm of the commutant of A_1 and A_2 is at most of order $\varepsilon^{1/2^N}$.

3. DETERMINING FUNCTIONS

Let A_1 and A_2 be bounded operators on a Hilbert space H. Recall that the *proper part* of the projective joint spectrum $\sigma(A_1, A_2, I)$, or just the *proper joint* *spectrum* $\sigma_{\mathrm{p}}(A_1, A_2)$ is the following set:

$$\sigma_{\mathrm{p}}(A_1, A_2) = \{(x, y) \in \mathbb{C}^2 : (x, y, -1) \in \sigma(A_1, A_2, I)\}.$$

It was shown in [27] that if A_1 and A_2 are compact, then $\sigma_{\mathrm{p}}(A_1, A_2)$ is an analytic set of codimension one in $\mathbb{C}^2$. The following explicit construction of the analytic function locally determining this set was given there. We present it for tuples of self-adjoint operators: the setting we consider in this paper.

Suppose that $A_1, \dots, A_m$ is a tuple of compact self-adjoint operators on a Hilbert space H. Choose a small $\varepsilon > 0$ and finite rank self-adjoint operators $K_1, \dots, K_m$ such that $\|A_j - K_j\| < \varepsilon$. If $(w_1, \dots, w_m) \in \mathbb{C}^m$ satisfy $\sum_{j=1}^{m} |w_j| < \frac{1}{\varepsilon}$, then the operator $I - \sum_{j=1}^{m} w_j(A_j - K_j)$ is invertible and we have

$$\begin{aligned}\sum_{j=1}^{m} w_j A_j - I &= \sum_{j=1}^{m} w_j K_j - I + \sum_{j=1}^{m} w_j(A_j - K_j) \\ &= \Big(I - \sum_{j=1}^{m} w_j(A_j - K_j)\Big)\Big(\sum_{l=1}^{m} w_l \Big(I - \sum_{j=1}^{m} w_j(A_j - K_j)\Big)^{-1} K_l - I\Big).\end{aligned}$$

Thus, $(w_1, \dots, w_m) \in \sigma_{\mathrm{p}}(A_1, \dots, A_m)$ if and only if the operator

$$\sum_{l=1}^{m} w_l \Big(I - \sum_{j=1}^{m} w_j(A_j - K_j)\Big)^{-1} K_l - I$$

is not invertible. Since $\sum_{l=1}^{m} w_l \Big(I - \sum_{j=1}^{m} w_j(A_j - K_j)\Big)^{-1} K_l$ is of finite rank, there is a finite dimensional subspace L of H such that this operator vanishes on the complement to this subspace and is represented by an $n \times n$ matrix on this subspace. Therefore, this operator is not invertible if and only if

$$\det\Big(\sum_{l=1}^{m} w_l \Big(I - \sum_{j=1}^{m} w_j(A_j - K_j)\Big)^{-1} K_l - I\Big) = 0. \tag{3.1}$$

The left-hand side of (3.1) is clearly an analytic function of $w_1, \dots, w_m$ in the domain $\Big\{\sum_{j=1}^{m} |w_j| < \frac{1}{\varepsilon}\Big\}$ and (3.1) determines $\sigma_{\mathrm{p}}(A_1, \dots, A_m)$ in this domain. We call this function a *determining function* of the proper projective spectrum. Thus a different choice of the finite rank approximations leads to a determining function with the same divisor of zeros in $\Big\{(w_1, \dots, w_m) \in \mathbb{C}^m : \sum_{j=1}^{m} |w_j| < \frac{1}{\varepsilon}\Big\}$.

If A_1 and A_2 are not compact, the joint spectrum is not necessarily an analytic set. For example, if $A_1 = I$ is the identity operator, the joint spectrum is a cone with vertex at $(1, 0)$ that consists of lines $\{x + \lambda y = 1\}, \lambda \in \sigma(A_2)$. Thus, if the cardinality of $\sigma(A_2)$ is infinite, the joint spectrum is not analytic at $(1,0)$. Nevertheless, essentially the same argument we used above to show the analyticity of the joint spectrum in the compact case, establishes the following local result.

Let A be a bounded operator acting on H, and let λ be an isolated spectral point of A. Recall that λ is said to have multiplicity k if for a contour γ in the resolvent set that contains λ as the only spectral point of A

$$P_\lambda = \frac{1}{2\pi \mathrm{i}} \int_\gamma (wI - A)^{-1} \mathrm{d}w, \tag{3.2}$$

is a rank k projection (not necessarily orthogonal).

THEOREM 3.1. *Let A_1 and A_2 be bounded operators on H, with A_1 normal, and $\lambda \neq 0$ be an isolated point of $\sigma(A_1)$ of finite multiplicity. Then $\sigma_{\mathrm{p}}(A_1, A_2)$ is an analytic set in a neighborhood of $(\frac{1}{\lambda}, 0)$.*

Proof. The spectral decomposition of A_1 is in the form

$$A_1 = \lambda P_1 + \int_{\sigma(A_1)\setminus\{\lambda\}} z \mathrm{d}E(z),$$

where the operator P_1 is the finite rank orthogonal projection of H onto the eigenspace of A_1 with eigenvalue λ and $\mathrm{d}E$ is the spectral measure on the rest of $\sigma(A_1)$. Since λ is an isolated spectral point of A_1, if (x,y) is close to $(\frac{1}{\lambda}, 0)$, the operator

$$\widetilde{A}(x,y) = x \int_{\sigma(A_1)\setminus\{\lambda\}} z \mathrm{d}E(z) + yA_2 - I$$

is inverible. Therefore, such a point (x,y) belongs to the joint spectrum if and only if the operator

$$B(x,y) = xP_1 \widetilde{A}(x,y)^{-1} - I$$

is not invertible. Since $xP_1\widetilde{A}(x,y)$ has finite rank n which is equal to the rank of P_1, the pairs (x,y) for which $B(x,y)$ is not invertible are zeros of a determinant of an $n \times n$ matrix, whose coefficients are analytic functions of (x,y), and the result follows. ∎

If the multiplicity of an isolated spectral point $\lambda \in \sigma(A_1)$ is equal to one, the local analyticity of the joint spectrum holds even without A_1 being normal.

THEOREM 3.2. *Let A_1 and A_2 be operators on H and $\lambda \neq 0$ be an isolated spectral point of A_1 of multiplicity one. Then there exists $\rho > 0$ such that in $\Delta_\rho(\frac{1}{\lambda}, 0)$ the proper joint spectrum $\sigma_{\mathrm{p}}(A_1, A_2)$ is a nonsingular analytic set.*

Proof. If ρ is small enough and $(x,y) \in \Delta_\rho(\frac{1}{\lambda},0)$, the operator $A(x,y) = xA_1 + yA_2$ has an isolated spectral point $\lambda(x,y)$ close to 1 that also has multiplicity one, so the projection

$$P(x,y) = \frac{1}{2\pi \mathrm{i}} \int_\gamma (wI - A(x,y))^{-1} \mathrm{d}w$$

has rank one, cf. p. 13 of [14], and the range of $P(x,y)$ consists of eigenvectors of $A(x,y)$ with eigenvalue $\lambda(x,y)$. The joint spectrum of A_1 and A_2 consists of those pairs (x,y) for which $\lambda(x,y) = 1$. Let e be the unit eigenvector of A_1 with eigenvalue λ. Then for $(x,y) \in \Delta_\rho(\frac{1}{\lambda},0)$ we have that $P(x,y)e$ is close to e, and, therefore, $P(x,y)e \neq 0$. Now, $\lambda(x,y) = 1$ if and only if $A(x,y)P(x,y)e = P(x,y)e$ and that happens if and only if

$$\langle (A(x,y)P(x,y) - P(x,y))e, e\rangle = 0. \tag{3.3}$$

Equation (3.3) determines $\sigma_{\mathrm{p}}(A_1, A_2)$ near the point $(\frac{1}{\lambda},0)$ and it is easily seen that the left-hand side is analytic in x and y. Now we write down explicitly the Taylor decomposition of this function in terms of $\Delta x = x - \frac{1}{\lambda}$ and y. We have

$$\begin{aligned}
&A(x,y)P(x,y) - P(x,y)\\
&\quad= \frac{1}{2\pi \mathrm{i}} \int_\gamma (w-1)(wI - A(x,y))^{-1} \mathrm{d}w\\
&\quad= \frac{1}{2\pi \mathrm{i}} \int_\gamma (w-1)\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1}\Big(I - (\Delta x A_1 + yA_2)\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1}\Big)^{-1} \mathrm{d}w\\
&\quad= \sum_{j=0}^{\infty} \frac{1}{2\pi \mathrm{i}} \int_\gamma (w-1)\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1}\Big[(\Delta x A_1 + yA_2)\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1}\Big]^j \mathrm{d}w\\
&\quad= \sum_{k,m=0}^{\infty} (\Delta x)^k y^m \Big(\frac{1}{2\pi \mathrm{i}} \int_\gamma (w-1)\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1} \mathcal{D}_{k,m}(w)\mathrm{d}w\Big),
\end{aligned}$$

where

$$\mathcal{D}_{k,m}(w) = \sum_\alpha \prod_{l=1}^{k+m} \mathcal{S}_{\alpha_l}(w),$$

with summation taken over all sequences $\alpha = (\alpha_1, \ldots, \alpha_{k+m})$ of zeros and ones of length $(k+m)$ having k zeros and m ones, and

$$\mathcal{S}_0 = A_1\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1}, \quad \mathcal{S}_1 = A_2\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1}.$$

Thus we have

$$\sigma_{\mathrm{p}}(A_1, A_2) \cap \Delta_\rho(\tfrac{1}{\lambda},0) = \{(x,y) \in \Delta_\rho(\tfrac{1}{\lambda},0) : \mathcal{F}(x,y) = 0\},$$

where

$$\mathcal{F}(x,y)=\sum_{k,m=0}^{\infty}\Big(x-\frac{1}{\lambda}\Big)^k y^m\Big(\frac{1}{2\pi\mathrm{i}}\int_\gamma(w-1)\Big\langle\Big(wI-\frac{1}{\lambda}A_1\Big)^{-1}\mathcal{D}_{k,m}(w)e,e\Big\rangle\mathrm{d}w\Big).$$

Obviously, $\mathcal{F}(x,y)$ is a nontrivial analytic function, so the joint spectrum is an analytic set in $\Delta_\rho(\frac{1}{\lambda},0)$. Further, it follows directly from the Taylor decomposition above that

$$\begin{aligned}\frac{\partial\mathcal{F}}{\partial x}|_{x=1/\lambda,y=0}&=\frac{1}{2\pi\mathrm{i}}\int_\gamma(w-1)\Big\langle\Big(wI-\frac{1}{\lambda}A_1\Big)^{-1}A_1\Big(wI-\frac{1}{\lambda}A_1\Big)^{-1}e,e\Big\rangle\mathrm{d}w\\&=\frac{\lambda}{2\pi\mathrm{i}}\int_\gamma\frac{\mathrm{d}w}{w-1}=\lambda\neq0,\end{aligned}$$

and, therefore, the zero set of $\mathcal{F}$ is nonsingular near $(\frac{1}{\lambda},0)$. ∎

4. NECESSARY CONDITIONS FOR AN ALGEBRAIC CURVE IN THE JOINT SPECTRUM

Let A_1 and A_2 be self-adjoint operators and $\lambda\neq0$ be an isolated point of $\sigma(A_1)$ such that:

(a) $\sigma_\mathrm{p}(A_1,A_2)$ in a neighborhood $\Delta_\rho(\frac{1}{\lambda},0)$ of $(\frac{1}{\lambda},0)$ is an algebraic curve given by a polynomial equation $\mathcal{R}(x,y)=0$ of degree k, where $\mathcal{R}$ is a polynomial with real coefficients, that is

$$\sigma_\mathrm{p}(A_1,A_2)\cap\Delta_\rho(\tfrac{1}{\lambda},0)=\{(x,y)\in\Delta_\rho(\tfrac{1}{\lambda},0):\mathcal{R}(x,y)=0\};$$

(b) $(\frac{1}{\lambda},0)$ belongs to only one reduced component of this curve and is a nonsingular point on this reduced component;

(c) the axis $\{y=0\}$ is not tangent to this reduced component of the curve at $(\frac{1}{\lambda},0)$.

Here for a curve, or more generally, for a hypersurface defined by a polynomial $G=G_1^{r_1}\cdots G_m^{r_m}$ (with each polynomial G_i irreducible and G_i not associate with G_j for $i\neq j$), the *components* of that hypersurface are defined by the polynomials $G_i^{r_i}$ (thus they are irreducible but not necessarily reduced), the *reduced components* are defined by the polynomials G_i, and the exponent r_i is called the *multiplicity* of the reduced component defined by G_i.

REMARK 4.1. It is a standard exercise that when A_1 and A_2 are matrices and Γ is a reduced component of $\sigma_\mathrm{p}(A_1,A_2)$ of multiplicity r, then at each nonsingular point (x,y) of Γ the matrix xA_1+yA_2 has eigenvalue 1 of multiplicity exactly r.

Now, we write

$$\mathcal{R}(x,y)=\sum_{j=0}^{k}R_j(x,y),\quad\text{where } R_j=\sum_{m=0}^{j}r_m^j x^m y^{j-m},\text{ and } R_0=-1. \tag{4.1}$$

Passing to a smaller neighborhood if necessary, we may assume that:

(i) the reduced component containing $(\frac{1}{\lambda}, 0)$ of the curve $\{\mathcal{R}(x,y) = 0\}$ has no singular points in $\Delta_\rho(\frac{1}{\lambda}, 0)$;

(ii) there is $0 < \rho' < \rho$ such that for $(x,y) \in \Delta_{\rho'}(\frac{1}{\lambda}, 0)$ the complex line $\{(\tau x, \tau y) : \tau \in \mathbb{C}\}$ has (up to multiplicity) exactly one point of intersection with $\{\mathcal{R}(x,y) = 0\}$ that lies in $\Delta_\rho(\frac{1}{\lambda}, 0)$.

Let $(x,y) \in \Delta_{\rho'}(\frac{1}{\lambda}, 0)$ and $(\tau x, \tau y) \in \{\mathcal{R}(x,y) = 0\}$. Then $\mathcal{R}(\tau x, \tau y) = 0$, and the equation in τ

$$\tau^k R_k(x,y) + \tau^{k-1} R_{k-1}(x,y) + \cdots + \tau R_1(x,y) - 1 = 0$$

has exactly one root, $\tau(x,y)$, in a neighborhood of 1. The corresponding eigenvalue $\mu(x,y) = \frac{1}{\tau(x,y)}$ of the operator $xA_1 + yA_2$ satisfies the equation

$$\mu^k - \mu^{k-1} R_1(x,y) - \cdots - R_k(x,y) = 0. \tag{4.2}$$

Of course, $\mu(x,y)$ is the only eigenvalue of $xA_1 + yA_2$ which lies at distance of order ρ from 1 and is an isolated point of the spectrum $\sigma(xA_1 + yA_2)$. It is also clear that if λ is a multiple spectral point of A_1, then $\mu(x,y)$ has the same multiplicity.

If both x and y are real, $xA_1 + yA_2$ is self-adjoint. Let $\zeta(x,y)$ be an eigenvector of $xA_1 + yA_2$ with eigenvalue $\mu(x,y)$. Then equation (4.2) implies

$$[(xA_1 + yA_2)^k - R_1(x,y)(xA_1 + yA_2)^{k-1} - \cdots - R_k(x,y)]\zeta(x,y) = 0. \tag{4.3}$$

Let $L(x,y) \subset H$ be the eigensubspace of $xA_1 + yA_2$ corresponding to $\mu(x,y)$, and let $P(x,y) : H \to L(x,y)$ be the orthogonal projection. For $0 < \delta < \tau$ write $\gamma = \{z \in \mathbb{C} : |z - 1| < \delta\}$. We have

$$P(x,y) = \frac{1}{2\pi \mathrm{i}} \int_\gamma (wI - (xA_1 + yA_2))^{-1} \mathrm{d}w. \tag{4.4}$$

It is readily seen that for $m = 0, 1, 2, \ldots$

$$(xA_1 + yA_2)^m P(x,y) = \frac{1}{2\pi \mathrm{i}} \int_\gamma w^m (wI - (xA_1 + yA_2))^{-1} \mathrm{d}w. \tag{4.5}$$

Equations (4.3) and (4.5) imply that for every (x,y) sufficiently close to $(\frac{1}{\lambda}, 0)$ the following identity holds:

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma \left[w^k - \sum_{j=1}^k R_j(x,y) w^{k-j} \right] (wI - (xA_1 + yA_2))^{-1} \mathrm{d}w = 0.$$

Write $\Delta x = x - \frac{1}{\lambda}$. If Δx and y are sufficiently small, the last relation implies

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma \Big[w^k - \sum_{j=1}^{k} R_j(x,y) w^{k-j} \Big] \Big(wI - \frac{1}{\lambda} A_1 \Big)^{-1} \times \sum_{m=0}^{\infty} \Big[(\Delta x A_1 + y A_2) \Big(wI - \frac{1}{\lambda} A_1 \Big)^{-1} \Big]^m \mathrm{d}w = 0. \tag{4.6}$$

If $\Delta x = 0$, the last relation turns into the following:

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma \Big[w^k - \sum_{j=1}^{k} w^{k-j} \sum_{n=0}^{j} r^j_{j-n} y^n x_1^{j-n} \Big] (wI - x_1 A_1)^{-1} \times \sum_{m=0}^{\infty} y^m \Big[A_2 \Big(wI - \frac{1}{\lambda} A_1 \Big)^{-1} \Big]^m \mathrm{d}w = 0.$$

Rearranging terms in the last equation, we obtain:

$$\begin{aligned}
&\sum_{m=0}^{k-1} \frac{y^m}{2\pi \mathrm{i}} \int_\gamma \Big\{ \Big(wI - \frac{1}{\lambda} A_1 \Big)^{-1} \times \Big(\Big(w^k \sum_{j=1}^{k} w^{k-j} \frac{r^j_j}{\lambda^j} \Big) \Big[A_2 \Big(wI - \frac{1}{\lambda} A_1 \Big)^{-1} \Big]^m \\
&\quad - \sum_{n=1}^{m} \Big(\sum_{j=n}^{k} w^{k-j} \frac{r^j_{j-n}}{\lambda^{j-n}} \Big) \Big[A_2 (wI - \frac{1}{\lambda} A_1)^{-1} \Big]^{m-n} \Big) \Big\} \mathrm{d}w \\
&\quad + \sum_{m=k}^{\infty} \frac{y^m}{2\pi \mathrm{i}} \int_\gamma \Big\{ \Big(wI - \frac{1}{\lambda} A_1 \Big)^{-1} \times \Big(\Big(w^k - \sum_{j=1}^{k} w^{k-j} \frac{r^j_j}{\lambda^j} \Big) \Big[A_2 \Big(wI - \frac{1}{\lambda} A_1 \Big)^{-1} \Big]^k \\
&\quad - \sum_{n=1}^{k} \Big(\sum_{j=n}^{k} w^{k-j} \frac{r^j_{j-n}}{\lambda^{j-n}} \Big) \Big[A_2 (wI - x_1 A_1)^{-1} \Big]^{k-n} \Big) \\
&\quad \times \Big[A_2 \Big(wI - \frac{1}{\lambda} A_1 \Big)^{-1} \Big]^{m-k} \Big\} \mathrm{d}w = 0.
\end{aligned} \tag{4.7}$$

Since (4.7) holds for every y in a neighborhood of the origin, it implies

$$\begin{aligned}
&\frac{1}{2\pi \mathrm{i}} \int_\gamma \Big\{ (wI - x_1 A_1)^{-1} \Big(\Big(w^k - \sum_{j=1}^{k} w^{k-j} \frac{r^j_j}{\lambda^j} \Big) \Big[A_2 \Big(wI - \frac{1}{\lambda} A_1 \Big)^{-1} \Big]^m \\
&\quad - \sum_{n=1}^{m} \Big(\sum_{j=n}^{k} w^{k-j} \frac{r^j_{j-n}}{\lambda^{j-n}} \Big) \Big[A_2 \Big(wI - \frac{1}{\lambda} A_1 \Big)^{-1} \Big]^{m-n} \Big) \Big\} \mathrm{d}w = 0,
\end{aligned} \tag{4.8}$$

for $1 \leqslant m \leqslant k-1$, and

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma} \Big\{ \Big(wI - \frac{1}{\lambda}A_1\Big)^{-1} \times \Big(\Big(w^k - \sum_{j=1}^{k} w^{k-j} \frac{r_j^j}{\lambda^j}\Big) \Big[A_2\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1}\Big]^k$$

$$- \sum_{n=1}^{k} \Big(\sum_{j=n}^{k} w^{k-j} \frac{r_{j-n}^j}{\lambda^{j-n}}\Big) \Big[A_2\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1}\Big]^{k-n}\Big)$$

$$\times \Big[A_2\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1}\Big]^{m-k}\Big\} \mathrm{d}w = 0 \tag{4.9}$$

for $m \geqslant k$. The integrands in (4.8) and (4.9) are operator-valued holomorphic functions in the punctured disk $\{w \in \mathbb{C} : 0 < |w-1| < \delta\}$ with poles at one. We denote these integrands by $\Psi_m(w), m \geqslant 1$. Thus, (4.8) and (4.9) imply the following result.

THEOREM 4.2. *Suppose that A_1 and A_2 are self-adjoint operators acting on a separable Hilbert space H, with $\lambda \in \sigma(A_1)$ an isolated point, and an algebraic curve determined by a polynomial equation* (4.1) *lies in $\sigma_{\mathrm{p}}(A_1, A_2)$ and satisfies conditions* (a)–(c) *above. Then the integrands $\Psi_m(w)$ of* (4.8) *and* (4.9) *satisfy the equation*

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma} \Psi_m(w) \mathrm{d}w = 0, \quad m \geqslant 1,$$

which is equivalent to

$$\mathcal{R}es_{w=1}(\Psi_m) = 0, \quad m \geqslant 1. \tag{4.10}$$

REMARK 4.3. If the operators A_1 and A_2 are not self-adjoint, in general, the result of Theorem 4.2 does not hold since the range of the operator (4.4) does not necessarily consist of eigenvectors of $xA_1 + yA_2$. However, if λ is an isolated spectral point of A_1 of multiplicity one, then for (x, y) sufficiently close to $(\frac{1}{\lambda}, 0)$, the operator $P(x, y)$ is a rank one projection (not necessarily orthogonal), and its range is an eigensubspace of $xA_1 + yA_2$, so the result of Theorem 4.2 is valid in this case too.

It follows directly from (4.8) and (4.9) that Ψ_m has a pole of order at most $m+1$ at $w = 1$. We will now obtain the expression of the residue of Ψ_m at 1. Let

$$A_1 = \lambda P_1 + \int_{\sigma(A_1)\setminus\{\lambda\}} z \mathrm{d}E(z) \tag{4.11}$$

be the spectral decomposition of A_1 with P_1 being the orthogonal projection on the eigenspace of A_1 corresponding to eigenvalues λ. If δ is small enough, we have

$$\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1} = \frac{1}{w-1}P_1 + \int_{\sigma(A_1)\setminus\{\lambda\}} \frac{\mathrm{d}E(z)}{w - (z/\lambda)}$$

$$= \frac{1}{w-1}P_1 - \int\limits_{\sigma(A_1)\setminus\{\lambda\}} \Big(\sum_{m=0}^{\infty} \Big(\frac{\lambda}{z-\lambda}\Big)^{m+1}(w-1)^m\Big)\mathrm{d}E(z)$$

$$= \frac{1}{w-1}P_1 - \sum_{m=0}^{\infty}(w-1)^m\Big(\int\limits_{\sigma(A_1)\setminus\{\lambda\}} \Big(\frac{\lambda}{z-\lambda}\Big)^{m+1}\mathrm{d}E(z)\Big). \tag{4.12}$$

Write

$$T(A_1) = T = \int\limits_{\sigma(A_1)\setminus\{\lambda\}} \frac{\lambda}{z-\lambda}\mathrm{d}E(z), \tag{4.13}$$

then

$$\int\limits_{\sigma(A_1)\setminus\{\lambda\}} \Big(\frac{\lambda}{z-\lambda}\Big)^{m+1}\mathrm{d}E(z) = T^{m+1},$$

so (4.12) can be written as

$$\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1} = \frac{1}{w-1}P_1 - \sum_{m=0}^{\infty}(w-1)^m T^{m+1}, \quad \text{and} \tag{4.14}$$

$$A_2\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1} = \frac{1}{w-1}A_2P_1 - \sum_{m=0}^{\infty}(w-1)^m A_2T^{m+1}. \tag{4.15}$$

The following result follows from Theorem 4.2 and equations (4.14) and (4.15).

THEOREM 4.4. *Under the conditions of Theorem* 4.2 *the associated integrands* $\Psi_m(\lambda)$ *determined by* (4.8) *and* (4.9) *are holomorphic in* $\{w \in \mathbb{C} : |w-1| < \delta\}$.

Proof. It follows from (4.8) and (4.9) that

$$\Psi_m(w) = \Psi_{m-1}(w)\Big[A_2\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1}\Big] - \Big(\sum_{j=m}^{k} w^{k-j}\frac{r^j_{j-m}}{\lambda^{j-m}}\Big)\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1}, \quad 2 \leqslant m \leqslant k,$$

$$\Psi_m(w) = \Psi_{m-1}(w)\Big[A_2\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1}\Big], \quad m \geqslant k+1. \tag{4.16}$$

Relations (4.14), (4.15), and (4.16) imply that if Ψ_{m-1} is holomorphic, then Ψ_m has pole of order at most one at $\lambda = 1$, and, therefore, by (4.10) Ψ_m is holomorphic. Thus, it suffices to show that $\Psi_1(w)$ is holomorphic at $w = 1$. We have

$$\Psi_1(w) = \Big(wI - \frac{1}{\lambda}A_1\Big)^{-1}\Big(w^k - \sum_{j=1}^{k} w^{k-j}\frac{r^j_j}{\lambda^j}\Big)\Big[A_2\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1}\Big]$$

$$- \Big(\sum_{j=1}^{k} w^{k-j}\frac{r^j_{j-1}}{\lambda^{j-1}}\Big)\Big(wI - \frac{1}{\lambda}A_1\Big)^{-1} = \widetilde{\Psi}_1(w) + \widetilde{\widetilde{\Psi}}_1(w). \tag{4.17}$$

Observe that $\mathcal{P}(w) = w^k - \sum_{j=1}^{k} w^{k-j}\frac{r_j^j}{\lambda^j}$ satisfies $\mathcal{P}(1) = -\mathcal{R}(\frac{1}{\lambda},0) = 0$, and, therefore, $\mathcal{P}(w) = (w-1)\mathcal{Q}(w)$, where $\mathcal{Q}$ is a polynomial of degree $k-1$. Now relations (4.14) and (4.15) show that both $\widetilde{\Psi}_1$ and $\widetilde{\widetilde{\Psi}}_2$ have poles of order at most one at $w=1$, and relation (4.10) implies that Ψ_1 is holomorphic at $w=1$. ∎

5. LINE IN THE SPECTRUM

Now, suppose that, as in the previous section, A_1 and A_2 are self-adjoint, that $\lambda \neq 0$ is an isolated spectral point of A_1, and that $\sigma_{\mathrm{p}}(A_1, A_2) \cap \Delta(\frac{1}{\lambda}, 0)$ coincides, up to multiplicity, with a line segment $\{(x,y) \in \Delta_\rho(\frac{1}{\lambda},0) : \lambda x + ay = 1\}$ where $a \neq 0$. Passing to $\frac{A_1}{\lambda}$ and $\frac{A_2}{a}$ if necessary, we may assume that $\lambda = a = 1$, that is,

$$\{x+y=1\} \cap \Delta_\rho(1,0) = \sigma_{\mathrm{p}}(A_1, A_2) \cap \Delta_\rho(1,0)$$

up to multiplicity. Coming back to relation (4.1), here we have $k=1, r_1^0 = r_1^1 = 1, \mu_1 = 1, x_1 = 1$. Let us write down the functions Ψ_1 and Ψ_2 in this particular case. Equations (4.9), (4.14), and (4.15) imply

$$\begin{aligned}\Psi_1(w) = &\Big(\frac{1}{w-1}P_1 - \sum_{m=0}^{\infty}(w-1)^m T^{m+1}\Big)\Big((w-1)\Big[\frac{1}{w-1}A_2P_1 \\ &- \sum_{m=0}^{\infty}(w-1)^m A_2T^{m+1}\Big] - I\Big),\end{aligned} \tag{5.1}$$

$$\begin{aligned}\Psi_2(w) &= \Psi_1(w)\Big[A_2(wI - A_1)^{-1}\Big] \\ &= \Big(\frac{1}{w-1}P_1 - \sum_{m=0}^{\infty}(w-1)^m T^{m+1}\Big)\Big((w-1)\Big[\frac{1}{w-1}A_2P_1 \\ &\quad - \sum_{m=0}^{\infty}(w-1)^m A_2T^{m+1}\Big] - I\Big)\Big[\frac{1}{w-1}A_2P_1 - \sum_{m=0}^{\infty}(w-1)^m A_2T^{m+1}\Big].\end{aligned} \tag{5.2}$$

It follows from (4.10), (5.1), and (5.2) that

$$\mathcal{R}\mathrm{es}_{\lambda=1}(\Psi_1) = P_1A_2P_1 - P_1 = 0, \tag{5.3}$$

$$\mathcal{R}\mathrm{es}_{\lambda=1}(\Psi_2) = P_1A_2TA_2P_1 - P_1(A_2P_1 - I)A_2T - T(A_2P_1 - I)A_2P_1 = 0. \tag{5.4}$$

The last two equations imply

$$P_1A_2TA_2P_1 = 0. \tag{5.5}$$

REMARK 5.1. Coming back to the beginning of this section, suppose that

$$\{\lambda x + ay = 1\} \cap \Delta_\rho(\tfrac{1}{\lambda},0) = \sigma_{\mathrm{p}}(A_1, A_2) \cap \Delta_\rho(\tfrac{1}{\lambda},0)$$

up to multiplicity. Then the operators $\frac{A_1}{\lambda}$ and $\frac{A_2}{a}$ satisfy (5.3) and (5.4). Since the projections P_j and the operator T for A_1 and $\frac{A_1}{a}$ are the same, we obtain

$$P_1 A_2 P_1 = aP_1. \tag{5.6}$$

Equation (5.5) stays the same.

Now we use (5.5) to establish necessary and sufficient conditions for a common eigenvector in the case when at least one of the operators A_1, A_2 is invertible.

LEMMA 5.2. *Let A_1, A_2 be self-adjoint, 1 be an isolated spectral point of A_1, and assume that A_1 is invertible. If there is $\rho > 0$ such that*

$$\{x+y=1\} \cap \Delta_\rho(1,0) = \sigma_{\mathrm{p}}(A_1, A_2) \cap \Delta_\rho(1,0)$$

up to multiplicity, then the following are equivalent:

(i) *A_1 and A_2 have an n-dimensional common eigensubspace, where $n = \operatorname{rank}(P_1)$, and the whole line $\{x+y=1\}$ is in $\sigma_{\mathrm{p}}(A_1, A_2)$;*

(ii) *there is $\rho' > 0$ such that the line segment $\{x+y=1\} \cap \Delta_{\rho'}(1,0)$ agrees with $\sigma_{\mathrm{p}}(A_1^{-1}, A_2) \cap \Delta_{\rho'}(1,0)$ up to multiplicity;*

(iii) *there is ρ'' such that the plane segment $\{x+y+z=1\} \cap \Delta_{\rho''}(1,1,0)$ agrees with $\sigma_{\mathrm{p}}(A_1, A_1^{-1}, A_2) \cap \Delta_{\rho''}(1,1,0)$ up to multiplicity.*

Proof. The implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (ii) are obvious. Thus, it suffices to prove (ii) $\Rightarrow$ (i).

Suppose that (ii) holds. Let L_1 be the eigensubspace of A_1 with eigenvalue one. Choose an orthonormal basis of L_1: $e_1, \dots$. Equation (4.13) implies that in our case for every $\xi \in H$

$$T(A_1)\xi = T\xi = \int\limits_{\sigma(A_1)\setminus\{1\}} \frac{1}{z-1} \mathrm{d}E(z)(\xi).$$

Therefore, using that A_2 is self-adjoint we have for every j

$$\begin{aligned} P_1 A_2 T A_2 P_1 e_j &= \sum_m \Big(\int\limits_{\sigma(A_1)\setminus\{1\}} \frac{1}{z-1} \langle \mathrm{d}E(z) A_2 e_j, A_2 e_m \rangle \Big) e_m \\ &= \sum_m \Big(\int\limits_{\sigma(A_1)\setminus\{1\}} \frac{1}{z-1} \langle \mathrm{d}E(z) A_2 e_j, \mathrm{d}E(z) A_2 e_m \rangle \Big) e_m. \end{aligned}$$

Equation (5.5) implies that for every pair j, m

$$\int\limits_{\sigma(A_1)\setminus\{1\}} \frac{1}{z-1} \langle \mathrm{d}E(z) A_2 e_j, \mathrm{d}E(z) A_2 e_m \rangle = 0.$$

In particular, when $j = m$ we obtain

$$\int\limits_{\sigma(A_1)\setminus\{1\}} \frac{1}{z-1} \left\| \mathrm{d}E(z) A_2 e_j \right\|^2 = 0. \tag{5.7}$$

We now apply all preceding considerations to the pair (A_1^{-1}, A_2). First we observe that

$$A_1^{-1} = P_1 + \int\limits_{\sigma(A_1)\setminus\{1\}} \frac{1}{z} \mathrm{d}E(z).$$

Hence,

$$P_1 A_2 \widetilde{T} A_2 P_1 = 0,$$

where

$$\widetilde{T} = T(A_1^{-1}) = \int\limits_{\sigma(A_1)\setminus\{1\}} \frac{z}{1-z} \mathrm{d}E(z). \tag{5.8}$$

In a similar way the last two relations yield

$$\int\limits_{\sigma(A_1)\setminus\{1\}} \frac{z}{1-z} \left\|\mathrm{d}E(z)A_2 e_j\right\|^2 = 0. \tag{5.9}$$

Adding (5.7) and (5.9) we obtain

$$\int\limits_{\sigma(A_1)\setminus\{1\}} \left\|\mathrm{d}E(z)A_2 e_j\right\|^2 = 0.$$

This means that $A_2 e_j \in L_1$ for every j. Thus, L_1 is invariant under A_2. Since the restriction of A_1 to L_1 is the identity operator, the joint spectrum of $A_2|_{L_1}$ and the identity of L_1 contains a cone with vertex at $(1,0)$ that contains every line of the family $\{x + \frac{y}{a} = 1 : a \in \sigma(A_2|_{L_1})\}$, and, of course, this cone lies in $\sigma_{\mathrm{p}}(A_1, A_2)$. Since the intersection of $\sigma_{\mathrm{p}}(A_1, A_2)$ with a neighborhood of $(1,0)$ is a line segment, we conclude that the spectrum of $A_2|_{L_1}$ consists of a single point, and, since A_2 is self-adjoint, this means that L_1 is an eigenspace for A_2. ∎

Since eigenvectors of an operator A and its scalar multiple are the same, the following result is a straightforward corollary to Lemma 5.2.

LEMMA 5.3. *Let A_1, A_2 be self-adjoint, $\lambda \neq 0$ be an isolated point of $\sigma(A_1)$, and A_1 be invertible. If there exist $a \neq 0$ and $\rho > 0$ such that, up to multiplicity, $\{\lambda x + ay = 1\} \cap \Delta_\rho(\frac{1}{\lambda}, 0) = \sigma_{\mathrm{p}}(A_1, A_2) \cap \Delta_\rho(1,0)$, then the following are equivalent:*

(i) *A_1 and A_2 have a common eigensubspace of dimension equal to the rank of P_1;*

(ii) *there is ρ' such that, up to multiplicity,*

$$\sigma_{\mathrm{p}}(A_1^{-1}, A_2) \cap \Delta_{\rho'}(\lambda, 0) = \left\{\frac{x}{\lambda} + ay = 1\right\} \cap \Delta_{\rho'}(\lambda, 0);$$

(iii) *there is $\rho'' > 0$ such that, up to multiplicity,*

$$\left\{\lambda x + \frac{y}{\lambda} + az = 1\right\} \cap \Delta_{\rho''}(\lambda, \tfrac{1}{\lambda}, 0) = \sigma_{\mathrm{p}}(A_1, A_1^{-1}, A_2) \cap \Delta_{\rho''}(\lambda, \tfrac{1}{\lambda}, 0).$$

We will use the result of Lemma 5.3 to give a necessary and sufficient condition for a common eigenvector for an arbitrary pair of self-adjoint operators.

To this end, for any self-adjoint operator A we consider the following family of perturbations:

$$A(\varepsilon,\lambda) = (1+\varepsilon)A - \lambda\varepsilon I, \quad \varepsilon \in \mathbb{R}, \varepsilon \neq -1. \tag{5.10}$$

REMARK 5.4. It is easily seen that for every $\varepsilon, \lambda \in \mathbb{R}$ the operator $A(\varepsilon,\lambda)$ is self-adjoint. Furthermore, if λ is an isolated spectral point of A, then it is an isolated spectral point of $A(\varepsilon,\lambda)$ for every $\varepsilon \neq -1$; and the line segment $\{\lambda x + ay = 1\} \cap \Delta_\rho(\frac{1}{\lambda}, 0)$ is in $\sigma_{\mathrm{p}}(A_1, A_2)$ if and only if it is in $\sigma_{\mathrm{p}}(A_1(\varepsilon,\lambda), A_2(\varepsilon, a))$. It is also straightforward that the eigensubspace of $A(\varepsilon,\lambda)$ corresponding to eigenvalue λ is either empty or is the same for all $\varepsilon \neq -1$. We further remark that if $\lambda \neq 0$, then there exists ε such that $A(\varepsilon,\lambda)$ is invertible. Indeed, the spectral mapping theorem, cf. Chapter 7, Section 3, Theorem 11 of [11], implies that $\sigma(A(\varepsilon,\lambda)) = (1+\varepsilon)\sigma(A) - \lambda\varepsilon$. Thus $0 \in \sigma(A(\varepsilon,\lambda))$ if and only if $\frac{\lambda\varepsilon}{1+\varepsilon} \in \sigma(A)$. Since λ is an isolated point of $\sigma(A)$, if $\varepsilon \in \mathbb{R}$ and $|\varepsilon|$ is big enough, zero is not in the spectrum of $A(\varepsilon,\lambda)$, that is $A(\varepsilon,\lambda)$ is invertible.

Before we proceed further, we pause to observe an elementary result about the general behavior of joint spectra under linear change of coordinates. Let

$$\mathbf{C} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

be a complex-valued matrix. For operators $A_1, \ldots, A_n$ write

$$B_k = c_{k1}A_1 + \cdots + c_{kn}A_n, \quad k = 1, \ldots, n. \tag{5.11}$$

We have the following proposition.

PROPOSITION 5.5. $\sigma_{\mathrm{p}}(A_1, \ldots, A_n) \supseteq \mathbf{C}^{\mathrm{T}} \sigma_{\mathrm{p}}(B_1, \ldots, B_n)$.

The proof is straightforward.

COROLLARY 5.6. *If* $\mathbf{C}$ *is invertible, then*

$$\sigma_{\mathrm{p}}(A_1, \ldots, A_n) = \mathbf{C}^{\mathrm{T}} \sigma_{\mathrm{p}}(B_1, \ldots, B_n). \tag{5.12}$$

Thus, coming back to our pair of self-adjoint operators A_1 and A_2 from the beginning of this section, by making a linear change of coordinates (which amounts to replacing A_2 by $A_2 + \delta A_1$ for a sufficiently small real δ) we can always reduce to the case when $a \neq 0$, hence the following result is an immediate corollary to Lemma 5.3.

THEOREM 5.7. *Let* A_1, A_2 *be self-adjoint operators on* H, *let* λ *be an isolated spectral point of* $\sigma(A_1)$, *and suppose that in some neighborhood* $\Delta_\rho(\frac{1}{\lambda}, 0)$ *of* $(\frac{1}{\lambda}, 0)$ *the joint spectrum* $\sigma_{\mathrm{p}}(A_1, A_2)$ *coincides up to multiplicity with a line segment* $\{(x,y) \in \Delta_\rho(\frac{1}{\lambda}, 0) : \lambda x + ay = 1\}$. *The following are equivalent:*

(i) *the eigenspace of* A_1 *corresponding to the eigenvalue* λ *is also an eigensubspace for* A_2;

(ii) *there exist* $\varepsilon \in \mathbb{R}, \varepsilon \neq -1$ *and* $\rho' > 0$ *such that* $A_1(\varepsilon, \lambda)$ *is invertible and the line segment* $\{(x,y) \in \Delta_{\rho'}(\lambda, 0) : \frac{x}{\lambda} + ay = 1\}$ *coincides up to multiplicity with* $\sigma_{\mathrm{p}}((A_1(\varepsilon,\lambda))^{-1}, A_2(\varepsilon, a)) \cap \Delta_{\rho'}(\lambda, 0)$;

(iii) *there exist* $\varepsilon \in \mathbb{R}, \varepsilon \neq -1$ *and* $\rho'' > 0$ *such that* $A_1(\varepsilon, \lambda)$ *is invertible and the plane segment* $\{(x,y,z) \in \Delta_{\rho''}(\frac{1}{\lambda}, \lambda, 0) : \lambda x + \frac{1}{\lambda} y + az = 1\}$ *coincides with* $\sigma_{\mathrm{p}}(A_1(\varepsilon,\lambda), A_1(\varepsilon,\lambda)^{-1}, A_2(\varepsilon, a)) \cap \Delta_{\rho''}(\frac{1}{\lambda}, \lambda, 0)$ *up to multiplicity.*

As a direct corollary to Theorem 5.7 we obtain the following result for an n-tuple of self-adjoint operators.

THEOREM 5.8. *Let* $A_1, \dots, A_n$ *be self-adjoint, let* $\lambda \neq 0$ *be an isolated point of* $\sigma(A_1)$*, and suppose there exists* $\rho > 0$ *such that, up to multiplicity,*

$$\begin{aligned}\{\lambda x_1 + a_2 x_2 + \cdots + a_n x_n = 1\} &\cap \Delta_\rho(\tfrac{1}{\lambda}, 0, \dots, 0) \\ &= \sigma_{\mathrm{p}}(A_1, \dots, A_n) \cap \Delta_\rho(\tfrac{1}{\lambda}, 0, \dots, 0).\end{aligned}$$

The following are equivalent:

(i) *the eigensubspace of* A_1 *corresponding to eigenvalue* λ *is an eigensubspace for each of the operators* $A_2, \dots, A_n$;

(ii) *there exist an* $\varepsilon \in \mathbb{R}, \varepsilon \neq 1$ *and* $\rho' > 0$ *such that* $A_1(\varepsilon, \lambda)$ *is invertible and*

$$\begin{aligned}\{\tfrac{1}{\lambda} x_1 + a_2 x_2 + \cdots + a_n x_n = 1\} &\cap \Delta_\rho(\lambda, 0, \dots, 0) \\ = \sigma_{\mathrm{p}}(A_1(\varepsilon,\lambda)^{-1}, A_2(\varepsilon, a_2), \dots, &A_n(\varepsilon, a_n)) \cap \Delta_{\rho'}(\lambda, 0, \dots, 0),\end{aligned}$$

up to multiplicity.

Proof. Obviously (i) implies (ii).

Suppose that (ii) holds. Since the line segments $\{(x_1, x_j) \in \Delta_\rho(\frac{1}{\lambda}, 0) : \lambda x_1 + a_j x_j = 1\}$ and $\{(x_1, x_j) \in \Delta_\rho(\lambda, 0) : \frac{1}{\lambda} x_1 + a_j x_j = 1\}$ coincide with the sets $\sigma_{\mathrm{p}}(A_1(\varepsilon,\lambda), A_j(\varepsilon, a_j)) \cap \Delta_\rho(\frac{1}{\lambda}, 0)$ and $\sigma_{\mathrm{p}}(A_1(\varepsilon,\lambda)^{-1}, A_j(\varepsilon, a_j)) \cap \Delta_{\rho'}(\lambda, 0)$ respectively for all $j = 2, \dots, n$, it follows from Theorem 5.7 that the eigenspace of A_1 that corresponds to the eigenvalue λ is an eigenspace of A_j for all $j = 2, \dots, n$, and (i) holds. ∎

6. SPECTRAL ALGEBRAIC CURVES, EXTERIOR POWERS, COMMON REDUCING SUBSPACES

It is clear that if operators A_1 and A_2 have a common reducing subspace of dimension n, the joint spectrum $\sigma_{\mathrm{p}}(A_1, A_2)$ contains an algebraic curve of order n. Our example from the Introduction shows that in general the converse is not true. In this section we use results of the previous section to establish necessary and sufficient conditions under which the presence of an algebraic curve in the joint spectrum implies the existence of a common reducing subspace. As our conditions are expressed in terms of joint spectra of exterior products, we begin by recalling some basic facts about exterior products of Hilbert spaces. For more details we refer the reader to Chapter V.1 of [30] and Chapter X.7 of [28].

For any $n \geqslant 1$ the nth tensor power $\overset{n}{\bigotimes} H$ of a Hilbert space H has inner product given by

$$\langle x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n \rangle = \langle x_1, y_1 \rangle \cdots \langle x_n, y_n \rangle.$$

The nth *exterior power* $\overset{n}{\bigwedge} H$ of H is defined as the quotient of $\overset{n}{\bigotimes} H$ modulo the subspace generated by all elements of the form

$$x_1 \otimes \cdots \otimes x \otimes x \otimes \cdots \otimes x_n.$$

The image of a simple tensor $x_1 \otimes \cdots \otimes x_n$ in $\overset{n}{\bigwedge} H$ is denoted by $x_1 \wedge \cdots \wedge x_n$. We consider $\overset{n}{\bigwedge} H$ as a subspace of $\overset{n}{\bigotimes} H$ via the canonical "antisymmetrizing map"

$$x_1 \wedge \cdots \wedge x_n \longmapsto \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

In particular, $\overset{n}{\bigwedge} H$ inherits via this map a Hilbert space structure from $\overset{n}{\bigotimes} H$, and it is straightforward to compute that its inner product satisfies

$$\langle x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_n \rangle = \det \begin{bmatrix} \langle x_1, y_1 \rangle & \cdots & \langle x_1, y_n \rangle \\ \vdots & & \vdots \\ \langle x_n, y_1 \rangle & \cdots & \langle x_n, y_n \rangle \end{bmatrix}.$$

Therefore if $\{e_1, \ldots, e_n, \ldots\}$ is an orthonormal basis of H we obtain that the set $\{e_{i_1} \wedge \cdots \wedge e_{i_n} : 1 \leqslant i_1 < \cdots < i_n\}$ is an orthonormal basis of $\overset{n}{\bigwedge} H$.

When A is a linear operator on H it induces a linear operator $\overset{n}{\bigwedge} A$ on $\overset{n}{\bigwedge} H$ via the formula $\left[\overset{n}{\bigwedge} A\right](v_1 \wedge \cdots \wedge v_n) = Av_1 \wedge \cdots \wedge Av_n$. Thus $\overset{n}{\bigwedge} A$ is just the restriction of $\overset{n}{\bigotimes} A$ to $\overset{n}{\bigwedge} H$ considered as a subspace of $\overset{n}{\bigotimes} H$. It is immediate that if A is of finite rank or self-adjoint then so is $\overset{n}{\bigwedge} A$; and when A is a bounded we get $\left\| \overset{n}{\bigwedge} A \right\| \leqslant \|A\|^n$. In particular, compactness of A implies compactness of $\overset{n}{\bigwedge} A$.

When A is self-adjoint and compact and $\lambda_1, \ldots$ are the eigenvalues of A, with $e_1, \ldots$ being a corresponding eigenbasis, then the orthogonal set $\{e_{i_1} \wedge \cdots \wedge e_{i_n} : 1 \leqslant i_1 < \cdots < i_n\}$ is an eigenbasis with $\{\lambda_{i_1} \cdots \lambda_{i_n}\}$ as the corresponding multiset of eigenvalues for the compact self-adjoint $\overset{n}{\bigwedge} A$.

DEFINITION 6.1. Let A be a self-adjoint operator. We say that a finite multiset $L = \{\lambda_1, \ldots, \lambda_n\}$ is a *spectral multiset* for A if each λ_i is an isolated point in the spectrum of A of finite multiplicity, and the number of times it occurs in L is at most its multiplicity as an eigenvalue of A. We say that a spectral multiset for A is *generic* if $\lambda = \lambda_1 \cdots \lambda_n$ is isolated and of multiplicity 1 in the spectrum of $\overset{n}{\bigwedge} A$.

REMARK 6.2. A straightforward consequence of the definition is that if $L = \{\lambda_1, \dots, \lambda_n\}$ is a generic spectral multiset for a self-adjoint operator A, then each eigenvalue λ_i of A has multiplicity equal to the number of times it occurs in L.

REMARK 6.3. Suppose A is a self-adjoint operator with countable spectrum, and $L = \{\lambda_1, \dots, \lambda_n\}$ is a spectral multiset for A such that each eigenvalue λ_i of A has multiplicity equal to the number of times it occurs in L. Then there is an open subset $U \subset \mathbb{R}$ such that $\mathbb{R} \setminus U$ is countable, and such that for every $\delta \in U$ the operator $A + \delta I$ is invertible and has $L + \delta = \{\lambda_1 + \delta, \dots, \lambda_n + \delta\}$ as a generic spectral multiset. Indeed, since $A + \delta I$ is invertible for $\delta \in \mathbb{R} \setminus -\sigma(A)$, and since the spectrum of $\bigwedge^n (A + \delta I)$ is a subset of the spectrum of $\bigotimes^n (A + \delta I)$, it suffices to show that for every δ in the set

$$\{\delta \in \mathbb{R} : (\lambda_1 + \delta) \cdots (\lambda_n + \delta) \neq (\mu_1 + \delta) \cdots (\mu_n + \delta) \\ \text{for } \mu_i \in \sigma(A) \text{ such that } \{\mu_1, \dots, \mu_n\} \neq L \text{ as multisets}\}$$

the point $(\lambda_1 + \delta) \cdots (\lambda_n + \delta)$ is isolated in the spectrum of $\bigotimes^n (A + \delta I)$. But, as the spectrum of a tensor product of operators is the product of their spectra, cf. [2], and each $\lambda_i + \delta$ is isolated in the spectrum of $A + \delta I$, this follows from the compactness of spectra by a standard argument.

The assumptions in the following theorem describe what we consider to be a "general position" setting.

THEOREM 6.4. *Let A_1, A_2 be self-adjoint operators with A_1 invertible. Consider a generic spectral multiset $L = \{\lambda_1, \dots, \lambda_n\}$ for A_1, and let $\lambda = \lambda_1 \cdots \lambda_n$. Suppose also that for some $a \neq 0$ and $\rho > 0$ the line segments*

$$\{\lambda x + ay = 1\} \cap \Delta_\rho(\tfrac{1}{\lambda}, 0) \quad \textit{and} \quad \{\tfrac{1}{\lambda} x + ay = 1\} \cap \Delta_\rho(\lambda, 0)$$

are inside $\sigma_{\mathrm{p}}(\bigwedge^n A_1, \bigwedge^n A_2)$ and $\sigma_{\mathrm{p}}(\bigwedge^n A_1^{-1}, \bigwedge^n A_2)$, respectively.

Then the eigenspace of A_1 corresponding to the eigenvalues $\lambda_1, \dots, \lambda_n$ is invariant under A_2.

Proof. Since λ is isolated and simple in the spectrum of $\bigwedge^n A_1$ it follows that the joint spectrum $\sigma_{\mathrm{p}}(\bigwedge^n A_1, \bigwedge^n A_2)$ is nonsingular at the point $(\frac{1}{\lambda}, 0)$, in particular this point belongs to no component other than the line $\lambda x + ay = 1$. By Lemma 5.3 the operators $\bigwedge^n A_1$ and $\bigwedge^n A_2$ have a common unit eigenvector v (of eigenvalue λ for $\bigwedge^n A_1$ and eigenvalue a for $\bigwedge^n A_2$). Since L is generic v must be of the form $v = e_1 \wedge \cdots \wedge e_n$, where each e_i is a unit eigenvector for A_1 of eigenvalue λ_i. Now we show that $\operatorname{span}(e_1, \dots, e_n)$ is invariant under A_2. Indeed, let e be any other eigenvector for A_1, and consider the column vector

$$w = \begin{bmatrix} \langle A_2 e_1, e\rangle \\ \vdots \\ \langle A_2 e_n, e\rangle \end{bmatrix}.$$

For $i = 1, \ldots, n$ set $v_i = e_1 \wedge \cdots \wedge e_{i-1} \wedge e \wedge e_{i+1} \wedge \cdots \wedge e_n$ and note that each v_i is orthogonal to v. As $\bigwedge^n A_2(v) = av$, it follows that

$$a = \left\langle \bigwedge^n A_2(v), v \right\rangle = \langle A_2 e_1 \wedge \cdots \wedge A_2 e_n, e_1 \wedge \cdots \wedge e_n\rangle$$

and therefore $\det Y = a \neq 0$, where Y is the matrix

$$Y = \begin{bmatrix} \langle A_2 e_1, e_1\rangle & \cdots & \langle A_2 e_1, e_n\rangle \\ \vdots & & \vdots \\ \langle A_2 e_n, e_1\rangle & \cdots & \langle A_2 e_n, e_n\rangle \end{bmatrix}.$$

In particular the linear system of equations

$$Y \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = w$$

has a unique solution given, according to Cramer's rule, by the formula

$$x_i = \frac{\det Y_i(w)}{a}$$

for each $i = 1, \ldots, n$, where $Y_i(w)$ is obtained by replacing with w the ith column of the matrix Y. But since

$$\det Y_i(w) = \left\langle \bigwedge^n A_2(v), e_1 \wedge \cdots \wedge e \wedge \cdots \wedge e_n \right\rangle = a\langle v, v_i\rangle = 0$$

we see that $x_i = 0$ for each i, and therefore $w = 0$. ∎

Recall that a self-adjoint operator A on an infinite dimensional separable Hilbert space H belongs to the class $\mathcal{E}(H)$ when $A = K + aI$ with K a compact self-adjoint operator on H. In this case every point in $\sigma(A) \setminus a$ is isolated of finite multiplicity, and the point a is either an accumulation point, or isolated of infinite multiplicity.

For operators in the class $\mathcal{E}(H)$ we are now ready to address the question of when the presence of an algebraic curve in the proper joint spectrum indicates the existence of a common reducing subspace. Consider two self-adjoint operators $A = K_1 + aI$ and $B = K_2 + bI$ in $\mathcal{E}(H)$, and suppose that Γ is an algebraic curve of degree k which is a union of components of the proper joint spectrum $\sigma_{\mathrm{p}}(A, B)$. Note that the line $\{ax + by = 1\}$ is always in $\sigma_{\mathrm{p}}(A, B)$ and therefore carries no information about common reducing subspaces. We will refer to this line as the *accumulation line* of the joint spectrum. Thus, without loss of generality we can assume that it is not a reduced component of Γ. Therefore, by making a linear change of coordinates if necessary (which amounts to replacing A and B by appropriate linear combinations of A and B with real coefficients, hence does not affect the presence and degrees of algebraic curves or common reducing subspaces and their dimensions) we may also assume that Γ intersects the x-axis in k points (counted with multiplicities) $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_k}$ and the y-axis in k points $\frac{1}{\mu_1}, \dots, \frac{1}{\mu_k}$, and that each point $(\frac{1}{\lambda_i}, 0)$ belongs only to components of $\sigma_{\mathrm{p}}(A, B)$ contained in Γ; in particular, the multiset $L = \{\lambda_1, \dots, \lambda_k\}$ is a spectral mutiset for A and each eigenvalue λ_i of A has multiplicity equal to the number of times it occurs in L. Therefore, by Remark 6.3, if needed, we can make an additional suitable linear fractional change of coordinates of the form

$$u = \frac{x}{1 + \delta x}, \quad v = \frac{y}{1 + \delta x}$$

(which amounts to replacing A by $A + \delta I$) and also assume that A is invertible, and that L is a generic spectral multiset for A. Thus, for operators in the class $\mathcal{E}(H)$ we can always reduce the search for a common invariant subspace to the "general position" case considered in the following theorem, which is one of the main results in this paper.

THEOREM 6.5. *Let $A = K_1 + aI$ and $B = K_2 + bI$ be self-adjoint operators in the class $\mathcal{E}(H)$, with A invertible. Let Γ be an algebraic curve of degree k which is a union of components of the proper joint spectrum $\sigma_{\mathrm{p}}(A, B)$, and which does not have the accumulation line $\{ax + by = 1\}$ as a reduced component. Suppose that the x-axis (respectively the y-axis) intersects Γ in the k points (counted with multiplicity) $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_k}$ (respectively $\frac{1}{\mu_1}, \dots, \frac{1}{\mu_k}$) such that each point $(\frac{1}{\lambda_i}, 0)$ belongs only to components of $\sigma_{\mathrm{p}}(A, B)$ contained in Γ, and the multiset $L = \{\lambda_1, \dots, \lambda_k\}$ is a generic spectral multiset for A. Set $\lambda = \lambda_1 \cdots \lambda_k$ and $\mu = \mu_1 \cdots \mu_k$. The following are equivalent:*

(i) *the eigenspace for A corresponding to $\lambda_1, \dots, \lambda_k$ is invariant for B.*

(ii) *there exists $\rho > 0$ such that the line segments*

$$\{\lambda x + \mu y = 1\} \cap \Delta_\rho(\tfrac{1}{\lambda}, 0) \quad \text{and} \quad \left\{\frac{1}{\lambda} x + \mu y = 1\right\} \cap \Delta_\rho(\lambda, 0)$$

are contained in $\sigma_{\mathrm{p}}(\bigwedge^k A, \bigwedge^k B)$ and $\sigma_{\mathrm{p}}(\bigwedge^k A^{-1}, \bigwedge^k B)$, respectively.

(iii) *the lines*

$$\{\lambda x + \mu y = 1\} \quad \text{and} \quad \left\{\frac{1}{\lambda} x + \mu y = 1\right\}$$

are contained in $\sigma_{\mathrm{p}}(\bigwedge^k A, \bigwedge^k B)$ and $\sigma_{\mathrm{p}}(\bigwedge^k A^{-1}, \bigwedge^k B)$, respectively.

Proof. The implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are straightforward, and (ii) implies (i) by Theorem 6.4. ∎

The case when H is a finite-dimensional Hilbert space is of classical interest in algebraic geometry. In that setting every linear operator is bounded and of finite rank, hence we can take $\mathcal{E}(H)$ to be the space of all linear operators on H.

Now the statement of our last theorem can be somewhat simplified and slightly expanded as follows.

Suppose that $\dim_{\mathbb{C}} H = N$, and let H^* denote the space dual to H. Then we have the canonical isomorphism

$$\bigwedge^k H^* \longrightarrow \bigwedge^{N-k} H \otimes \bigwedge^N H^*,$$

which, together with the inner product induced isomorphism $\bigwedge^k H \to \bigwedge^k H^*$, allows us to consider in a natural way $\bigwedge^{N-k} A$ as acting on the space $\bigwedge^k H$ for any linear operator A on H. In particular, we can consider the proper joint spectrum $\sigma_{\mathrm{p}}(\bigwedge^{N-k} A, \bigwedge^k B)$. When A is self-adjoint, a standard exercise in multilinear algebra shows that the Sylvester expansion formula for the determinant transforms into the equality $\left(\bigwedge^{N-k} A\right)\left(\bigwedge^k A\right) = (\det A)I$. Thus, for an invertible self-adjoint A we have $\bigwedge^{N-k} A = \det(A) \bigwedge^k A^{-1}$, and therefore Theorem 6.5 implies immediately the following result.

THEOREM 6.6. *With assumptions and notation as in Theorem* 6.5*, suppose in addition that* $a = b = 0$ *and* $\dim_{\mathbb{C}} H = N$*. The following are equivalent:*

(i) *the eigenspace for A corresponding to $\lambda_1, \ldots, \lambda_k$ is invariant for B;*

(ii) *for some $\rho > 0$ the line segments*

$$\{\lambda x + \mu y = 1\} \cap \Delta_\rho(\tfrac{1}{\lambda}, 0) \quad and \quad \left\{\frac{1}{\lambda} x + \mu y = 1\right\} \cap \Delta_\rho(\lambda, 0)$$

are contained in $\sigma_{\mathrm{p}}(\bigwedge^k A, \bigwedge^k B)$ and $\sigma_{\mathrm{p}}(\bigwedge^k A^{-1}, \bigwedge^k B)$, respectively;

(iii) *for some $\rho > 0$ the line segments*

$$\{\lambda x + \mu y = 1\} \cap \Delta_\rho(\tfrac{1}{\lambda}, 0) \quad and \quad \left\{\frac{\det A}{\lambda} x + \mu y = 1\right\} \cap \Delta_\rho(\tfrac{\lambda}{\det A}, 0)$$

are contained in $\sigma_{\mathrm{p}}(\bigwedge^k A, \bigwedge^k B)$ and $\sigma_{\mathrm{p}}(\bigwedge^{N-k} A, \bigwedge^k B)$, respectively;

(iv) *the lines*

$$\{\lambda x + \mu y = 1\} \quad and \quad \left\{\frac{\det A}{\lambda} x + \mu y = 1\right\}$$

are contained in $\sigma_{\mathrm{p}}(\bigwedge^k A, \bigwedge^k B)$ and $\sigma_{\mathrm{p}}(\bigwedge^{N-k} A, \bigwedge^k B)$, respectively.

Finally, we note that Theorem 2.3 from Section 2 is obtained from the result above in the same way that Theorem 5.8 is obtained from Theorem 5.7.

7. SPECTRAL CONTINUITY AND COMMON "ALMOST EIGENVECTORS"

Spectral continuity is well-known in the classical spectral theory, see [7]. In our case it implies that if $A_{1,n} \to A_1$ and $A_{2,n} \to A_2$ in operator norm topology as $n \to \infty$, where all $A_{1,n}, A_{2,n}, A_1, A_2$ are in $\mathcal{E}(H)$, then $\sigma_{\mathrm{p}}(A_{1,n}, A_{2,n})$ converges to $\sigma_{\mathrm{p}}(A_1, A_2)$ in Hausdorff topology uniformly on compact subsets of $\mathbb{C}^2$. In particular, this implies that if A_1 and A_2 have a common eigenvector, $A_{1,n}$ and $A_{2,n}$ have a common "almost eigenvector" (we define it below) and $\sigma_{\mathrm{p}}(A_{1,n}, A_{2,n})$ contains an irreducible component that converges to a line in Hausdorff topology uniformly on compacts as $n \to \infty$. In this section we prove results that establish the converse: under certain natural assumptions local closeness of $\sigma_{\mathrm{p}}(A_1, A_2)$ to a line implies existence of a common almost eigenvector.

DEFINITION 7.1. We say that a non-zero vector ξ is an ε-eigenvector (almost eigenvector) of an operator A if there is $\lambda \in \mathbb{C}$ such that $\|A\xi - \lambda\xi\| \leqslant \varepsilon\|\xi\|$.

Since the distance from the vector $A\xi$ to the line $\{\lambda\xi : \lambda \in \mathbb{C}\}$ is equal to $\left\|A\xi - \frac{\langle A\xi,\xi\rangle}{\|\xi\|^2}\xi\right\|$, we come to an equivalent definition of an ε-eigenvector: *ξ is an ε-eigenvector of A* if

$$\left\|A\xi - \frac{\langle A\xi, \xi\rangle}{\|\xi\|^2}\xi\right\| \leqslant \varepsilon\|\xi\|. \tag{7.1}$$

It immediately follows from (7.1) that ξ is an eigenvector of A in the traditional sense, if and only if it is an ε-eigenvector for all $\varepsilon > 0$. More generally, if $\lambda \in \sigma(A)$, then for every $\varepsilon > 0$ there exists an ε-eigenvector ξ such that $\|A\xi - \lambda\xi\| \leqslant \varepsilon\|\xi\|$.

Of course, every vector ξ is an ε-eigenvector with the appropriate choice of ε to be equal to the lefthand side of (7.1), but this is quite meaningless. The notion of an ε-eigenvector is meaningful when ε is small. In this case being an ε-eigenvector means that $A\xi$ lies in a small aperture cone that has the line $\{\lambda\xi : \lambda \in \mathbb{C}\}$ as the symmetry axis.

Our next result is a generalization of Theorem 5.7 to the case of common almost eigenvectors for compact operators. Let Γ be an analytic curve that passes through $(x, y) \in \mathbb{C}^2$ and let $\rho > 0$. We will use the following notation:

$$\Gamma_\rho(x, y) = \Gamma \cap \Delta_\rho(x, y).$$

If ε is close to zero and $A_1(\varepsilon)$ and $A_2(\varepsilon)$ are close to A_1 and A_2 respectively, then spectral continuity implies that locally $\sigma_{\mathrm{p}}(A_1, A_2)$ is close to $\sigma_{\mathrm{p}}(A_1(\varepsilon), A_2(\varepsilon))$. For this reason in the next theorem without loss of generality we may assume that A_1 is invertible. To simplify the notation we will also use rescaling, if necessary, so that the point $(1, 0)$ is in the joint spectrum, of A_1 and A_2 and A_1 is invertible. Finally, recall that the operator A belongs to the class $\mathcal{E}(H)$, if it is represented as $A = K + \alpha I$ where K is compact and $\alpha \in \mathbb{R}$.

THEOREM 7.2. *Let $A_1, A_2 \in \mathcal{E}(H)$ such that $1 \in \sigma(A_1)$, and, therefore, the point $(1, 0)$ belongs to $\sigma_{\mathrm{p}}(A_1, A_2)$ and to $\sigma_{\mathrm{p}}(A_1^{-1}, A_2)$. Suppose that 1 is not an accumulation point of $\sigma(A_1)$ and $(1, 0)$ is not a singular point of either $\sigma_{\mathrm{p}}(A_1, A_2)$, or $\sigma_{\mathrm{p}}(A_1^{-1}, A_2)$. Let $\sigma_{\mathrm{p}}(A_1, A_2)$ and $\sigma_{\mathrm{p}}(A_1^{-1}, A_2)$ near $(1, 0)$ be zeros of analytic functions $f_1(x, y)$ and $f_2(x, y)$, respectively. If there exist $0 < \rho < 1$ and $0 < \varepsilon \ll \rho$ such that:*

(i) $d = 1 - (1 - \rho)\|A_1\| - \varepsilon\sqrt{2}\|A_2\| > 0$;

(ii) *the Hausdorff distances from* $\sigma_{\rm p}(A_1, A_2)_\rho(1,0)$ *and* $\sigma_{\rm p}(A_1^{-1}, A_2)_\rho(1,0)$ *to the line* $\{x + \beta y = 1\}$ *are less than* ε *(*β *is a real number);*

(iii) $\frac{\partial f_j}{\partial x} + \beta \frac{\partial f_j}{\partial y} \neq 0$ *in* $\Delta_\rho(1,0)$, $j = 1, 2$*;*

then A_1 *and* A_2 *have a common* δ*-eigenvector, where* $\delta = D\sqrt{\varepsilon}$*, and* D *is a constant independent of* β*.*

Proof. First we observe that conditions (i) and (ii) imply that $|\beta|$ has an upper bound expressed in terms of ρ, ε and the norms of A_1 and A_2. Indeed, suppose that $|\beta| > 1$. Without loss of generality we may assume that $\beta > 0$. It is shown below that there is a point $(1-\rho, \tau) \in \sigma_{\rm p}(A_1, A_2)$. The distance from this point to $\{x + \beta y = 1\}$ is equal to $\frac{|\beta\tau - \rho|}{\sqrt{1+\beta^2}}$. Condition (ii) implies

$$\frac{\rho}{\beta} - \varepsilon\sqrt{1 + \frac{1}{\beta^2}} \leqslant \tau \leqslant \frac{\rho}{\beta} + \varepsilon\sqrt{1 + \frac{1}{\beta^2}},$$

so that

$$|\tau| \leqslant \frac{\rho}{\beta} + \varepsilon\sqrt{2}.$$

Since the operator $(1-\rho)A_1 + \tau A_2 - I$ is not invertible, we have

$$\begin{aligned} 1 &\leqslant \|(1-\rho)A_1 + \tau A_2\| \leqslant (1-\rho)\|A_1\| + |\tau|\|A_2\| \\ &\leqslant (1-\rho)\|A_1\| + \left(\frac{\rho}{\beta} + \varepsilon\sqrt{2}\right)\|A_2\|. \end{aligned}$$

This implies

$$\beta \leqslant \frac{\rho\|A_2\|}{1 - (1-\rho)\|A_1\| - \varepsilon\sqrt{2}\|A_2\|} = \frac{\rho\|A_2\|}{d}. \tag{7.2}$$

Now we approximate the compact parts of A_1 and A_2 by finite rank operators with simple spectra (that is every non-zero eigenvector has multiplicity one) resulting in operators $\widetilde{A}_1$ and $\widetilde{A}_2$. We can find $\tau \in \mathbb{R}$ as close to zero as we want, such that $\widehat{A}_1 = \widetilde{A}_1 + \tau I$ is invertible. Since $\widehat{A}_1$ and $\widetilde{A}_2$ are close to A_1 and A_2 respectively, the spectral continuity implies that $\sigma_{\rm p}(\widehat{A}_1, \widetilde{A}_2)$ is close to $\sigma_{\rm p}(A_1, A_2)$ in the bidisk $\{|x-1| \leqslant \rho, |y| \leqslant \rho\}$. Also if $\widehat{A}_1$ and $\widetilde{A}_2$ are close enough to A_1 and A_2 respectively, then δ-eigenvectors for $\widehat{A}_1$ and $\widetilde{A}_2$ are 2δ-eigenvectors for A_1 and A_2. It is clear that $\sigma_{\rm p}(\widehat{A}_1, \widetilde{A}_2)$ and $\sigma_{\rm p}(\widehat{A}_1^{-1}, \widetilde{A}_2)$ are algebraic sets. Finally, the points of intersection of $\sigma_{\rm p}(\widehat{A}_1, \widetilde{A}_2)$ and $\sigma_{\rm p}(\widehat{A}_1^{-1}, \widetilde{A}_2)$ with the x-axis that are close to one, are regular points, and the distances between the line $\{x + \beta y = 1\}$ and $\sigma_{\rm p}(\widehat{A}_1, \widetilde{A}_2)$ and $\sigma_{\rm p}(\widehat{A}_1^{-1}, \widetilde{A}_2)$ is less than 2ε. We denote the polynomials that determine $\sigma_{\rm p}(\widehat{A}_1, \widetilde{A}_2)$ and $\sigma_{\rm p}(\widehat{A}_1^{-1}, \widetilde{A}_2)$ by $\mathcal{R}(x,y)$ and $\mathcal{S}(x,y)$. As the rank of the approximating operators increases and τ approaches zero, the polynomials $\mathcal{R}$ and $\mathcal{S}$ approach f_1 and f_2, respectively; this follows from the direct expression of the defining function given by equation (3.1). Thus, the condition (ii) holds for these polynomials with $\varepsilon_1 = 2\varepsilon$. Again rescaling with a coefficient close to one we may assume that both $\sigma_{\rm p}(\widehat{A}_1, \widetilde{A}_2)$ and $\sigma_{\rm p}(\widehat{A}_1^{-1}, \widetilde{A}_2)$ pass through $(1,0)$.

Next we note that since $\varepsilon \ll \rho$, the orthogonal projection in $\mathbb{C}^2$ onto $\{x + \beta y = 1\}$ of each curve $\sigma_{\rm p}(\widehat{A}_1, \widetilde{A}_2)$ and $\sigma_{\rm p}(\widehat{A}_1^{-1}, \widetilde{A}_2)$ contains the disk of radius $\frac{\rho}{\sqrt{2}}$ centered at $(1,0)$. Indeed, again we may assume $\beta \geqslant 0$. Let us change the coordinates to

$$u = \frac{x-1}{\sqrt{1+\beta^2}} + \frac{\beta y}{\sqrt{1+\beta^2}}, \quad v = \frac{\beta(x-1)}{\sqrt{1+\beta^2}} - \frac{y}{\sqrt{1+\beta^2}}. \tag{7.3}$$

It is easily seen that the bidisk $\{|x-1| \leqslant \rho, |y| \leqslant \rho\}$ contains the bidisk $\Delta_\rho(\beta) = \left\{|u| \leqslant \frac{\rho\sqrt{1+\beta^2}}{1+\beta}, |v| \leqslant \frac{\rho\sqrt{1+\beta^2}}{1+\beta}\right\}$. Since $\frac{\sqrt{1+\beta^2}}{1+\beta} \geqslant \frac{1}{\sqrt{2}}$, the bidisk $\Delta_\rho = \left\{|u| \leqslant \frac{\rho}{\sqrt{2}}, |v| \leqslant \frac{\rho}{\sqrt{2}}\right\}$ is in $\{|x-1| \leqslant \rho, |y| \leqslant \rho\}$. In the (u,v)-coordinates $\sigma_{\rm p}(\widehat{A}_1, \widetilde{A}_2)$ and $\sigma_{\rm p}(\widehat{A}_1^{-1}, \widetilde{A}_2)$ are zeros of the polynomials $\widetilde{\mathcal{R}}(u,v) = \mathcal{R}\Big(\frac{u+\beta v}{\sqrt{1+\beta^2}} + 1, \frac{\beta u - v}{\sqrt{1+\beta^2}}\Big)$ and $\widetilde{\mathcal{S}}(u,v) = \mathcal{S}\Big(\frac{u+\beta v}{\sqrt{1+\beta^2}} + 1, \frac{\beta u - v}{\sqrt{1+\beta^2}}\Big)$, respectively. Suppose $(u_0, v_0) \in \Delta_\rho$ and $\widetilde{\mathcal{R}}(u_0, v_0) = 0$, that is $(u_0, v_0) \in \sigma_{\rm p}(\widehat{A}_1, \widetilde{A}_2)$. The distance from this point to the line $\{x + \beta y = 1\} = \{u = 0\}$ is equal to $|u_0|$. Hence, $|u_0| \leqslant 2\varepsilon$. Consider the functions $\phi_v(u) = \widetilde{\mathcal{R}}(u,v)$. Since $\phi_{v_0}(u_0) = 0$, Hurwitz's theorem (see, for example, p. 231 of [13]) implies that if v_1 is close to v_0, ϕ_{v_1} has zero u_1 close to u_0. Since $|u_0| \leqslant 2\varepsilon \ll \frac{\rho}{\sqrt{2}}$, we conclude that $|u_1| < \frac{\rho}{\sqrt{2}}$, and, therefore, $(u_1, v_1) \in \Delta_\rho$, and $(0, v_1)$ belongs to the projection of $\sigma_{\rm p}(\widehat{A}_1, \widetilde{A}_2)$ onto $\{u = 0\}$. A similar argument applied to $\sigma_{\rm p}(\widehat{A}_1^{-1}, \widetilde{A}_2)$ finishes the proof of the claim.

We now return to relations (4.16) and (4.17) to express residues of Ψ_1 and Ψ_2 explicitedly in terms of derivatives of the determining polynomial. It is easy to check that in our case these relations for $\sigma_{\rm p}(\widehat{A}_1, \widetilde{A}_2)$ yield

$$\mathcal{R}\mathrm{es}\Psi_1(\lambda)|_{\lambda=1} = \frac{\partial \mathcal{R}}{\partial x}|_{(1,0)} P_1 \widetilde{A}_2 P_1 - \frac{\partial \mathcal{R}}{\partial y}|_{(1,0)} P_1 = 0, \tag{7.4}$$

$$\begin{aligned} \mathcal{R}\mathrm{es}\Psi_2(\lambda)|_{\lambda=1} = -\frac{\partial \mathcal{R}}{\partial x}|_{(1,0)} P_1 \widetilde{A}_2 T \widetilde{A}_2 P_1 - \frac{1}{2}\Big(\frac{\partial^2 \mathcal{R}}{\partial x^2}\left(\frac{\frac{\partial \mathcal{R}}{\partial y}}{\frac{\partial \mathcal{R}}{\partial x}}\right)^2 \\ - 2\frac{\partial^2 \mathcal{R}}{\partial x \partial y}\left(\frac{\frac{\partial \mathcal{R}}{\partial y}}{\frac{\partial \mathcal{R}}{\partial x}}\right)^2 + \frac{\partial^2 \mathcal{R}}{\partial y^2}\Big)|_{(1,0)} P_1 = 0, \end{aligned} \tag{7.5}$$

where, as in Section 3, P_1 is the orthogonal projection on the eigenspace of $\widehat{A}_1$ corresponding to the eigenvalue one, and T is defined by (4.13).

Equation (7.4) together with condition (iii) of this theorem give $\frac{\partial \mathcal{R}}{\partial x}|_{(1,0)} \neq 0$, and, hence, this derivative does not vanish in a neighborhood of $(1,0)$. By the implicit function theorem the relation $\mathcal{R}(x,y) = 0$ determines x as an analytic function of y in a neighborhood of $(1,0)$. In terms of this function $x(y)$ equations

(7.4) and (7.5) can be written as

$$P_1\widetilde{A}_2P_1 = -x'(0)P_1, \tag{7.6}$$

$$P_1\widetilde{A}_2T\widetilde{A}_2P_1 = -\frac{x''(0)}{2}P_1. \tag{7.7}$$

As it was done before, we denote by $e_1, e_2, \dots$ an orthonormal eigenbasis for $\widehat{A}_1$ with $\widehat{A}_1(e_1) = e_1, \widehat{A}_1(e_j) = \mu_j e_j, \mu_j \neq 1, j = 2, \dots$ and P_j being the orthogonal projection onto a subspace spanned by e_j. In this basis relation (7.7) can be written as

$$\sum_{j=2}^{\infty} \frac{|\langle \widetilde{A}_2 e_1, e_j\rangle|^2}{\mu_j - 1} = -\frac{x''(0)}{2}. \tag{7.8}$$

Our next step is to show that $x''(0)$ is small. Let us once again pass to the coordinates (7.3). We have $\frac{\partial\widetilde{\mathcal{R}}}{\partial u} = \frac{1}{\sqrt{1+\beta^2}}\frac{\partial\mathcal{R}}{\partial x} + \frac{\beta}{\sqrt{1+\beta^2}}\frac{\partial\mathcal{R}}{\partial y} \neq 0$ for every $(u,v) \in \left\{|u| \leqslant \frac{\rho}{\sqrt{2}}, |v| \leqslant \frac{\rho}{\sqrt{2}}\right\}$. Applying the implicit function theorem we see that equation $\widetilde{\mathcal{R}}(u,v) = 0$ determines u as an analytic function of v in a neighborhood of every point $v \in \left\{|v| \leqslant \frac{\rho}{\sqrt{2}}\right\}$. Since this function is globally continuous in $\left\{|v| \leqslant \frac{\rho}{\sqrt{2}}\right\}$, by the monodromy theorem, see p. 161 of [13], u is holomorphic in the whole disk $\left\{|v| \leqslant \frac{\rho}{\sqrt{2}}\right\}$. The above argument showed that $|u(v)| \leqslant 2\varepsilon$ for every $v \in \left\{|v| \leqslant \frac{\rho}{\sqrt{2}}\right\}$. Now the Cauchy theorem implies that

$$|u'(0)| \leqslant \frac{4\varepsilon}{\rho}, \tag{7.9}$$

$$|u''(0)| \leqslant \frac{4\sqrt{2}\varepsilon}{\rho^2}. \tag{7.10}$$

A straightforward computation shows that

$$\frac{\mathrm{d}^2x}{\mathrm{d}y^2} = \frac{(1+\beta^2)^{3/2}\frac{\mathrm{d}^2u}{\mathrm{d}v^2}}{(1-\beta\frac{\mathrm{d}u}{\mathrm{d}v})^3}. \tag{7.11}$$

Equations (7.2), (7.9), (7.10), and (7.11) yield

$$|x''(0)| \leqslant C\varepsilon \tag{7.12}$$

where

$$C = \frac{4\sqrt{2}(1 + (\frac{\rho\|A_2\|}{d})^2)^{3/2}\rho}{(\rho - \frac{4\rho\|A_2\|}{d}\varepsilon)^3} \tag{7.13}$$

is a constant independent of β. Now equations (7.8) and (7.12) give

$$\left|\sum_{j=2}^{\infty} \frac{|\langle \widetilde{A}_2 e_1, e_j\rangle|^2}{\mu_j - 1}\right| \leqslant \frac{C}{2}\varepsilon. \tag{7.14}$$

Applying a similar argument to the pair $\widehat{A}_1^{-1}, \widetilde{A}_2$ and using (5.8) we obtain

$$\left|\sum_{j=2}^{\infty} \frac{\mu_j |\langle \widetilde{A}_2 e_1, e_j\rangle|^2}{\mu_j - 1}\right| \leqslant \frac{C}{2}\varepsilon. \tag{7.15}$$

Equations (7.14) and (7.15) result in

$$\sum_{j=2}^{\infty} |\langle \widetilde{A}_2 e_1, e_j\rangle|^2 \leqslant C\varepsilon.$$

Set $\lambda = \langle \widetilde{A}_2 e_1, e_1\rangle$. The last relation implies

$$\|\widetilde{A}_2 e_1 - \lambda e_1\| \leqslant \sqrt{C\varepsilon}, \tag{7.16}$$

which means that e_1 is $\widetilde{\delta}$-eigenvector of $\widetilde{A}_2$, where $\widetilde{\delta} = \sqrt{C\varepsilon}$. Thus, e_1 is a common $\widetilde{\delta}$-eigenvector of $\widehat{A}_1$ and $\widetilde{A}_2$. Therefore, as it was mention above, e_1 is a δ-eigenvector of A_1 and A_2 with $\delta = 2\widetilde{\delta}$. We are done. ∎

It was mentioned in the proof of Theorem 7.2 that the polynomial $\mathcal{R}$ determining the spectrum of $\widehat{A}_1$ and $\widetilde{A}_2$ converges uniformly on compacts in a neighborhood of $(1,0)$ to the function f determining $\sigma_{\mathrm{p}}(A_1, A_2)$ as the finite rank approximations of compact parts converge to those of A_1 and A_2. Thus, we have the following corollary to the proof of Theorem 7.2.

COROLLARY 7.3. *Suppose that $A_1, A_2 \in \mathcal{E}(H)$, with $(1,0) \in \sigma_{\mathrm{p}}(A_1, A_2)$. Suppose further that $f(x,y)$ is an analytic function that determines $\sigma_{\mathrm{p}}(A_1, A_2)$ near $(1,0)$ and $\frac{\partial f}{\partial x}|_{(1,0)} \neq 0$. If P_1 is the orthogonal projection onto the eigensubspace of A_1 corresponding to $\lambda = 1$ and T is given by* (4.13), *then*

$$P_1 A_2 P_1 = -x'(0) P_1, \tag{7.17}$$

$$P_1 A_2 T A_2 P_1 = -\frac{x''(0)}{2} P_1, \tag{7.18}$$

where $x(y)$ is the implicit function near $y = 0$ determined by $f(x,y) = 0, x(0) = 1$.

If in Theorem 7.2 the norm of A_2 is equal to $|\beta|$, then no condition on $\sigma_{\mathrm{p}}(A_1^{-1}, A_2)$ is necessary for the existence of a common almost eigenvector.

THEOREM 7.4. *Let A_1, A_2 be compact and $\|A_2\| = |\beta|$. Suppose that $\alpha \neq 0$ and $(\frac{1}{\alpha}, 0)$ belongs to $\sigma_{\mathrm{p}}(A_1, A_2)$. If there exist $\rho > 0$ and $0 < \varepsilon \ll \rho$ such that:*

(i) *the Hausdorff distance from $\sigma(A_1, A_2)_\rho(\frac{1}{\alpha}, 0)$ to the line $\{\alpha x + \beta y = 1\}$ does not exceed ε;*

(ii) *$\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y} \neq 0$ in the bidisk $\{|x - \frac{1}{\alpha}| \leqslant \rho : |y| \leqslant \rho\}$, where f is an analytic function that determines $\sigma_{\mathrm{p}}(A_1, A_2)$;*

then there is an eigenvector of A_1 that is $\left(\sqrt{\frac{8|\beta|(1+\beta^2)}{\rho - 4\beta\varepsilon}}\sqrt{\varepsilon}\right)$-eigenvector of A_2.

Proof. As we mentioned before, we can replace A_1 with $\frac{A_1}{\alpha}$, so that $\alpha = 1$. Also, as in Theorem 7.2 using an arbitrary small perturbation we may assume that eigenvalue $\lambda = 1$ of A_1 has multiplicity one. Condition (ii) implies that in the bidisk $\{|x-1| \leqslant \rho : |y| \leqslant \rho\}$ the joint spectrum $\sigma_{\mathrm{p}}(A_1, A_2)$ is nonsingular and, therefore, is a smooth analytic curve Γ. Using condition (i) and the argument with passing to the coordinates (7.3) similar to the one that was used in the proof of Theorem 7.2 and the fact that $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\frac{\mathrm{d}u}{\mathrm{d}v}+\beta}{\beta\frac{\mathrm{d}u}{\mathrm{d}v}-1}$ we show that $|-x'(0)-\beta| \leqslant \frac{4(1+\beta^2)\varepsilon}{\rho-4\beta\varepsilon}$. Thus, if e_1 is a unit eigenvector of A_1 with eigenvalue $\lambda = 1$, the relation (7.17) implies

$$|\langle A_2e_1, e_1\rangle| = |x'(0)| \geqslant |\beta| - \frac{4(1+\beta^2)\varepsilon}{\rho - 4\beta\varepsilon}.$$

Hence,

$$\begin{aligned} \|A_2e_1 - \langle A_2e_1, e_1\rangle e_1\|^2 &= \|A_2e_1\|^2 - |\langle A_2e_1, e_1\rangle|^2 \\ &\leqslant \beta^2 - \Big(|\beta| - \frac{4(1+\beta^2)\varepsilon}{\rho-4\beta\varepsilon}\Big)^2 \leqslant \frac{8|\beta|(1+\beta^2)\varepsilon}{\rho-4\beta\varepsilon}. \end{aligned} \tag{7.19}$$ ∎

REMARK 7.5. The condition $\|A_2\| = |\beta|$ can obviously be replaced with $|\|A_2\| - |\beta|| < \delta$. In this case there exists a common $\sqrt{2\beta\delta + \delta^2 + \frac{8\beta(1+\beta^2)\varepsilon}{\rho-4\beta\varepsilon}}$ -eigenvector.

8. NORM ESTIMATES FOR THE COMMUTANT OF A PAIR OF MATRICES

Under the assumptions of Theorem 7.4 we will now define a new operator close to A_2 that has a common eigenvector with A_1. Let as above e_1 be an eigenvector of A_1 with $\lambda = 1$. Write

$$\widehat{A}_2 = P_1A_2P_1 + (I-P_1)A_2(I-P_1).$$

Of course, e_1 is a common eigenvector of A_1 and $\widehat{A}_2$.

Let ξ be a unit vector orthogonal to e_1, that is $\|\xi\| = 1, (I-P_1)\xi = \xi$. We have

$$\|A_2\xi - \widehat{A}_2\xi\| = \|P_1A_2\xi\| = |\langle A_2e_1, \xi\rangle| = |\langle (A_2e_1 - \langle A_2e_1, e_1\rangle e_1), \xi\rangle| \leqslant C\sqrt{\varepsilon},$$

where $C = \sqrt{\frac{8\|A_2\|(1+\|A_2\|^2)}{\rho-4\|A_2\|\varepsilon}}$. For $\zeta = ce_1 + \sqrt{1-|c|^2}\xi$ with $\|\xi\| = 1, (I-P_1)\xi = \xi$ the last relation yields

$$\|(A_2-\widehat{A}_2)\zeta\| \leqslant |c|\|A_2e_1 - \langle A_2e_1, e_1\rangle e_1)\| + \sqrt{1-|c|^2}\|A_2\xi - \widehat{A}_2\xi\| \leqslant \sqrt{2}C\sqrt{\varepsilon}, \tag{8.1}$$

and, therefore,

$$\|A_2 - \widehat{A}_2\| \leqslant \sqrt{2}C\sqrt{\varepsilon}.$$

This gives us the following estimate for the norm of the commutant $[A_1, A_2]$:

$$\|[A_1, A_2]\| \leqslant \|[A_1, (A_2 - \widehat{A}_2]\| + \|[A_1, \widehat{A}_2]\| \leqslant \sqrt{2}C\sqrt{\varepsilon}\|A_1\| + \|[A_1^{(1)}, A_2^{(1)}]\|, \tag{8.2}$$

where $A_1^{(1)} = (I - P_1)A_1(I - P_1), A_2^{(1)} = (I - P_1)A_2(I - P_1)$ are the compressions of A_1 and A_2 to the orthocomplement to e_1.

REMARK 8.1. If the the point $(\frac{1}{\alpha}, 0)$ is not a singular point of the proper joint spectrum of A_1 and A_2 with $\|A_2\| = |\beta|$, and $\sigma_{\mathrm{p}}(A_1, A_2)_\rho(\frac{1}{\alpha}, 0)$ is at less than ε Hausdorff distance from the line $\{\alpha x + \beta y = 1\}$, the inequality (8.2) still holds. This follows from the fact that the pair $(\frac{A_1}{\alpha}, A_2)$ satisfies the conditions of Theorem 7.4.

Now we will use the relation (8.2) to estimate in Theorem 2.4 the norm of the commutant of a pair of self-adjoint $N \times N$ matrices in terms of the Hausdorff distance from the joint spectrum to a set of lines that imitates a joint spectrum of a pair of commuting matrices. Since our result is stable with respect to small perturbations, we assume that both matrices have simple spectra and the absolute values of their eigenvalues are different. Since the commutant of A_1 and A_2 is the same as the commutant of $A_1 + \alpha I$ and $A_2 + \beta I$ for every α, β, we also assume that A_1 and A_2 are invertible, that is all their eigenvalues are nontrivial.

Let $f(z)$ be analytic in the closed disk $\overline{\Delta_\rho(a)} = \{|z - a| \leqslant \rho\}$ and its derivative does not vanish there. Then f is locally univalent in $\overline{\Delta_\rho(a)}$. Write

$$\widetilde{\delta}(w) = \sup\{r : f \text{ is univalent in } \Delta_r(w)\}, \quad \delta(f, \rho, a) = \min\{\widetilde{\delta}(w) : w \in \Delta_\rho(a)\}. \tag{8.3}$$

REMARK 8.2. The above definition of $\delta(f, \rho, a)$ is slightly reminiscent of Bloch's constant B, cf. [23], but, of course, they are very different.

We are now ready to give the proof of Theorem 2.4.

Proof of Theorem 2.4. By Theorem 7.4 the eigenvector $e_{n(1)}$ of A_1 that corresponds to eigenvalues $\alpha_{n(1)}$ is a $\sqrt{\frac{8|\beta_1|(1+|\beta_1|^2)}{\rho - 4|\beta_1|\varepsilon}}\varepsilon$ -eigenvector of A_2, and relation (8.2) holds with P_1 being replaced with $P_{n(1)}$. Write $\varepsilon_1 = \varepsilon$. We want to estimate ε_2 such that the compression of $A_1^{(1)}$ and $A_2^{(1)}$ to $\operatorname{span}\{e_k : k = 1, \ldots, N, k \neq n(1)\}$ satisfy conditions (i) and (ii) of the present theorem with $\varepsilon_2, \frac{\rho}{2}$.

It follows from (7.19) that in the eigenbasis $e_1, \ldots, e_N$ of the matrix A_1 every entry of the $n(1)$-th row (and column) of the matrix A_2 except for the one on the main diagonal has absolute value that does not exceed $C_1\sqrt{\varepsilon_1}$, where $C_1 = \sqrt{\frac{8|\beta_1|(1+|\beta_1|^2)}{\rho - 4|\beta_1|\varepsilon_1}}$. Let

$$\mathcal{P}(x, y) = \det[xA_1 + yA_2 - I], \quad \mathcal{P}_1(x, y) = \det[xA_1^{(1)} + yA_2^{(1)} - I],$$

and let

$$d = \min\Big\{\Big|\alpha_{n(j)}\frac{\partial\mathcal{P}}{\partial x} + \beta_j\frac{\partial\mathcal{P}}{\partial y}\Big| : \Big|x - \frac{1}{\alpha_{n(j)}}\Big| \leqslant \varepsilon, |y| \leqslant \varepsilon, 1 \leqslant j \leqslant N\Big\}. \tag{8.4}$$

Of course, the determining polynomials $\mathcal{P}$ and $\mathcal{P}_1$ satisfy

$$\mathcal{P}(x,y) = (\alpha_{n(1)}x + \beta_1 y - 1)\mathcal{P}_1(x,y) + \mathcal{Q}(x,y), \tag{8.5}$$

where $\mathcal{Q}$ is a polynomial of degree N whose coefficients in absolute values do not exceed $NC_1|\beta_1|^{N-1}\sqrt{\varepsilon_1}$. Write

$$M = \max\Big\{\Big|\frac{1}{\alpha_j}\Big| + \rho + 1\Big\}.$$

We obviously have

$$|\mathcal{P}(x,y)| \leqslant (|\alpha_{n(j_1)}| + |\beta_1| + 1)M|\mathcal{P}_1(x,y)| + NC_1|\beta_1|^{N-1}M^N\sqrt{\varepsilon_1}.$$

Now, let $(x,y) \in \sigma_{\mathrm{p}}(A_1^{(1)}, A_2^{(1)}) \cap \{|x - \frac{1}{\alpha_{n(m)}}| \leqslant \frac{\rho}{2}, |y| \leqslant \frac{\rho}{2}\}$ for some $2 \leqslant m \leqslant N$. Then $|\mathcal{P}(x,y)| \leqslant NC_1|\beta_1|^{N-1}M^N\sqrt{\varepsilon_1}$. Write $f(t) = \mathcal{P}(x + t\alpha_{n(m)}, y + t\beta_m)$. Equation (8.4) implies that

$$|f'(t)| \geqslant d > 0 \tag{8.6}$$

for $|t| \leqslant \frac{\rho}{2}$, and, therefore, $f(t)$ is locally univalent in the disk $\{|t| \leqslant \frac{\rho}{2}\}$. By (8.3) f is univalent in the disk of radius $\tau = \min\{\delta(F_{x,y}, \rho, 0) : |x - \frac{1}{\alpha_m}| \leqslant \frac{\rho}{2}, |y| \leqslant \frac{\rho}{2}\}$, where $F_{x,y}(w) = \mathcal{P}(x + w\alpha_{n(m)}, y + w\beta_m)$, so that this radius does not depend on the point (x,y). By (8.6) $|f'(0)| \geqslant d$, so Koebe's $\frac{1}{4}$ theorem, cf. p. 150 of [19], implies that if ε_1 is small enough so that $NC_1|\beta_1|^{N-1}M^N\sqrt{\varepsilon_1} < \frac{d\tau}{4}$, the function f has a zero in $\{|t| \leqslant 4NC_1|\beta_1|^{N-1}M^N\sqrt{\varepsilon_1}\}$, and, hence, the distance from

$$\sigma_{\mathrm{p}}(A_1^{(1)}, A_2^{(1)}) \cap \Big\{\Big|x - \frac{1}{\alpha_{n(m)}}\Big| \leqslant \frac{\rho}{2}, |y| \leqslant \frac{\rho}{2}\Big\}$$

to $\sigma(A_1, A_2)$ does not exceed $4NC_1|\beta_1|^{N-1}M^N\sqrt{\varepsilon_1}$, and, therefore, the distance from

$$\sigma_{\mathrm{p}}(A_1^{(1)}, A_2^{(1)}) \cap \Big\{\Big|x - \frac{1}{\alpha_m}\Big| \leqslant \frac{\rho}{2}, |y| \leqslant \frac{\rho}{2}\Big\}$$

to the line $\{\alpha_{n(m)}x + \beta_m y = 1\}$ does not exceed $\varepsilon_2 = 5NC_1|\beta_1|^{N-1}M^N\sqrt{\varepsilon_1}$.

The fact that the eigenvalues of $A_2^{(1)}$ differ from those of A_2 by a magnitude of order $\sqrt{\varepsilon_1}$ follows directly from (8.1) and the fact that for two compact normal operators the distance between their spectra does not exceed the distance between them in the operator norm topology, cf. Proposition 1 of [15].

Fanally, it follows from (8.5) that the difference between $\alpha_{n(k)}\frac{\partial\mathcal{P}_1}{\partial x} + \beta_k\frac{\partial\mathcal{P}_1}{\partial y}$, $k \neq 1$ and $\alpha_{n(k)}\frac{\partial\mathcal{P}}{\partial x} + \beta_k\frac{\partial\mathcal{P}}{\partial y}$ is of order of ε_2, and, therefore, the $(N-1) \times (N-1)$-dimensional matrices $A_1^{(1)}$ and $A_2^{(1)}$ satisfy the conditions of this theorem with ε_2 and $\frac{\rho}{2}$. Continuing inductively we arrive at the claimed estimate. ∎

9. CONCLUDING REMARKS

Here we want to state and discuss three problems immediately related to the material of the preceding sections.

1. It was mentioned that, according to [27], if $A_1, \ldots, A_n$ are compact, then on every compact subset of $\mathbb{C}^n$ the joint spectrum $\sigma_{\mathrm{p}}(A_1, \ldots, A_n)$ has a global defining function. Our first problem is the following.

PROBLEM 9.1. *Describe those globally defined in $\mathbb{C}^n$ analytic sets that have spectral representation. In particular, which real globally defined analytic sets (that is, zeros of entire functions with real Taylor coefficients) have self-adjoint spectral representation?*

In the case of matrices the similar problem for general matrices was solved by Dickson [9] and for self-adjoint matrices by Helton and Vinnikov [17].

2. It is natural to ask whether a complete analog of Theorem 2.2 for algebraic curves of order higher than 1 is valid. More precisely, we are compelled to pose the following problem.

PROBLEM 9.2. *Suppose that A and B are self-adjoint, A is invertible, and the joint spectrum $\sigma_{\mathrm{p}}(A, B)$ contains a real algebraic curve Γ of order k that meets the x- and y-axes at points $(\alpha_1, 0), \ldots, (\alpha_k, 0)$ and $(0, \beta_1), \ldots, (0, \beta_k)$, respectively. Also suppose that all these points of intersection of Γ with the coordinate axes are isolated spectral points of the corresponding operators. If $\sigma_{\mathrm{p}}(A^{-1}, B)$ contains an algebraic curve Γ_1 of the same order k that meets the coordinate axes at points $(\frac{1}{\alpha_1}, 0), \ldots, (\frac{1}{\alpha_k}, 0)$ and $(0, \beta_1), \ldots, (0, \beta_k)$ respectively, does this imply that A and B have a common k-dimensional reducing subspace?*

3. The last problem we would like to mention is related to the norm estimate of the commutant of two matrices, or more generally two compact operators. The estimate given by Theorem 2.4 seems to be rather rough. Besides, this estimate does not allow any extension of the result of Theorem 2.4 to an infinite dimensional case. One possible way of improving the estimate is to consider that the proper spectrum is close to a set of lines not locally, but on a big compact subset of $\mathbb{C}^2$. Alternatively, we might impose the condition that the joint projective spectrum in $\mathbb{CP}^2$ is close to the set of projective lines in Fubini study metric. This leads us to the following problem.

PROBLEM 9.3. *Let A and B be self-adjoint compact operators acting on a separable Hilbert space H. Suppose that the distance from $\sigma(A, B, I)$ to a set of projective lines that contains $\{[x : y : 0]\}$ and satisfies the following condition:*

(C) *the intersection of this set of lines with the lines $\{[0 : y : z]\}$ and $\{[x : 0 : z]\}$ coincides with the inverse spectra $\sigma(A)^{-1}$ and $\sigma(B)^{-1}$, counting multiplicities;*

does not exceed ε in the Fubini study metric. Estimate the norm of the commutant $[A, B]$ in terms of ε.

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