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ANGULAR DERIVATIVES AND COMPACTNESS OF COMPOSITION OPERATORS ON HARDY SPACES

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ABSTRACT. Let D_0 be a simply connected subdomain of the unit disk and A be a compact subset of D_0. Let ϕ be a universal covering map for $D_0 \setminus A$. We prove that the composition operator C_ϕ is compact on the Hardy space H^p if and only if ϕ does not have an angular derivative at any point of the unit circle. This result extends a theorem of M.M. Jones.

KEYWORDS: *Composition operator, Hardy space, universal covering map, angular derivative, Green function, Lindelöf principle.*

MSC (2010): 47B33, 30H10, 30C80.

1. INTRODUCTION

A holomorphic self-map ϕ of the unit disk $\mathbb{D}$ induces a composition operator defined by the equation

$$C_\phi f = f \circ \phi,$$

for functions f holomorphic on the unit disk. By the Littlewood subordination principle, this operator is bounded on the classical Hardy space H^p, $0 < p \leqslant \infty$. The main theme in the study of composition operators is to find relations between geometric-analytic properties of ϕ and operator theoretic properties of C_ϕ. The basic results of this theory are presented in the books [3] and [14]. Here we are interested in the compactness of composition operators and how it is related to the angular derivatives of ϕ.

The connection between compactness and angular derivatives was discovered by J.H. Shapiro and P.D. Taylor [15]. They proved that if C_ϕ is compact on H^p for some p, then C_ϕ is compact on H^p for every p, and moreover, compactness of C_ϕ implies that ϕ does not have angular derivative at any point of the unit circle. The converse is not true; see Chapter 10 of [14] and references therein. If, however, ϕ is univalent (or boundedly valent), then it was proved by B.D. MacCluer and J.H. Shapiro [8] that nonexistence of the angular derivative of ϕ characterizes the compact composition operators C_ϕ. The general characterization of compact composition operators was discoverd by Shapiro [13]: C_ϕ is compact on H^p if and only if

$$\lim_{|w|\to 1-} \frac{N_\phi(w)}{\log(1/|w|)} = 0, \tag{1.1}$$

where N_ϕ denotes the Nevanlinna counting function for ϕ; see Subsection 2.1 for the definition.

M.M. Jones [6], [7] considered the case when ϕ is a universal covering map of $\mathbb{D}$ onto a finitely connected domain $D \subset \mathbb{D}$. He proved that C_ϕ is compact on H^p if and only if ϕ does not have angular derivative at any point of $\partial\mathbb{D}$. Moreover, if A is the union of the bounded complementary components (the "holes") of D, Jones considered a Riemann map ψ of $\mathbb{D}$ onto the simply connected domain $D_o = D \cup A$ and proved that if C_ϕ is compact on H^p, then so is C_ψ.

Jones used tools such as Fuchsian groups, Dirichlet fundamental polygons, and Poincaré series. We will use a different set of tools (Green functions, subordination, prime ends) to prove a stronger result.

THEOREM 1.1. *Let $D_o \subset \mathbb{D}$ be a simply connected domain. Let $D \subset D_o$ be a domain and assume that $D_o \setminus D$ is a compact subset of D_o. Let ϕ be a universal covering map of $\mathbb{D}$ onto D and let ψ be a Riemann map of $\mathbb{D}$ onto D_o. The following are equivalent:*

(i) *C_ϕ is compact on H^p, $0 < p \leqslant \infty$;*
(ii) *C_ψ is compact on H^p, $0 < p \leqslant \infty$;*
(iii) *ϕ does not have an angular derivative at any point of the unit circle;*
(iv) *ψ does not have an angular derivative at any point of the unit circle.*

The domain D in the theorem may be infinitely connected. The main assumption is that $D_o \setminus D$ is a compact subset of D_o; that is, the holes of D do not accumulate on the boundary of D_o. This assumption cannot be omitted. Indeed, consider the set of dyadic points

$$A = \left\{ w_{nk} = \left(1 - \frac{1}{2^n}\right) \exp\left(\frac{i\pi(2k-1)}{2^n}\right) : k = 1, 2, \dots, 2^n, \;\; n = 1, 2, \dots \right\}.$$

Let ϕ be a universal covering map of $\mathbb{D}$ onto $\mathbb{D} \setminus A$. Then ϕ does not have an angular derivative at any point $\zeta \in \partial\mathbb{D}$ because if it had, then by a theorem of Ch. Pommerenke ([11], p. 291, [13], p. 383) $\phi(\mathbb{D})$ would contain a small angular region with vertex at $\phi(\zeta)$; this cannot happen because every such region contains dyadic points. On the other hand, ϕ is an inner function (see p. 37 of [2]) and therefore C_ϕ is not compact on H^p (see p. 382 of [13]).

The proof of Theorem 1.1 is in Section 3 after some background material presented in Section 2.

2. BACKGROUND MATERIAL

2.1. GREEN AND NEVANLINNA FUNCTIONS. A bounded planar domain D possesses a Green function $g_D(z,w)$, $z,w \in D$, $z \neq w$; see [9], [12]. An important property of the Green function is the Lindelöf principle. Suppose that D and Ω are bounded domains in the complex plane. Let ϕ be a holomorphic function mapping D into Ω. If $w \in \Omega$, we denote by z_j the pre-images of w under ϕ with the usual convention that each pre-image is repeated as many times as its multiplicity. If $a \in D$, then for all $w \in \Omega \setminus \{\phi(a)\}$,

$$\sum_j g_D(z_j, a) \leqslant g_\Omega(w, \phi(a)). \tag{2.1}$$

Moreover, if ϕ is a universal covering map of D onto Ω, then equality holds in (2.1). Additional information about this result and its applications can be found in [1] and the references therein.

If $\phi : \mathbb{D} \to \mathbb{D}$ is holomorphic, the Nevanlinna counting function for ϕ is defined for all $w \in \mathbb{D} \setminus \{\phi(0)\}$ by:

$$N_\phi(w) = \begin{cases} \sum\limits_j \log(1/|z_j|) & w \in \phi(\mathbb{D}), \\ 0 & w \notin \phi(\mathbb{D}). \end{cases}$$

Since $g_{\mathbb{D}}(z,0) = -\log|z|$, the Lindelöf principle implies that

$$N_\phi(w) \leqslant g_{\phi(\mathbb{D})}(w, \phi(0))$$

with equality if ϕ is a universal covering map.

2.2. ANGULAR DERIVATIVES. Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. We say that ϕ has an angular derivative at a point $\zeta \in \partial\mathbb{D}$ if there is a point $\omega \in \partial\mathbb{D}$ such that the angular (= non-tangential) limit

$$\angle \lim_{z \to \zeta} \frac{\phi(z) - \omega}{z - \zeta}$$

exists (finitely). This limit, if it exists, is called the angular derivative of ϕ at ζ. The main theorem concerning the angular derivative is the Julia–Carathéodory theorem which asserts that the following three conditions are equivalent:

(a) ϕ has an angular derivative at ζ;
(b) ϕ has angular limit of modulus 1 at ζ and ϕ' has angular limit at ζ;
(c) the limit

$$\liminf_{z \to \zeta} \frac{1 - |\phi(z)|}{1 - |z|}$$

exists.

We refer to the books [3], [9], [10], [14] for presentations of the theory of the angular derivative.

2.3. SUBORDINATION. Let σ, ϕ be holomorphic functions on $\mathbb{D}$. We say that σ is subordinate to ϕ if there exists a holomorphic function $\omega : \mathbb{D} \to \mathbb{D}$ with the properties

$$|\omega(z)| \leqslant |z|, \quad \sigma(z) = \phi(\omega(z)) \quad z \in \mathbb{D}.$$

We will need two basic results (see Section 2.8 of [5]):

(a) If σ is holomorphic and maps $\mathbb{D}$ onto a domain D and ϕ is a universal covering map of $\mathbb{D}$ onto D with $\sigma(0) = \phi(0)$, then σ is subordinate to ϕ.

(b) If σ is subordinate to ϕ, then

$$\max_{|z|=r} |\sigma(z)| \leqslant \max_{|z|=r} |\phi(z)| \quad 0 < r < 1.$$

2.4. PRIME ENDS. We will use some basic facts about prime ends. We refer to Section 9.2 of [10] for a presentation of the theory of prime ends.

3. PROOF OF THEOREM 1.1

The implications (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) come from the result of Shapiro and Taylor. The implication (iv) $\Rightarrow$ (ii) follows from the result of MacCluer and Shapiro. Also, (ii) $\Rightarrow$ (i) holds by subordination; see Section 2.

Proof of (iii) $\Rightarrow$ (i). Since nonexistence of the angular derivative and compactness of composition operators are not affected by a composition with a conformal automorphism of $\mathbb{D}$, we may assume that $0 \in D$ and $\phi(0) = \psi(0) = 0$. By Shapiro's theorem, we need to show that

$$\lim_{|w| \to 1-} \frac{N_\phi(w)}{\log(1/|w|)} = 0. \tag{3.1}$$

Let (w_n) be a sequence in $\mathbb{D}$ with $|w_n| \to 1$. By the definition of the Nevanlinna counting function, we may assume that $w_n \in D_o$ for every n. Since the prime end compactification $\widehat{D}_o$ of D_o is a compact set, the sequence (w_n) has a subsequence converging to a point in $\widehat{D}_o$. The limit does not belong to D_o because $|w_n| \to 1$. Therefore, the limit is a prime end. We use now an elementary fact about sequences of real numbers: if every subsequence of a sequence (a_n) has a subsequence converging to a, then $a_n \to a$. Thus, to prove (3.1), we may assume that $w_n \to P$, where P is a prime end of D_o whose impression contains at least one point of the unit circle. We may further assume that this point is 1. By Carathéodory's theorem ([2], Chapter 9, [10], Section 9.2) we may extend the conformal map ψ to a function mapping $\overline{\mathbb{D}}$ onto the prime end compactification $\widehat{D}_o$ of D_o. We may assume that $\psi(1) = P$.

Set $A = D_o \setminus D$, the union of the holes of D. Since A is a compact subset of D_o, $\psi^{-1}(A)$ is a compact subset of $\mathbb{D}$. So there exists a positive number $r_o < 1$ such that the set $\Delta := \{z \in \mathbb{D} : |z - 1| < r_o\}$ does not intersect $\psi^{-1}(A)$. Set $\Omega = \psi(\Delta)$. Then $\Omega \subset D_o \setminus A = D$.

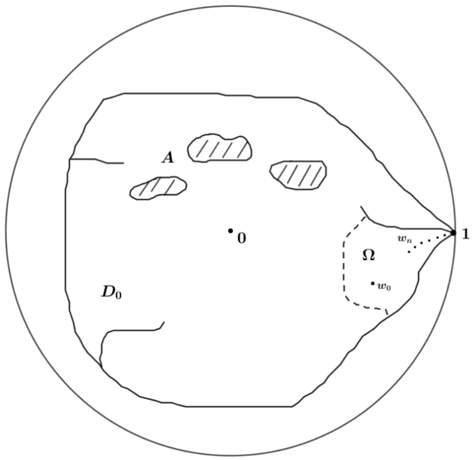

FIGURE 1. The simply connected domain $\Omega \subset D = D_o \setminus A$. The points w_o and w_n belong to Ω.

We may assume that Ω contains all the points w_n. Fix a point $w_o \in \Omega$ and let $z_o = \psi^{-1}(w_o) \in \Delta$. Let also $z_n := \psi^{-1}(w_n) \in \mathbb{D}$. For every w_n, let $z_{n,j}$, $j = 1, 2, \dots$ be the preimages of w_n under ϕ.

By the Lindelöf principle, the domain monotonicity and the conformal invariance of the Green function, for every w_n,

$$\begin{aligned} N_\phi(w_n) &= \sum_j \log(1/|z_{n,j}|) = \sum_j g_{\mathbb{D}}(z_{n,j}, 0) \\ &\leqslant g_D(w_n, 0) \leqslant g_{D_o}(w_n, 0) = g_{\mathbb{D}}(z_n, 0). \end{aligned} \tag{3.2}$$

By a boundary Harnack principle for the Green function (see Lemma 7 of [4]),

$$g_{\mathbb{D}}(z_n, 0) \leqslant C\, g_\Delta(z_n, z_o) = C\, g_\Omega(w_n, w_o), \tag{3.3}$$

where $C > 0$ is a constant independent of n. It follows from (3.2) and (3.3) that

$$N_\phi(w_n) \leqslant C\, g_\Omega(w_n, w_o). \tag{3.4}$$

The prime end P is determined by a null-chain of cross cuts of D_o. All but finitely many of these cross-cuts belong to Ω and therefore they determine a prime end of Ω which we continue to denote by P. Consider the conformal mapping σ of $\mathbb{D}$ onto Ω with $\sigma(0) = w_o$ and $\sigma(1) = P$. Choose a point $\zeta_o \in \mathbb{D}$ with $\phi(\zeta_o) = w_o$. Let τ be the conformal automorphism of $\mathbb{D}$ with $\tau(0) = \zeta_o$ and $\tau(1) = 1$. Then the function $\phi_1 = \phi \circ \tau$ is a universal covering map of $\mathbb{D}$ onto D with $\phi_1(0) = w_o$. Set $z'_n := \sigma^{-1}(w_n)$. By conformal invariance,

$$g_\Omega(w_n, w_o) = g_{\mathbb{D}}(z'_n, 0) = \log(1/|z'_n|) \leqslant 2(1 - |z'_n|). \tag{3.5}$$

Since $\sigma(\mathbb{D}) = \Omega \subset D = \phi_1(\mathbb{D})$ and $\sigma(0) = \phi_1(0) = w_o$, the function σ is subordinate to ϕ_1; see Subsection 2.3. Therefore,

$$\max_{|z|=r}|\sigma(z)| \leqslant \max_{|z|=r}|\phi_1(z)| \quad 0 < r < 1. \tag{3.6}$$

Let $z_n'' \in \mathbb{D}$ be such that

$$|z_n''| = |z_n'| \quad \text{and} \quad |\phi_1(z_n'')| = \max_{|z|=|z_n'|}|\phi_1(z)|. \tag{3.7}$$

Then, by (3.6), $|\sigma(z_n')| \leqslant |\phi_1(z_n'')|$. Hence, (3.4) and (3.5) yield

$$\frac{N_\phi(w_n)}{\log(1/|w_n|)} \leqslant 2C\,\frac{1-|z_n'|}{1-|\sigma(z_n')|} \leqslant 2C\,\frac{1-|z_n'|}{1-|\phi_1(z_n'')|} = 2C\,\frac{1-|z_n''|}{1-|\phi_1(z_n'')|}. \tag{3.8}$$

Consider the Carathéodory extension $\sigma : \widehat{\mathbb{D}} \to \widehat{\Omega}$ of the conformal mapping σ. Since $\sigma(1) = P$ and $w_n \to P$, we infer that

$$|z_n''| = |z_n'| = |\sigma^{-1}(w_n)| \to 1 \quad \text{as } n \to +\infty. \tag{3.9}$$

Now we use the assumption (iii) that ϕ does not have an angular derivative at any point of the unit circle. It follows that the same is true for the function $\phi_1 = \phi \circ \tau$. So, by the Julia–Carathéodory theorem (see Subsection 2.2),

$$\lim_{n\to+\infty} \frac{1-|\phi_1(z_n'')|}{1-|z_n''|} = +\infty. \tag{3.10}$$

It follows from (3.8) and (3.10) that

$$\lim_{n\to+\infty} \frac{N_\phi(w_n)}{\log(1/|w_n|)} = 0. \tag{3.11}$$

So (3.1) has been proved. ∎

Proof of (i) ⇒ (ii). Suppose that C_ϕ is compact. As above, to prove that C_ψ is compact it suffices to show that

$$\lim_{n\to+\infty} \frac{N_\psi(w_n)}{\log(1/|w_n|)} = 0, \tag{3.12}$$

where (w_n) is a sequence in D_o with $|w_n| \to 1$ and $w_n \to P$, where P is a prime end of D_o. Using the notation we set above, the conformal invariance of the Green function, and the boundary Harnack principle, we obtain

$$g_{D_o}(w_n, 0) = g_{\mathbb{D}}(z_n, 0) \leqslant C\, g_\Delta(z_n, z_o) = C\, g_\Omega(w_n, w_o). \tag{3.13}$$

It follows from (3.13) and the Lindelöf principle (equality case) that

$$\begin{aligned} \frac{N_\psi(w_n)}{\log(1/|w_n|)} &= \frac{g_{D_o}(w_n,0)}{\log(1/|w_n|)} \leqslant \frac{g_{D_o}(w_n,0)}{1-|w_n|} \\ &\leqslant C\,\frac{g_\Omega(w_n,w_o)}{1-|w_n|} \leqslant C\,\frac{g_D(w_n,w_o)}{1-|w_n|}. \end{aligned} \tag{3.14}$$

Consider the conformal automorhism τ of the unit disk defined above. Then $\phi_1 := \phi \circ \tau$ is a universal covering map of D with $\phi_1(0) = w_o$. For each n, let

$\zeta_{n,j}$ be the sequence of pre-images of w_n under ϕ_1. By the Lindelöf principle (see Subsection 2.1),

$$g_D(w_n, w_o) = \sum_j g_{\mathbb{D}}(\zeta_{n,j}, 0) = N_{\phi_1}(w_n). \tag{3.15}$$

Combining (3.14) and (3.15), we obtain

$$\frac{N_\psi(w_n)}{\log(1/|w_n|)} \leqslant C\,\frac{N_{\phi_1}(w_n)}{1-|w_n|} \leqslant 2C\,\frac{N_{\phi_1}(w_n)}{\log(1/|w_n|)}, \tag{3.16}$$

for all sufficiently large n. Since C_ϕ is compact, so is C_{ϕ_1}. Now (3.12) follows from Shapiro's theorem and (3.16). ∎

4. REMARKS

REMARK 4.1. If ϕ is a universal covering map of $\mathbb{D}$ onto a domain $\Omega \subset \mathbb{D}$ and $b : \mathbb{D} \to \mathbb{D}$ is an inner function with $\phi(0) = 0$, the equality holds in Lindelöf's principle for the function $\phi \circ b$; see Theorem 1.3 of [1]. Therefore

$$N_{\phi\circ b}(w) = g_\Omega(w, \phi(0)) \quad w \in \Omega \setminus \{\phi(0)\}. \tag{4.1}$$

This equality has the following consequences:

(1a) Let Ω_1, Ω_2 be two domains in $\mathbb{D}$. Suppose that the logarithmic capacity of the set $(\Omega_1 \setminus \Omega_2) \cup (\Omega_2 \setminus \Omega_1)$ is zero so that the Green functions of Ω_1, Ω_2 coincide. Let $\phi_1 : \mathbb{D} \to \Omega_1$, $\phi_2 : \mathbb{D} \to \Omega_2$ be the corresponding universal covering maps and let b be an inner function with $b(0) = 0$. Then $C_{\phi_1 \circ b}$ is compact on H^p if and only if $C_{\phi_2\circ b}$ is compact on H^p. This fact provides a short proof of Theorem 3 in [6].

(1b) If ϕ is a holomorphic self-map of $\mathbb{D}$, let $\|C_\phi\|_{\mathrm{e}}$ denote the essential norm of C_ϕ on H^p. By Shapiro's well-known formula [13],

$$\|C_\phi\|_{\mathrm{e}} = \limsup_{|w|\to 1-} \frac{N_\phi(w)}{\log(1/|w|)}. \tag{4.2}$$

By using (4.1), we see that for universal covering maps the essential norm is conformally invariant. More precisely: if g maps a domain $D_1 \subset \mathbb{D}$ conformally onto a domain $D_2 \subset \mathbb{D}$ and $\psi_1 : \mathbb{D} \to D_1$ is a universal covering maps, then

$$\|C_{\psi_1}\|_{\mathrm{e}} = \|C_{g\circ\psi_1}\|_{\mathrm{e}}. \tag{4.3}$$

REMARK 4.2. Suppose that D is a domain and $\phi : \mathbb{D} \to D$ is a universal covering map satisfying the assumptions of Theorem 1.1. The proof of Theorem 1.1 shows that there exists a constant $C > 0$, that depends only on the geometry of D, such that

$$\|C_\phi\|_{\mathrm{e}}^2 \leqslant C\ \sup\left\{\frac{1}{|\phi'(\zeta)|} : \zeta \in \partial\mathbb{D}\right\}; \tag{4.4}$$

cf. p. 386 of [13]. An inequality in the opposite direction holds for general ϕ; see p. 385 of [13].

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